QUANTUM FIELD THEORY EQUATIONS IN THE AXIOMATIC APPROACH. II.

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It is shown, within the framework of the axiomatic formulation of quantum field theory,[1,2] that when the values of some invariants are fixed, the indefinite quasilocal terms can be expressed in terms of R-functions. The invariant properties of the retarded matrix elements or v-functions are effectively used in this case. It is shown that on the mass shell the v-functions in the physical region of variation of the 4-momenta depend only on the invariant scalar products of the 4-vectors. The procedure for excluding the quasilocal terms yields explicit equations for an arbitrary matrix element (n-point diagram) in integro-differential and difference forms. The equations contain a finite number of arbitrary parameters, boundary values, or charges, equal to the number of amplitudes which do not vanish at infinity.

1. INTRODUCTION

A system of integro-differential equations was obtained earlier,[1,2] 1) within the framework of the axiomatic method in quantum field theory, for the S-matrix elements extrapolated off the mass shell only with respect to one of the external four-momenta. Such a system of equations follows from a somewhat modified (compared with the usual one) formulation of the fundamental axioms of the theory. One of these axioms is chosen to be the condition for minimal singularity of the commutator of the field and current operators at identical times. Microcausality, the existence of a unitary S matrix, and renormalizability of the theory are consequences of the initial axioms. The equations do not contain diverging expressions and, in the case of the iterational solutions, give the renormalized series of perturbation theory.

The main differences from other formulations of axiomatic equations in quantum field theory are as follows. First, the undetermined subtractional (or quasilocal) terms are eliminated from the equations by using the covariant properties of the matrix elements.2) Second, the number of arbitrary constants ("charges") which enter in the solution is limited, and is exactly equal to the number of amplitudes that do not vanish at an infinitely remote point in the momentum space of the invariant variables. It must be emphasized, that in the formulations of Lehmann, Simanzyk, and Zimmerman[4] and of Bogolyubov, Medvedev, and Polivanov[3] the solution of the corresponding axiomatic equations was determined in each order of perturbation theory accurate to a finite number of constants, that is, the exact solution for the nonrenormalizable theories contains, generally speaking, an infinite number of arbitrary parameters. Third, finite solutions of the equations correspond only to renormalizable variants of the theory (from the point of view of the Lagrangian formulation).

The equations of I were derived explicitly in integro-differential form only for the simplest matrix elements (including the six-point diagram). The advantages of the integro-differential form of notation become manifest, in particular, when solving equations by perturbation theory, when the supplementary conditions are automatically satisfied, and the region in which the invariants are specified (modified) is not limited—it is possible to integrate along any path in a space of real values of the invariant variables. On going beyond the framework of perturbation theory, the integro-differential notation in all probability loses some of its attractive features, and the supplementary conditions turn into too rigid and cumbersome obstacles when attempts are made to obtain approximate equations. This raises the important question of eliminating the quasilocal terms in the integral or differential formulation of the equations.3)

1)These papers are cited henceforth as I.

2)This is the most essential deviation from the formulation adopted in [3], where the departure off the mass shell is also carried out in accordance with one of the external four-momenta.

3)Such equations were investigated earlier by one of the authors [3].

The purpose of the present paper is to investigate the invariant properties of the matrix elements contained in the equations and to obtain in explicit form equations for an n-point diagram in differential (or integral) form with the quasilocal terms eliminated. In Sec. 2 we investigate questions involving the invariant variables on which the v-functions on the mass shell depend, the region of variation of these variables in the equations, and a convenient choice of independent invariance in the case of an arbitrary n-point diagram. In Sec. 3, to explain the gist of the method, we first derive difference equations for three-, four-, and five-point diagrams. We show the distinguishing features of the boundary conditions at the threshold and at an infinitely remote point. The method is then generalized to six-point and n-point diagrams. In Sec. 4 we discuss briefly the obtained equations and the prospects for their solution outside the framework of perturbation theory. The equations for the n-point diagram are written out in the Appendix.

In the following paper (III) we shall present material pertaining to the nonrelativistic limit of the axiomatic equations and its connection with the Schrödinger equation.

Work is now under way towards completing the derivation of analogous equations with participation of spinor particles.[6]

2. INVARIANT VARIABLES

1. We define an arbitrary v-function by the relation

$$v(p_1,\ldots,p_m)=(2\pi)^{-3m/2}\int\exp\left(i\sum_{i=1}^{m}p_ix_i\right)d^4x_1\ldots d^4x_m$$
$$\times\left\langle 0\left|\frac{\delta^m j(0)}{\delta\varphi_{in}(x_1)\ldots\delta\varphi_{in}(x_m)}\right|0\right\rangle \qquad (2.1)$$

The notation here is the same as in I. For the sake of generality we assume all the particle masses to be different.

