STATIONARY CURRENT DISTRIBUTION IN AN AXIALLY-SYMMETRIC CONDUCTOR WITH STRONG HALL EFFECT

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The current flow in a slightly deformed axially symmetric conductor is considered in the case of a strong Hall effect. It is shown by numerical calculations and by analytic methods that (1) in a ridged conductor the current is pushed out of the ridges, (2) in a cavity made up of crossed corrugations the current flows in a thin layer which skirts the walls. The current layers near the walls are calculated.

1. INTRODUCTION

A paper by one of the authors and Shubin[1] considered some electromagnetic effects in a slightly deformed conducting medium in the presence of a strong Hall effect. It was assumed that in such a medium Ohm's law can be written in the form[1)]

$$\mathbf{u} = -k(\mathbf{E} + c^{-1}[\mathbf{uH}]). \qquad (1)^*$$

Here $\mathbf{u}$ is the mean electron velocity, $\mathbf{E}$ and $\mathbf{H}$ are the field intensities, and k the electron mobility:

$$k \equiv \sigma / en. \qquad (2)$$

The conductivity σ and the electron concentration n are assumed constant. The electron velocity $\mathbf{u}$ is connected with the magnetic field $\mathbf{H}$ by the Maxwell equation:

$$\operatorname{rot} \mathbf{H} = -\gamma \mathbf{u}, \qquad \gamma = 4\pi en / c, \qquad (3)^\dagger$$

hence (1) is non-linear, and this makes the investigation of a number of phenomena substantially more complicated.

The set of equations (1) to (3) apply, with a certain degree of approximation, to the behavior of semiconductors, but may also allow conclusions about the behavior of a plasma.[2]

In the planar case when the magnetic field is at right angles to the plane of the current flow, the equation for the single field component H_z is identical with the corresponding equation in the absence of Hall effect.[1] This is due to the fact that in this case

$$\operatorname{rot} [\mathbf{u}\,\mathbf{H}] = 0, \qquad (4)$$

and therefore the Hall effect is brought about only by changes in the boundary conditions. On the other hand the current flow in an axially-symmetric conductor in the presence of only a field component H_φ leads to the differential equation[1]

$$\frac{\partial^2 I}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial I}{\partial r}\right) = \frac{k}{c}\left(\gamma\frac{\partial I}{\partial t} + \frac{2}{r^2}I\frac{\partial I}{\partial z}\right) \qquad (5)$$

for the function

$$I = -H_\varphi r. \qquad (6)$$

This equation differs from the usual skin-effect equation by the non-linear term.

Equation (5) is the simplest form of a skin-effect equation modified by the Hall effect and therefore deserves special attention. A preliminary discussion of (5), reported earlier,[1] showed that the presence of the non-linear term leads to stationary solutions which vary sharply with radius in the limit

$$\omega_e\tau = \sigma H / enc \to \infty. \qquad (7)$$

In the present paper we investigate the form in which these rapidly varying solutions appear in simple concrete geometries, viz., a full or hollow ridged axially symmetric conductor. This investigation was first carried out numerically and this helped us to visualize the over-all picture; the study was then extended analytically.

2. NUMERICAL INVESTIGATION

A. Method of Calculation

It is convenient for the numerical calculation and also for analyzing the current pattern to put

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1) Our notation differs from[1] by taking the electron charge as negative.

*[u H] = u × H.

†rot ≡ curl.

Eq. (5) into dimensionless form. On substituting

$$r = L\rho, \quad z = L\zeta, \quad t = 4\pi\sigma L^2 c^{-2}\tau, \tag{8a}$$

$$I = (encL/\sigma)\,Y, \tag{8b}$$

Eq. (5) takes the form

$$\frac{\partial Y}{\partial \tau} + \frac{2}{\rho^2} Y \frac{\partial Y}{\partial \zeta} = \frac{\partial^2 Y}{\partial \zeta^2} + \rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial Y}{\partial \rho}\right). \tag{9}$$

The properties of the medium, represented by the conductivity σ and the carrier density n, are no longer explicitly shown here. We choose L to be a characteristic radius of the conductor. Then the characteristic time

$$T = 4\pi\sigma L^2 / c^2 \tag{10}$$

is that connected with skin-effect. The quantity Y is connected with the Hall parameter

$$\omega_e\tau = \sigma|H| / enc \tag{11}$$

by the simple relation

$$Y = -\omega_e\tau\rho \operatorname{sign} H. \tag{12}$$

Equation (9) was solved numerically for the shapes shown in Figs. 1 and 2 with the following boundary conditions: on the corrugation crest we took

$$Y(\tau) = Y_0(1 - e^{-\tau/T}), \quad T = 0.1, \quad Y_0 = 10;\ 50; \tag{13}$$

and on the ζ axis or the corrugation bottom we put $Y \equiv 0$; the values on the end surfaces were connected by the cyclic condition

$$Y(\rho)\,|_{\zeta=0} = Y(\rho)\,|_{\zeta=\zeta_0}.$$

The computation was performed by a step-by-step calculation from the inside outwards; the stationary solution was found by following the time development.

