UNIQUENESS OF A CENTRALLY SYMMETRIC GRAVITATIONAL FIELD IN VACUUM

R. I. KHRAPKO

Moscow Aviation Institute

Submitted to JETP editor July 5, 1965.

J. Exptl. Theoret. Phys. (U.S.S.R.) 49, 1884-1886 (December, 1965)

It is shown that the system of coordinates usually used for describing a portion of the Friedmann world and the empty region adjacent to it do not satisfy the Lichnerowicz continuity conditions. The coordinate system admissible in the sense of Lichnerowicz has the metric found by Petrov,[6] and, for an empty region, such a system is equivalent to the Schwarzschild coordinate system.

1. IN the work of Oppenheimer and Snyder,[1] and also in more recent articles[2, 3] dealing with the question of the joining together of an empty spherically symmetric region with a portion of the Friedmann world (briefly—the Friedmann part), the method of a freely falling reference frame, which in the region occupied by matter is comoving, is used. However, as a simple argument indicates, such a reference frame is known not to satisfy the Lichnerowicz continuity conditions.

Let us consider for definiteness the case of expansion of the Friedmann part, and denote the radial component of the metric tensor by g_{RR}, and the radial and time coordinates by R and τ. Then it is obvious that $\partial g_{RR}/\partial\tau > 0$ inside the Friedmann part. At the same time, $\partial g_{RR}/\partial\tau < 0$ for the empty region, since free test particles, which contribute the system of coordinates in the present case, approach each other during motion along one radius from the center (if they never collide). The latter assertion can immediately be verified for "parabolic" motion with the aid of the Lemaitre metric. In view of the different sign of the derivative $\partial g_{RR}/\partial\tau$ inside and outside the Friedmann part, the appearance of a discontinuity in the metric coefficient g_{RR} itself is inevitable on the boundary, even if this discontinuity is absent at an isolated moment of time.

A more rigorous proof is based on the well known equation for the angular metric coefficient r^2 (the factor preceding $d\theta^2 + \sin^2\theta\, d\phi^2$):[4]

$$\tau - \Phi = f^{-1}\sqrt{fr^2 + Fr} + F(-f)^{-3/2} \arcsin\sqrt{-fr/F}.$$

Here $\Phi(R)$, f(R), and F(R) are functions having discontinuities in their derivatives on the boundary of the Friedmann part. Therefore, continuity of the derivative r' (even if it is observed at a certain moment of time) cannot be preserved as time passes. But this implies a discontinuity of the metric coefficient

$$g_{RR} = r'^2/(1+f).$$

Nevertheless, utilization of a freely falling coordinate system is permissible in the cases under consideration, and the physical conclusions obtained here are correct because, in spite of the discontinuity of the metric, this system describes a real continuous space-time.

The existence of a piecewise analytic transformation to a system of coordinates, covering the surface of the indicated discontinuity and satisfying the Lichnerowicz continuity conditions on it, serves as a proof of the last assertion. The approximate metric for such a system of coordinates is given in the article by Einstein and Straus;[5] the exact metric is given in the present article.

2. In order to determine the exact metric of the coordinate system, satisfying the Lichnerowicz continuity conditions on the boundary of the Friedmann part, we start, following Einstein and Straus,[5] from the interval

$$ds^2 = -A d\tau^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) + r^2 R^{-2} dR^2, \quad (1)$$

where $r(\tau, R)$ and $A(\tau, R)$ are the continuous functions, which are unknown for the empty region, but have for the Friedmann part the usual expressions (we confine ourselves to the case of a closed world):

$$r = a(\tau)R/(1 + R^2/4), \quad (2)$$

$$A = 1, \quad (3)$$

where $a(\tau)$ satisfies the differential equation

$$a\dot{a}^2 = a_0 - a.$$

Integration of the Einstein equations for the empty region leads to the relations[6]

$$r' = \pm\sqrt{r^4\nu^2 + r^2 - rr_0} / R, \qquad (4)$$

$$A = \dot{r}^2 / \nu^2 r^2, \qquad (5)$$

in which the plus sign corresponds to the case of a "star," and the minus sign corresponds to the case of a "vacuole."