The microcausality condition means that the integrand in (2.1) vanishes when $x_i^2 < 0$ or $x_{i0} > 0$ for any $1 \le i \le m$. Consequently, the function $v(p_1, \ldots, p_m)$ is a complete analog in p-space of the Wightman function

$$w_{m+1}(\xi_1,\ldots,\xi_m)\equiv\langle 0|\varphi(x_1)\ldots\varphi(x_{m+1})|0\rangle,$$
$$\xi_j=x_j-x_{j+1},\quad 1\leqslant j\leqslant m+1.$$

The function $v(p_1, \ldots, p_m)$ admits of a unique analytic continuation in the tubular region

$$p_i\to q_i=p_i-i\eta_i,\quad \eta_i^2>0,\quad \eta_{i0}>0,$$
$$-\infty<p_i<+\infty,\quad i=1,\ldots,m. \qquad (2.2)$$

From the Hall-Wightman theorem[7] it follows that $v(q_1, \ldots, q_m)$ is a function of only the scalar products q_iq_j

$$v(q_1,\ldots,q_m)=v(\{q_iq_j\}). \qquad (2.3)$$

The region of holomorphism of $v(\{q_iq_j\})$ is the region of variation of the scalar products, when the vectors $q_1, \ldots, q_m$ vary in the tubular region (2.2).

The arbitrary v function (2.1) can be defined on the mass shell as follows:

$$v(\mathbf{p}_1,\ldots,\mathbf{p}_k|\mathbf{p}_{k+1},\ldots,\mathbf{p}_m)\equiv\langle\mathbf{p}_1,\ldots,\mathbf{p}_k|j(0)|\mathbf{p}_{k+1},\ldots,\mathbf{p}_m\rangle$$
$$=\lim v(q_1,\ldots,q_m),$$
$$\mathbf{q}_i\to\mathbf{p}_i,\quad q_{i0}\to+\sqrt{\mathbf{p}_i^2+m_i^2}-i\varepsilon\equiv E_i,\quad i=1,\ldots,k,$$
$$\mathbf{q}_j\to-\mathbf{p}_j,\quad q_{j0}\to-\sqrt{\mathbf{p}_j^2+m_j^2}-i\varepsilon\equiv E_j,$$
$$j=k+1,\ldots,m. \qquad (2.4)$$

Thus, all the v-functions (2.4) on the mass shell are different boundary values of a single analytic function $v(q_1, \ldots, q_m)$.

We shall show that, owing to the quadratic properties of (2.3), the v-function on the mass shell depends only on m_i^2 and on the scalar products p_ip_j, and does not depend on the sign of the different combinations of the type $E_i \pm E_j$. We go in the limit to real values in the analyticity region of (2.2), putting

$$q_i=p_i-i\eta_i,\quad \eta_i=\varepsilon p_i,\quad p_{i0}=+\sqrt{\mathbf{p}_i^2+m_i^2},$$
$$i=1,\ldots,k;$$
$$q_j=-p_j-j\eta_j,\quad \eta_j=\varepsilon p_j,\quad p_{j0}=+\sqrt{\mathbf{p}_j^2+m_j^2},$$
$$j=k+1,\ldots,m;\quad \varepsilon\to+0.$$

Then, recognizing $p_ip_{i'} > 0$ and $p_jp_{j'} > 0$, we obtain

$$\lim_{\varepsilon\to+0}q_iq_{i'}=p_ip_{i'}(1-i\varepsilon)^2=p_ip_{i'}-i0,\quad i\geqslant i'=1,\ldots,k;$$
$$\lim_{\varepsilon\to+0}q_jq_{j'}=p_jp_{j'}(1+i\varepsilon)^2=p_jp_{j'}+i0,$$
$$j\geqslant j'=k+1,\ldots,m;$$
$$\lim_{\varepsilon\to+0}q_iq_j=-p_ip_j(1+i\varepsilon)(1-i\varepsilon)=-p_ip_j,\quad i\neq j. \qquad (2.5)$$

Consequently, the v-function on the mass shell depends only on the scalar products $p_ip_{i'}$, $p_jp_{j'}$, and p_ip_j, is regular at the point $-p_ip_j$, and is a generalized function of $p_ip_{i'}$, and $p_jp_{j'}$.

Let us explain this result using as an example the four-point diagram:

$$v(\mathbf{p}_1,\mathbf{p}_2|\mathbf{p}_3)\equiv\langle\mathbf{p}_1\mathbf{p}_2|j(0)|\mathbf{p}_3\rangle=v(m_1^2-i0,\ m_2^2-i0,$$
$$m_3^2+i0,\ p_1p_2-i0,\ -p_1p_3,\ -p_2p_3),$$

$$v(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3|) \equiv \langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 | j(0) | 0\rangle = v(m_1^2 - i0,\ m_2^2 - i0,\ m_3^2 - i0,\ p_1p_2 - i0,\ p_1p_3 - i0,\ p_2p_3 - i0). \quad (2.6)$$

So long as we are not interested in the analytic properties of v-functions, we can simply write

$$v(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|) = v_+(p_1p_2,\ p_2p_3,\ p_1p_3) \equiv v_+(s, u, t),$$
$$v(\mathbf{p}_1\mathbf{p}_2|\mathbf{p}_3) = v_-(p_1p_2,\ p_2p_3,\ p_1p_3) \equiv v_-(s,\ u,\ t),$$
$$s = p_1p_2, \quad u = p_2p_3, \quad t = p_1p_3,$$

that is, we can assume that they depend only on the invariant scalar products s, u, and t.