B. Results of Computation

The stationary current pattern was first obtained for the ridged conductor shown in Fig. 1.

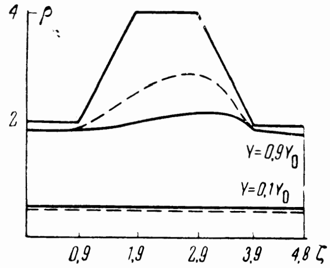

FIG. 1. Lines of constant Y (ρ, ζ) (stream lines) in a continuous ridged conductor. The upper lines correspond to $0.9Y_0$ and the lower ones to $0.1Y_0$. The full curves are for $Y_0 = 50$, the broken ones for $Y_0 = 10$.

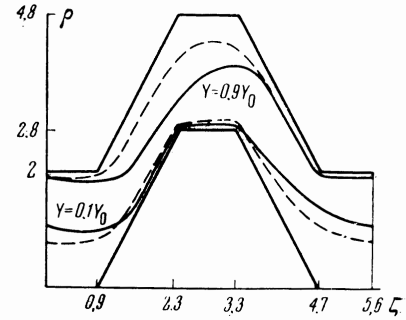

FIG. 2. Lines of constant $Y(\rho, \zeta)$ (stream lines) in a hollow corrugated conductor. The upper lines correspond to $0.9Y_0$, the lower ones to $0.1Y_0$. The full curves for $Y_0 = 50$, the broken ones for $Y_0 = 10$.

The pattern of the stream lines depends on the magnitude of the total current Y_0 which flows through the conductor, and with increasing Y_0 the current was pushed out of the ridges. Hence, as $\omega_e\tau \to \infty$, current which flows in the corrugated conductor is itself no longer corrugated. We then considered the current pattern in a hollow tube in which both the external and the internal surface were corrugated. The dimensions of the corrugations were chosen such (Fig. 2) that in the limit as $\omega_e\tau \to \infty$ the current could not confine itself to a straight cylindrical shell. With rising Y_0 the stream lines show a tendency to form thin layers which skirt the tube wall. On the basis of these results it is plausible that in the limit as $Y_0 \to \infty$ (or equivalently as $\omega_0\tau \to \infty$) the layer will become infinitesimally thin and have the shape shown in Fig. 3. In the picture of stream lines shown in Fig. 2 it is noticeable that the most widely spread out parts of the current layer are those parallel to the ζ axis.

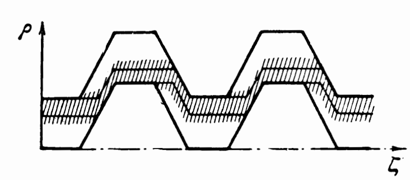

FIG. 3

In the analytic discussion below we shall concentrate on two problems: firstly, to explain the fact that in the limit as $\omega_e\tau \to \infty$ the current is pushed out of the ridges, and secondly to determine the thickness of the current layers that skirt the walls of the conductor when $\omega_e\tau \to \infty$ in the case of overlapping corrugations. The case $\omega_e\tau \approx 1$ is analytically very complicated while the case $\omega_e\tau \ll 1$ has already been considered.[1]

3. ANALYTICAL MODEL OF THE FLOW PATTERN

A. Elimination of the Current from the Ridges

Following [1], we linearize (9) by letting

$$Y = Y_0 + Y_1 + \dots \tag{14}$$

However, whereas in [1] the leading term in the solution of (9) was taken as

$$Y_0 = \text{const}, \tag{15}$$

which excluded the treatment of current in a solid conductor, we now take a different solution of (9), which represents a uniform current distribution in a cylindrical conductor:

$$Y_0 = {}^1/_2 a\rho^2, \qquad a = \text{const}. \tag{16}$$

In that case the equation for stationary flow is, to first order,

$$\frac{\partial^2 Y_1}{\partial \zeta^2} + \rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial Y_1}{\partial \rho}\right) = a\frac{\partial Y_1}{\partial \zeta}. \tag{17}$$