Subsequent integration of Eq. (4) gives

$$r = r_0 / [{}^1/_3 - 4\wp(\pm \ln R + \lambda, r_0^2\nu)], \qquad (6)$$

where r_0 is a constant, and $\nu(\tau)$ and $\lambda(\tau)$ are arbitrary functions determined by joining the solutions together; $\wp$ is the Weierstrass elliptic function, $\pm \ln R + \lambda$ is its argument, and $r_0^2\nu$ defines its invariant.

Now let us carry out the joining of the functions (2) and (3) with the functions (5) and (6) at the boundary value $R = R_b$ for all values of the time τ. We require fulfillment of the equations

$$\frac{a(\tau) R_b}{1 + R_b^2/4} = [r]_{R_b} \equiv \frac{r_0}{{}^1/_3 - 4\wp(\pm \ln R_b + \lambda, r_0^2\nu)}, \qquad (7)$$

$$1 = [\dot{r}^2 / \nu^2 r^2]_{R_b}. \qquad (8)$$

Then $\lambda(\tau)$ is determined from Eq. (7), and from Eq. (8) we have

$$\nu^2 = (a_0 - a) / a^3. \qquad (9)$$

Upon satisfaction of (7) and (8), the derivatives of the metric tensor automatically turn out to be continuous. In fact, it can immediately be verified that

$$\left[\frac{\partial}{\partial R} \frac{a^2 R^2}{(1 + R^2/4)^2}\right]_{R_b} = \left[\frac{\partial}{\partial R} r^2\right]_{R_b}, \qquad (10)$$

provided that

$$r_0 = a_0 R_b^3 (1 + R_b^2/4)^{-3}, \qquad (11)$$

and then

$$0 = \left[\frac{\partial}{\partial R} \frac{\dot{r}^2}{\nu^2 r^2}\right]_{R_b}. \qquad (12)$$

3. For the empty region, the system of coordinates τ, R with the metric (1) admits a transformation to the Schwarzschild coordinate system t, r:

$$ds^2 = -\frac{r - r_0}{r} dt^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + \frac{r}{r - r_0} dr^2.$$

In order to determine (in quadratures) the function $t(\tau, R)$, we consider the equations relating the components of the metric tensors:

$$-\frac{r - r_0}{r} \dot{t}t' + \frac{r}{r - r_0} \dot{r}r' = 0,$$

$$-\frac{r - r_0}{r} \dot{t}^2 + \frac{r}{r - r_0} \dot{r}^2 = -\frac{\dot{r}^2}{r^2\nu^2},$$

$$-\frac{r - r_0}{r} t'^2 + \frac{r}{r - r_0} r'^2 = \frac{r^2}{R}.$$

This system of three equations with two unknown functions $\dot{t}$ and t' is consistent, as one can easily verify. Its solution

$$\dot{t} = -\dot{r}r' / r(r - r_0)\nu, \qquad t' = -r^3\nu / 2R(r - r_0)$$

satisfies the equation $(t')^{\cdot} = (\dot{t})'$, which is directly verified by the use of the differential consequences of Eq. (4). This proves the existence of a function $t(\tau, R)$ and thus removes the doubt expressed by Unt[7] with regard to the validity of Birkhoff's theorem in the case of a pulsating central body.

[1] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

[2] I. D. Novikov, Vestnik, Moscow State University, Series III, No. 5 (1962).

[3] I. D. Novikov, Astronom. zh. **40**, 772 (1962), Soviet Astronomy A. J. **7**, 587 (1963).

[4] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Fizmatgiz, 1960.

[5] A. Einstein and E. G. Straus, Revs. Modern Phys. **17**, 120 (1945).

[6] A. Z. Petrov, JETP **44**, 1525 (1963), Soviet Phys. JETP **17**, 1026 (1963).

[7] V. Unt, Issledovaniya po teoreticheskoi fizike (Investigations in Theoretical Physics), Tartu, 1962.

Translated by H. H. Nickle