2. Let us find the region of the independent variation of the invariants for real values of the momenta. It is easy to show that any three 4-momenta p_1, p_2, and p_3 are connected by the inequality

$$\Phi(\xi, \eta, \zeta) \equiv \xi^2 + \eta^2 + \zeta^2 - 2\xi\eta\zeta - 1 \leqslant 0,$$
$$\xi = \frac{p_1p_2}{m_1m_2} = \frac{s}{m_1m_2}, \quad \eta = \frac{p_2p_3}{m_2m_3} = \frac{u}{m_2m_3}, \quad \zeta = \frac{p_1p_3}{m_1m_3} = \frac{t}{m_1m_3}. \quad (2.7)$$

For equal masses ($m_1 = m_2 = m_3 \equiv m$) we have

$$s^2 + u^2 + t^2 - \frac{2\,sut}{m^2} - m^4 \leqslant 0. \quad (2.7')$$

3. Let us consider the question of the choice of independent invariants in many-point diagrams. The amplitude of the n-point diagram $v(p_1, \ldots, p_k \mid p_{k+1}, \ldots, p_{n-1})$ depends on $(n-1)$ four-momenta. We can set up altogether $(n-1)\times(n-2)/2$ scalar products of these momenta. Only $(3n-9)$ invariants will be independent. In four-dimensional space there exists between any five vectors a geometrical connection that leads to the vanishing of the determinant $\det p_ip_k$ $(i, k = 1, \ldots, 5)$. The independent invariants can be constructed on the basis of any three vectors, say, p_1, p_2, and p_3. Indeed, the 3n—9 invariants are

$$\begin{array}{r|l} p_1p_2,\ \ p_1p_3, \ldots, p_1p_{n-1} & n-2 \\ p_2p_3, \ldots, p_2p_{n-1} & n-3 \\ p_3p_4, \ldots, p_3p_{n-1} & n-4 \end{array} \quad (2.8)$$

independent, since any fifth'order determinant $\det p_ip_k (i, k = 1, \ldots, 5)$ contains at least one invariant which does not enter in (2.8). All the remaining invariants are expressed in terms of these independent invariants with the aid of a determinant made up of p_1, p_2, p_3, p_i, and p_k (i and k fixed)

$$\begin{vmatrix} m_1^2, & (p_1p_2) & (p_1p_3) & (p_1p_i) & (p_1p_k) \\ (p_2p_1) & m_2^2 & (p_2p_3) & (p_2p_i) & (p_2p_k) \\ (p_3p_1) & (p_3p_2) & m_3^2 & (p_3p_i) & (p_3p_k) \\ (p_ip_1) & (p_ip_2) & (p_ip_3) & m_i^2 & (p_ip_k) \\ (p_kp_1) & (p_kp_2) & (p_kp_3) & (p_kp_i) & m_k^2 \end{vmatrix} = 0. \quad (2.9)$$

This equation is of second-order in p_ip_k. Consequently, every ten invariants (p_1p_2), (p_1p_3), (p_2p_3), (p_1p_i), (p_2p_i), (p_3p_i), (p_2p_k), (p_3p_k), (p_ip_k) are connected by a single condition. It can be written in the universal form

$$(p_ip_k) \equiv s_{ik} = f(s_{12}, s_{23}, s_{13}, s_{1i}, s_{2i}, s_{3i}, s_{1k}, s_{2k}, s_{3k}). \quad (2.10)$$

From this we see, in particular, that the dependent invariants are functions of not all the independent ones, but of nine only.

3. DIFFERENCE EQUATIONS

In I we obtained a formal solution for the v-functions, starting from the commutation relations for equal times. This solution is a system of m equations for each function

$$v(\mathbf{p}_1, \ldots | \ldots, \mathbf{p}_m) \equiv v(1, 2, \ldots | \ldots, m) \equiv v_m,$$
$$v(1, \ldots, k|k+1, \ldots, m) = R_i(1, \ldots | \ldots, m) + K_i(1, \ldots, 0_i \ldots | \ldots, m). \quad (3.1)$$

Here

$$R_i(1, \ldots, k|k+1, \ldots, m) = i(2\pi)^{-3/2}\int d^4x e^{ip_ix} \times \langle \mathbf{p}_1, \ldots, 0_i, \ldots, \mathbf{p}_k | \theta(-x_0)[j(x), j(0)]_- | \mathbf{p}_{k+1}, \ldots, \mathbf{p}_m\rangle, \quad (3.2a)$$

where 0_i denotes the absence of a particle with momentum i, if i = 1, ..., k. Alternately,

$$R_i(1, \ldots, k|k+1, \ldots, m) = i(2\pi)^{-3/2}\int d^4x\, e^{-ip_ix} \times \langle \mathbf{p}_1, \ldots, \mathbf{p}_k | \theta(-x_0)[j(x), j(0)]_- | \mathbf{p}_{k+1}, \ldots, 0_i, \ldots \mathbf{p}_m\rangle, \quad (3.2b)$$

if i = k + 1, ..., m. K_i are arbitrary functions that do not depend on p_i. This solution is formal, because it contains undetermined, generally speaking divergent, K_i-terms (or quasilocal terms). The presence of m different equations (3.1) for each v_m-function, and also the covariant properties of these functions, make it possible to exclude from the equations the undetermined K_i-terms. This was first done in I by formally differentiating the system (3.1) with respect to the invariant variables, using as an example simple v-functions (including a six-point diagram). We present here a difference method for eliminating[4)] the K_i-terms and deriving of a difference (or integral) form of the equations. These equations are, naturally, solutions of the corresponding integro-differential equations in I, with allowance for the additional conditions in the region of variation (2.7) of the independent invariants.

The equations for the lower v-functions (three-,

[4)]See also [5].

four-, and five-point diagrams) are of greatest practical interest. We shall therefore derive them in greater detail, and write out the right side of the four-point diagram in explicit form in terms of other v-functions. We consider for simplicity identical particles.