The use of perturbation theory in this case means that we consider only the case of a weakly corrugated conductor with a surface given by

$$\rho = \rho_0 + \rho_1 \cos k\zeta, \qquad \rho_1 / \rho_0 \ll 1. \tag{18}$$

Inserting in (17) the expression

$$Y_1 = \operatorname{Re}[e^{ik\zeta}Q(\rho)] \tag{19}$$

and using the condition of regularity on the ζ axis we find

$$Y_1 = \rho \operatorname{Re}\{e^{ik\zeta}I_1[\rho(\alpha + i\beta)]\},$$

$$\alpha + i\beta \equiv (k^2 + 2iak)^{1/2}, \qquad a > 0. \tag{20}$$

We see that the Hall effect is strong when

$$k^2 \lesssim 2a|k|. \tag{21}$$

Assuming for simplicity that $k \gg 1$ and $\rho_0 = 1$, we can use in the neighborhood of $\rho \sim \rho_0$ the asymptotic form of the Bessel function:

$$I_n(x) \approx (2\pi x)^{-1/2}e^x. \tag{22}$$

Using the fact that the surface of the corrugated cylinder (18) has the property

$$Y = \text{const}, \tag{23}$$

we can find an expression for $Y = Y_0 + Y_1$ in terms of ρ_1.

In the limiting cases in which the Hall effect is either negligible ($a \to 0$) or very large ($a \to \infty$), the expression for Y takes the forms

$$Y = a({}^1/_2\rho^2 - \rho_1\sqrt{\rho}e^{-k(1-\rho)}\cos k\zeta), \qquad a \to 0, \tag{24}$$

$$Y = a({}^1/_2\rho^2 - \rho_1\sqrt{\rho}\exp[-\sqrt{ka}(1-\rho)] \times \cos[k\zeta - \sqrt{ka}(1-\rho)]), \qquad a \to \infty. \tag{25}$$

We now construct the lines of constant Y, e.g., for

$$\rho = 1 - \nu / k, \qquad \nu = 0; 0.1; 0.2; \dots, \tag{26}$$

and we then see easily that for $a \to 0$ the lines are practically parallel (Fig. 4a), whereas in the case $a \to \infty$ the distance between the lines with $\nu = 0$ and $\nu = 0.1$ varies appreciably. In this case the lines $\nu = 0.1$, $\nu = 0.2$ etc. turn out to be practically straight (Fig. 4b). Thus we account analytically for the pushing-out of the current from the ridges of the corrugation. Obviously the difference in the behavior of the lines of constant Y for $a \to 0$ and $a \to \infty$ arises simply from the difference in the exponents in (24) and (25).

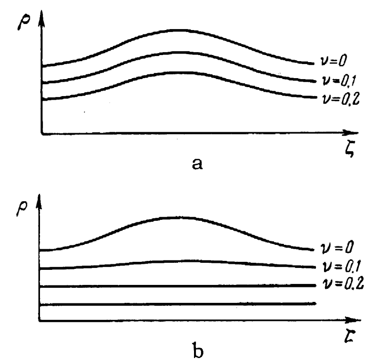

FIG. 4

It is easy to see that these arguments remain valid also in the case $k \lesssim 1$, as in the numerical example.

B. Structure of the Current Layer Near a Wall, for $\omega_e\tau \gg 1$

It was already noted (Fig. 3) that the electron current in skirting the wall is compressed close to it and forms when $\omega_e\tau \gg 1$ an extremely thin layer. We consider this effect first using as an example the flow near a conical wall (Fig. 5).

$$\rho - \alpha\zeta = \text{const}. \tag{27}$$

Using the fact that as $\omega_e\tau \to \infty$ the stream lines duplicate the geometry of the wall with great accuracy, we can write in the neighborhood of some point P of the wall

$$Y = Y_0 f\left(\frac{\rho - \alpha\zeta}{\delta_P}\right), \tag{28}$$

where $Y \gg 1$ is a characteristic current ampli-

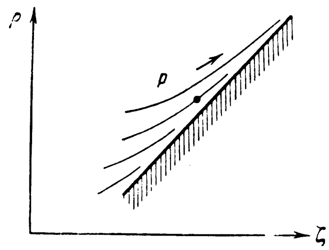

FIG. 5

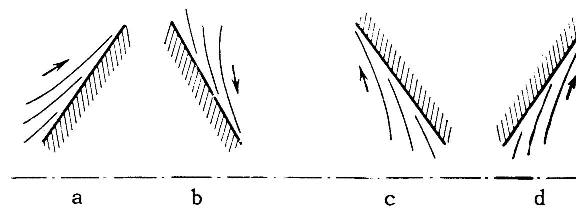

FIG. 6

tude and δ_P is the effective thickness of the current-carrying layer near P. We insert (28) in (9) and remember that f, f′, and f″ [2)] are of order of magnitude unity, and Y_0 and $1/\delta_P$ are much greater than unity. We then obtain to the leading order in δ_P the following equation for f in the neighborhood of P

$$f''(1+\alpha^2) = -\frac{2\alpha Y_0 \delta_P}{\rho_P{}^2} f'f. \tag{29}$$

The quantity ρ must in this equation be treated as a parameter that does not depend on the argument of the function f. In view of what we know about the orders of magnitude we can find from (29) the value of δ_P:

$$\delta_P = \frac{(1+\alpha^2)\rho}{2|\alpha|\omega_e\tau} = \frac{(1+\alpha^2)\rho^2}{2|\alpha|Y_0}. \tag{30}$$

We see that δ_P is inversely proportional to $\omega_e\tau$ and to α, in agreement with the results of the numerical work. The cases $\alpha = 0$ or ∞ call for a separate investigation.