Three-point diagram. We have, generally speaking, three different v-functions:

$$v(12|0) \equiv v(\mathbf{p}_1, \mathbf{p}_2|) = \langle \mathbf{p}_1, \mathbf{p}_2 | j(0) | 0\rangle,$$

$$v(|12) \equiv v(|\mathbf{p}_1, \mathbf{p}_2) = \langle 0 | j(0) | \mathbf{p}_1, \mathbf{p}_2\rangle,$$

$$v(1|2) \equiv v(\mathbf{p}_1 | \mathbf{p}_2) = \langle \mathbf{p}_1 | j(0) | \mathbf{p}_2\rangle. \qquad (3.3)$$

Since $v^*(|12) = v(12|0)$ we can confine ourselves to an analysis of only two: $v(12|)$ and $v(1|2)$. We denote

$$v_+(s) \equiv v(12|), \quad v_-(s) \equiv v(1|2) = v_+(-s),$$

as is seen from (2.5);[5] $R_i(12|) = R_i^+(s)$, $R(1|2) \equiv R_i^-(s) = R_i^+(-s)$, $s = E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2$ is the only invariant on which the three-point diagram depends.

The invariant form of (3.1) is:[6]

$$v_\pm(s) = \begin{cases} R_1^\pm(s) + K_1^\pm \\ R_2^\pm(s) + K_2 \end{cases} \quad s \geqslant m^2. \qquad (3.4)$$

$K_i^\pm$ are constants that satisfy the conditions $K_1^- = K_1^+$ and $K_2^- = K_2^+$. These conditions follow from relations (3.7) of [2]. The difference equations are obtained in elementary fashion:

$$v_\pm(s) - v_\pm(s_0) = R_i^\pm(s) - R_i^\pm(s_0), \quad i = 1, 2. \quad (3.5)$$

What is specially singled out is the threshold value $s_0 = m^2$. At this point $v_\pm(f)$ do not have imaginary parts.[7] If $v_\pm(s)$ do not vanish when $s \to \infty$, then the theory, at first glance, will contain two arbitrary constants, say $\lambda_\pm \equiv v_\pm(m^2)$. However, the commutation relations for different times (or the microcausality conditions) will be satisfied only if (see (3.4))

$$\lambda_+ - \lambda_- = R_1^+(m^2) - R_1^-(m^2). \qquad (3.6)$$

Thus, the three-point diagram can introduce into the theory only one arbitrary parameter. If $v_\pm(s) \to 0$ as $s \to \infty$, then we get from (3.7) ($s_0 = m^2$)

$$\lambda_\pm = R_i^\pm(m^2) - R_i^\pm(\infty), \quad i = 1, 2. \qquad (3.7)$$

Equations (3.6) and (3.7) in this case can be regarded, to some degree, as conditions that determine the admissible ("eigen") values of the interaction constants $\lambda_\pm$ in order for the three-point diagrams to have solutions that decrease at infinity.

We note that for a three-point diagram it is simpler to use in lieu of (3.5) the dispersion relations, in those cases when the latter can be rigorously proved within the framework of the axiomatic approach.

Four-point diagram. In this case, since $v^*(12|3) = v(3|12)$ and $v^*(123|) = v(|123)$, there are two different v-functions:[8]

$$v(123|) = \langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 | j(0) | 0\rangle = v_+(s, u, t),$$
$$v(12|3) = \langle \mathbf{p}_1, \mathbf{p}_2 | j(0) | \mathbf{p}_3\rangle = v_-(s, u, t).$$

Each of them, in accordance with (2.6), depends on three invariants $s = p_1p_2$, $u = p_2p_3$, and $t = p_1p_3$. When $p_4 \equiv p_1 + p_2 - p_3$ lies on the mass shell, that is, $p_4 = +\sqrt{(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)^2 + m^2}$, the function $v(12|3)$ describes the real process of scattering of particles (1 and 2) in (3 and 4), while the invariants s, u, and t are connected by the relation[9]

$$s - t - u + m^2 = 0.$$

Equations (3.1) can now be written in the form

$$v_\pm(s, u, t) = \begin{cases} R_1^\pm(s, u, t) + K_1^\pm(u), \\ R_2^\pm(s, u, t) + K_2^\pm(t), \\ R_3^\pm(s, u, t) + K_3^\pm(s), \end{cases} \qquad (3.8)$$

where

$$R_i^+(s, u, t) = R_i(123|), R_i^-(s, u, t) = R_i(12|3),$$
$$i = 1, 2, 3.$$

In addition

$$K_3^+(s) = K_3^-(s). \qquad (3.8a)$$

Eliminating $K_i^\pm$ from (3.8), we get

$$v_\pm(s, u, t) - v_\pm(s_0, u, t_0) = R_1^\pm(s, u, t) - R_1^\pm(s_0, u, t_0),$$
$$v_\pm(s, u, t) - v_\pm(s_0, u_0, t) = R_2^\pm(s, u, t) - R_2^\pm(s_0, u_0, t),$$
$$v_\pm(s, u, t) - v_\pm(s, u_0, t_0) = R_3^\pm(s, u, t) - R_3^\pm(s, u_0, t_0). \qquad (3.9)$$

The arbitrariness in the choice of the points s_0, u_0,

[5] The relation $v_+(s) = v_-(-s)$ does not denote that we assume them to be a single analytic function, although in the case of a three-point diagram and identical masses there is a rigorous proof of this fact.

[6] In fact, owing to symmetry and invariance, $v(12|) = v(21|)$ and $v^*(1|2) = v(2|1)$, we get $R_1^+ = R_2^+$ and $R_1^- = R_2^-$, so that for $v_\pm(s)$ we get only one equation each.

[7] $v_-(s) = v^*_-(s)$ in the entire physical region.

[8] For an n-point diagram the number of different v-functions is $[n + \sin(\pi n/2)]/2$.

[9] For convenience we have not chosen Mandelstam variables.

and t_0 in each equation of (3.9) is limited here by the physical region (2.7). Combining these equations, we obtain a set of difference equations of the type

$$v_\pm(s,u,t) = v_\pm(s_0,u_0,t_0) + R_1^\pm(s,u,t) - R_1(s_0,u,t_0) + R_2^\pm(s_0,u,t_0) - R_2^\pm(s_0,u_0,t_0);$$

$$v_\pm(s,u,t) = v_\pm(s_0,u_0,t_0) + R_1^\pm(s,u,t) - R_1^\pm(s_0,u,t) + R_2^\pm(s_0,u,t) - R_2^\pm(s_0,u_0,t) + R_3^\pm(s_0,u_0,t) - R_3^\pm(s_0,u_0,t_0)$$

etc. These and similar equations have major shortcomings: first, $v_\pm(s_0, u_0, t_0)$, which is the boundary value or the arbitrary constants of the equation, is in general a complex quantity; second, because of (2.7), the right sides of the functions $v_\pm(s, u, t)$ are not defined in the entire physical region of s, u, and t. For example, the region of variation of the invariant u in $R_i(s_0, u, t_0)$ is narrower than the regional variation of this invariant in v(s, u, t).

Exceptions are the equations with boundary value $v_\pm(s_0, u_0, t_0)$ at points that are physically singled-out, that is, at the threshold ($s_0 = u_0 = t_0 = m^2$) or at infinity ($s, u, t \to \infty$). We note that the threshold is the only point in the physical region (2.7) where the amplitude is pure real.[10)]

If we fix, for example, $s = m^2 \equiv 1$, then the physical region (2.7) degenerates into a straight line:

$$u^2 + t^2 - 2tu \leqslant 0 \quad \text{or} \quad (u-t)^2 \leqslant 0,$$

that is, u = t. Therefore, every subtraction must of necessity fix two invariants. Using this circumstance and the symmetry of the R-functions, $R_1^\pm(s, u, t) = R_2^\pm(s, t, u)$, we obtain three different equations ($\lambda_\pm \equiv v_\pm(1,1,1)$):

$$v(s,u,t) = \begin{cases} \lambda_\pm + R_1^\pm(s,u,t) - R_1^\pm(u,u,1) \\ \quad + R_2^\pm(u,u,1) - R_2^\pm(1,1,1), \\ \lambda_\pm + R_2^\pm(s,u,t) - R_2^\pm(1,t,t) \\ \quad + R_3^\pm(1,t,t) - R_3^\pm(1,1,1), \\ \lambda_\pm + R_3^\pm(s,u,t) - R_3^\pm(s,1,s) \\ \quad + R_1^\pm(s,1,s) - R_1(1,1,1). \end{cases} \tag{3.10}$$

From (3.8) and (3.8a) it follows also that

$$\lambda_+ - \lambda_- = R_3^+(1,1,1) - R_3^-(1,1,1).$$

Thus, in the case when $v_\pm(s, u, t)$ do not vanish as s, u, $t \to \infty$, only one arbitrary constant enters into theory, λ_+ or λ_-. If $v_\pm(s, u, t) \to 0$ while s, u, t tend to infinity, then $\lambda_\pm$ can be expressed with the aid of (3.10) in terms of the $R_i^\pm$ functions, for example

$$\lambda_\pm = R_3^\pm(\infty,\infty,1) - R_3^\pm(\infty,\infty,\infty) + R_1(1,1,1) - R_1(\infty,\infty,1).$$

We see from (2.7) that one invariant cannot tend to infinity. In order not to violate (2.7), at least two invariants must tend simultaneously to infinity. This fact also strongly limits the number of different difference equations with boundary condition at infinity. In order not to clutter up the text, we shall not write these out. We present for illustration one of the R-functions in the right side of (3.10). For example,