The choice of the expression (30) for δ_P allows (29) to be written in the form

$$f'' = -f'f\theta, \qquad \theta \equiv \operatorname{sign}\alpha. \tag{31}$$

Equation (31) has the first integral

$$f' + {}^1\!/_2 f^2\theta = \text{const.} \tag{32}$$

This equation has an important singular property. Consider a free current layer which is not pushed against a wall. Then outside of this layer $f' \to 0$. However, this fact is not described by Eq. (32). We must therefore conclude, in line with Fig. 3, that there can be no free current layers which flow at an angle to the ζ axis.

In principle there are four possible cases of the formation of current layers near walls (Fig. 6). It is easy to see that cases b and d of Fig. 6 are incompatible with (32). This agrees with the results of the numerical work, where boundary layers are formed only when the electrons "surge" towards the wall. Consider now case a of Fig. 6, assuming, as is the case in the computed example, the boundary conditions

$$f = 0 \text{ when } \xi = 0, \qquad f' = 0 \text{ when } f = 1. \tag{33}$$

Then

$$f' + {}^1\!/_2 f^2 = {}^1\!/_2, \tag{34}$$

$$f = \frac{e^\xi - 1}{e^\xi + 1}, \tag{35}$$

i.e., $f \to 1$ as $\xi \to \infty$.

In case c of Fig. 6 we can take as boundary conditions

$$f = 1 \text{ when } \xi = 0, \qquad f' = 0 \text{ when } f = 0. \tag{36}$$

Then Eq. (32) takes the form

$$f' = {}^1\!/_2 f^2 \tag{37}$$

and consequently

$$f = (1-\xi)^{-1}, \qquad \xi < 0, \tag{38}$$

i.e., $f \to 0$ as $\xi \to -\infty$.

The great difference in behavior of the current-carrying layer in the cases of Figs. 6a and 6c presents considerable interest.

The calculations of the boundary current layers, presented in this section and based on Eq. (31), remain valid when the current skirts not a straight but a curved profile. This is caused by the local character of (31) and, in the general case, α should be understood to be the slope of the tangent to the profile.

C. Structure of Layers Parallel to the ζ Axis.

For the discussion of such layers we must distinguish between the behavior of the current near the axis of the conductor, where $Y \to 0$, and in the region in which $Y \gg 1$. Near the axis of the conductor the Hall effect does not arise and here the current distribution is well described by the usual equation of the Laplace type

[2)] The primes denote derivatives with respect to $\xi \equiv (\rho - \alpha\zeta)/\delta_P$.

$$\Delta^* \equiv \frac{\partial^2 Y}{\partial \zeta^2} + \rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial Y}{\partial \rho}\right) = 0. \tag{39}$$

Hence, the penetration of current into the region near the axis will be very strong.

In the region of large Y, on the other hand, Eq. (9) takes the form

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial Y}{\partial \rho}\right) = \frac{2}{\rho^2} Y \frac{\partial Y}{\partial \zeta}. \tag{40}$$

By linearizing this equation near $Y = Y_1$ we can see that Y must vary very rapidly with ρ in the limit as $Y \to \infty$. This justifies for the main part the current picture shown in Fig. 3.

Finally we want to discuss the possible importance of these effects. In a hot plasma $\omega_e \tau$ may be very large, therefore the elimination of the current from the ridges of a corrugated plasma cylinder may considerably slow up the development of the narrower part of the cylinder, at least in those cases in which the build-up time of the narrow neck is greater than the flow time of the current into the ridges. The pushing of the current from the ridges should also affect the diffusion of plasma in a magnetic field, since a magnetic field ceases to confine the plasma in the ridges.

In semiconductors (such as InSb), which cannot carry heavy currents, the effects described above would require the use of external magnetic fields (for example a field $H \sim 1000$ Oe gives for InSb $\omega_e \tau \sim 1$). This could be realized, for example, by using a hollow corrugated rod and placing inside it a metallic conductor carrying a heavy current. In that case the current passing through the semiconducting rod could be quite weak.

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[1] A. I. Morozov and A. P. Shubin, JETP **46**, 710 (1964), Soviet Phys. JETP **19**, 484 (1964).

[2] T. F. Volkov, Yadernyĭ sintez (Nuclear Fusion) **4**, 305 (1964).

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