$$R_1^-(s,u,t) = (2\pi)^{3/2} \sum_n \frac{1}{n!} \int \frac{d\mathbf{k}_1 \dots d\mathbf{k}_n}{2\omega_1 \dots 2\omega_n}$$
$$\times \Big\{ v(\mathbf{p}_2|\mathbf{k}_1,\dots,\mathbf{k}_n) v(\mathbf{k}_1,\dots,\mathbf{k}_n|\mathbf{p}_3)$$
$$\times \left(\frac{\delta(\Sigma\mathbf{k}_i - \mathbf{p}_1 - \mathbf{p}_2)}{-\Sigma\omega_i + E_1 + E_2 - i\varepsilon} - \frac{\delta(\Sigma\mathbf{k}_i + \mathbf{p}_1 + \mathbf{p}_3)}{\Sigma\omega_i - E_3 + E_1 - i\varepsilon} \right)$$
$$+ v(|\mathbf{p}_3, \mathbf{k}_1,\dots,\mathbf{k}_n) v(\mathbf{k}_1,\dots,\mathbf{k}_n\mathbf{p}_2|)$$
$$\times \left(\frac{\delta(\Sigma\mathbf{k}_i + \mathbf{p}_3 - \mathbf{p}_1)}{-\Sigma\omega_i - E_3 + E_1 - i\varepsilon} - \frac{\delta(\Sigma\mathbf{k}_i + \mathbf{p}_1 + \mathbf{p}_2)}{\Sigma\omega_i + E_1 + E_2 - i\varepsilon} \right)$$
$$+ v(\mathbf{p}_2|\mathbf{p}_3\mathbf{k}_1,\dots,\mathbf{k}_n) v(\mathbf{k}_1,\dots,\mathbf{k}_n|)$$
$$\times \left(\frac{\delta(\Sigma\mathbf{k}_i - \mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3)}{-\Sigma\omega_i + E_1 + E_2 - E_3 - i\varepsilon} - \frac{\delta(\Sigma\mathbf{k}_i + \mathbf{p}_1)}{\Sigma\omega_i + E_1 - i\varepsilon} \right)$$
$$+ v(|\mathbf{k}_1,\dots,\mathbf{k}_n) v(\mathbf{k}_1,\dots,\mathbf{k}_n\mathbf{p}_2|\mathbf{p}_3)$$
$$\times \left(\frac{\delta(\Sigma\mathbf{k}_i - \mathbf{p}_1)}{-\Sigma\omega_i + E_1 - i\varepsilon} - \frac{\delta(\Sigma\mathbf{k}_i + \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)}{\Sigma\omega_i + E_1 + E_2 - E_3 - i\varepsilon} \right) \Big\}.$$

Here $\omega_i \equiv +(\mathbf{k}_i^2 + m^2)^{1/2}$.

$$\Sigma\omega_i \equiv \sum_{i=1}^n \omega_i, \quad \Sigma\mathbf{k}_i \equiv \sum_{i=1}^n \mathbf{k}_i.$$

Five-point diagram. We confine ourselves to a consideration of only

$$v(12|34) \equiv \langle \mathbf{p}_1, \mathbf{p}_2 | j(0) | \mathbf{p}_3, \mathbf{p}_4 \rangle = v(s_{ij}), \quad i < j = 1, 2, 3, 4,$$
$$s_{ij} = p_i p_j = E_i E_j - \mathbf{p}_i \mathbf{p}_j.$$

The condition (2.7) relates the following invariants:

$$\Phi(s_{12}, s_{13}, s_{23}) \leqslant 0, \quad \Phi(s_{12}, s_{14}, s_{24}) \leqslant 0,$$
$$\Phi(s_{23}, s_{24}, s_{34}) \leqslant 0, \quad \Phi(s_{13}, s_{14}, s_{34}) \leqslant 0. \tag{3.11}$$

We shall consider difference equations with boundary condition at the threshold, that is, all $s_{ij} = m^2 = 1$. If, for example, $s_{12} = 1$, then we get from

[10)]The proof of this statement for equal masses is elementary. In the case of unequal masses it becomes generally speaking incorrect.

(3.11) $s_{13} = s_{23}$ and $s_{14} = s_{24}$. Thus, even the first subtraction fixes all the invariants connected with the momentum p_1. Analogously, the succeeding subtractions fix all the invariants connected with momentum p_i (i —number of the corresponding R_i function). One of the difference equations for the five-point diagram has the form ($\lambda_5 \equiv v|_{s_{ij}=1}$)

$$\begin{aligned} v(s_{ij}) &= \lambda_5 + R_1(s_{12}, s_{13}, s_{23}, s_{14}, s_{24}, s_{34}) \\ &- R_1(1, s_{23}, s_{23}, s_{24}, s_{34}) + R_3(1, s_{23}, s_{23}, s_{24}, s_{24}, s_{34}) \\ &- R_3(1, 1, 1, s_{24}, s_{24}, s_{24}) + R_4(1, 1, 1, s_{24}, s_{24}, s_{24}) \\ &- R_4(1, 1, 1, 1, 1, 1). \end{aligned} \tag{3.12}$$

We emphasize that in the case of a five-point diagram all the v-functions must vanish at infinity; otherwise the system of equations will not have finite solutions. The proof of this statement by perturbation theory is obvious (since $L_{int} \sim \lambda_5\varphi^5/5!$ is the nonrenormalizable variant of the scalar theory). A proof which does not use perturbation theory will be published elsewhere.

Six-point diagram. Dependent invariants appear among the scalar products $s_{ij} = p_i p_j$ for the first time starting with the six-point diagram. We shall develop a method of eliminating the K_i-terms, using a six-point diagram as an example, and then generalize this method to the case of an arbitrary n-point diagram.

We write the system (3.1) for one of the six-point diagrams:

$$v(123|45) = \begin{cases} R_1(123|45) + K_1(23|45), & (3.13a) \\ R_2(123|45) + K_2(13|45), & (3.13b) \\ R_3(123|45) + K_3(12|45), & (3.13c) \\ R_4(123|45) + K_4(123|5), & (3.13d) \\ R_5(123|45) + K_5(123|4). & (3.13d) \end{cases}$$

$$p_6 = p_1 + p_2 + p_3 - p_4 - p_5, \qquad p_6^2 \neq m^2.$$

The number of invariant scalar products made up of kinematically independent momenta p_i (i = 1, ..., 5) is equal to 10. They are connected by a single relation of the type (2.9). We take it, for example, in the form

$$s_{45} = f(s_{12}, s_{13}, s_{23}, s_{14}, s_{24}, s_{34}, s_{15}, s_{25}, s_{35}), \tag{3.14}$$

i.e., we assume the invariant s_{45} to be dependent. Then K_1, K_2, and K_3 depend implicitly on all nine invariants, so that the simple subtractional procedure described for the three-, four-, and five-point diagram does not hold: the sequence of fixing the invariants and elimination of the K-terms is not arbitrary. As before, we choose as the boundary value $v(123|45)|_{s_{ij}=1} \equiv \lambda_6$. At first we fix $s_{45} = 1$ and eliminate the K_4-term from (3.13d). We then get from (2.7) $s_{14} = s_{15}$, $s_{24} = s_{25}$, and $s_{34} = s_{35}$. We next put $s_{35} = 1$, $s_{13} = s_{15}$, $s_{23} = s_{25}$, etc. As a result we obtain the equation

$$v\begin{pmatrix} 12\,13\,14\,15 \\ 23\,24\,25 \\ 34\,35 \\ 45 \end{pmatrix} = \lambda_6 + R_4\begin{pmatrix} 12\,13\,14\,15 \\ 23\,24\,25 \\ 34\,35 \\ 45 \end{pmatrix} - R_4\begin{pmatrix} 12\,13\,15\,15 \\ 23\,25\,25 \\ 35\,35 \\ 1 \end{pmatrix} + R_3\begin{pmatrix} 12\,13\,15\,15 \\ 23\,25\,25 \\ 35\,35 \\ 1 \end{pmatrix}$$
$$- R_3\begin{pmatrix} 12\,15\,15\,15 \\ 25\,25\,25 \\ 1\;\;1 \\ 1 \end{pmatrix} + R_2\begin{pmatrix} 12\,15\,15\,15 \\ 25\,25\,25 \\ 1\;\;1 \\ 1 \end{pmatrix} - R_2\begin{pmatrix} 15\,15\,15\,15 \\ 1\;\;1\;\;1 \\ 1\;\;1 \\ 1 \end{pmatrix} + R_1\begin{pmatrix} 15\,15\,15\,15 \\ 1\;\;1\;\;1 \\ 1\;\;1 \\ 1 \end{pmatrix}$$
$$- R_1\begin{pmatrix} 1\;\;1\;\;1\;\;1 \\ 1\;\;1\;\;1 \\ 1\;\;1 \\ 1 \end{pmatrix}. \tag{3.15}$$

For convenience we have arranged all the invariants $s_{ij} \equiv (i, j)$ in the form of a matrix. Equation (3.15) contains R_4 and R_5 with $s_{45} = 1$, $s_{14} = s_{15}$, $s_{24} = s_{25}$, and $s_{34} = s_{35}$. It is obvious that this does not impose additional limitations on the region of variation of the independent invariants, since (3.14) is satisfied identically (two colums in the corresponding determinant (2.9) are identical).

In the Appendix we write out in explicit form the equations for an n-point diagram in both the difference and integro-differential forms, as well as the supplementary conditions.

4. DISCUSSION

We have thus succeeded in showing that the undetermined quasilocal terms are eliminated from the equations or, more accurately, that they can be expressed in terms of R-functions for fixed values of certain invariants. The theory contains, in addition to the masses, a finite number of constants.

In our opinion, it is of interest to pursue the following investigations:

1. Search for simplest solvable models, especially in a space with a smaller number of dimensions. It must be emphasized here that hopes for finding an exact solution in a real four-dimensional space are practically nil, since the simplest model is equivalent (from the point of view of the Lagrangian formulation) to the renormalized variants

of the interaction of scalar particles with $L_{int}(x) \sim \lambda_3\varphi^3(x)$ or $\sim \lambda_4\varphi^4(x)$.

2. Derive and analyze, outside the framework of perturbation theory, an approximate system of equations which would retain to the maximum degree the main symmetry properties of the initial exact system.

The principal difficulty in this case lies in the following: any cutoff of the system with respect to the number of intermediate-state particles violates the rigorous microcausality conditions,[11] and consequently also the conditions for relativistic invariance. To be sure, we can first write out exact equations in invariant form, and then carry out the cutoff. We obtain equations that are invariant in form, but differ in accordance with the system of coordinates in which the relativization of the exact equations is carried out. It is not excluded that this arbitrariness can be used to minimize the discarded terms and by the same token choose the most suitable coordinate frame. This difficulty is general in any axiomatic approach in quantum field theory. It can be apparently circumvented only by finding in lieu of the "particle-number" variables more convenient quantum numbers which characterize the possible states of the interacting particles.

3. Solve the equations for small "threshold" energies. In this area we can hope to obtain in the nearest future results that are directly connected with experiment. Thus, the equation for the interaction of two particles with masses μ and M (for example, mesons and nucleons) goes over in the static approximation $M \to \infty$ into equations of the Low-type on the mass shell,[9] but with a number of additional terms representing the contribution of $\pi\pi$ interaction and different resonances to the processes of scattering and photoproduction of pions on nucleons at low energies.

We note also that the possibility of expressing quasilocal terms by means of the initial amplitudes can be utilized also in approximate equations that follow from the single dispersion relations, especially to refine the Chew-Low-Nambu-Goldberger equation,[10] in which no account is taken of $\pi\pi$ interaction.

4. The nonrelativistic limit and comparison with the Schrödinger equation. An investigation of this question is of interest both from the general point of view (the presence of an additional class of solutions, limitations on the admissible character of the potentials, etc.[12]), and from the point of view of practical applications: it is desired to develop a method for obtaining at low energies approximate equations in which elementary and complex particles (bound states) are considered on par and one can take the relativistic corrections consistently into account. Of special interest, from our point of view, is the derivation and proof of equations for nuclear reactions ("axiomatic" theory of nuclear reactions).

5. The problem of bound states in the axiomatic approach. At first glance, the axiomatic approach does not distinguish between "elementary" and "complex" particles. However, the equations admit of a different treatment and, in particular, of formulation of the problem of construction of all particles from the fundamental primary field (in analogy with Heisenberg's ideas). "Complex" particles appear as pole singularities in the amplitudes of the transitions of the primary particles, at certain fixed values of the invariants and parameters that enter into the theory (masses and "charges" of the primary particles). The amplitudes of transitions in which complex particles participate are constructed by means of a definite procedure of analytic continuation from the amplitudes of the primary particles.

The unitary S-matrix is made up in this case of the complete aggregate of all the amplitudes. This raises the very important and fundamental question: are there any finite solutions for the initial system if the primary particles are fermions? Within the framework of perturbation theory, the axiomatic equations do not have finite solutions (nonrenormalizability of the four-fermion interaction). If no such solutions exist, then we must either forego the idea of existence of a primary field and primary particles, or our notions concerning the interaction at small distances should be radically reviewed.[13]

More detailed results of an investigation of these questions will be published in other papers.

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[11]This is brought about by the fact that such a cutoff is equivalent to the assumption that the current operator j(x) represents a functional polynomial of finite degree in the in-operators. As is well known[8], such currents do not commute outside the cone, with the exception of the trivial case corresponding to a unit S-matrix. This is connected with the fact that the microcausality reflects the properties of interacting particles at arbitrarily small distances, when the number of particles itself increases without limit.

[12]See also [11].

[13]In this connection, the ideas of a curved p-space, intensely developed recently by I. E. Tamm, are very attractive. In Tamm's approach the four-fermion interaction is renormalizable.

APPENDIX I

EQUATION FOR n-POINT DIAGRAM IN DIFFERENCE FORM

The derivation of the equations (the elimination of quasilocal terms) is carried out by the same method as for the six-point diagram. If we choose (2.8) as the independent invariants, then one of the equations with boundary condition at the threshold will take the form

$$v\begin{pmatrix} 12\ 13\ldots(1,n-1) \\ 23\ldots(2,n-1) \\ \vdots \\ (n-2,n-1)\end{pmatrix} = \lambda_n + R_{n-2}\begin{pmatrix} 12\ 13\ldots(1,n-1) \\ 23\ldots(2,n-1) \\ \vdots \\ (n-2,\ n-1)\end{pmatrix} - R_{n-2}\begin{pmatrix} 12\ 13\ (1,n-1)\ (1,n-1) \\ (2,n-1)\ (2,n-1) \\ \vdots \\ (n-3,n-1)\ (n-3,n-1) \\ 1 \end{pmatrix} + \ldots$$

$$\ldots + R_1\begin{pmatrix} (1,n-1)\ \ldots\ (1,n-1) \\ 1\ \ldots\ 1 \\ \vdots \\ 1\end{pmatrix} - R_1\begin{pmatrix} 1\ 1\ \ldots 1 \\ 1\ \ldots 1 \\ \vdots \\ 1\end{pmatrix}.$$

Here $\lambda_n \equiv r$ for all $s_{ij} = 1$.

APPENDIX 2

EQUATIONS FOR n-POINT DIAGRAM IN INTEGRO-DIFFERENTIAL FORM

We choose the independent invariants as in (2.8). Since the K-term is an (n − 1)-point diagram, for some arbitrary choice of the independent invariants in each of the K-terms the equations take the form

$$\frac{\partial v}{\partial s_{ij}} = \frac{\partial R_i}{\partial s_{ij}} + \sum_a \frac{\Delta_a{}^i}{\Delta^i}\,\frac{\partial s_a}{\partial s_{ij}},$$

where s_{ij} are all the independent invariants; $s_a = \{s_{45}, s_{56}, \ldots, s_{n-2,\ n-1}\}$, and

$$\Delta^i = \begin{vmatrix} \frac{\partial s_{45}}{\partial s_{i4}}, & \ldots, & \ldots, & \ldots & \\ \frac{\partial s_{45}}{\partial s_{i5}}, & \frac{\partial s_{56}}{\partial s_{i5}}, & \ldots, & \ldots & \\ \cdot & \frac{\partial s_{56}}{\partial s_{i6}}, & \frac{\partial s_{67}}{\partial s_{i6}}, & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ \ldots, & \ldots, & \ldots, & \frac{\partial s_{n-3,\ n-2}}{\partial s_{i,\ n-2}}, & \frac{\partial s_{n-2,\ n-1}}{\partial s_{i,\ n-2}} \end{vmatrix} = \frac{\partial s_{45}}{\partial s_{i4}}\cdots\frac{\partial s_{n-2,\ n-1}}{\partial s_{i;\ n-2}},$$

Δ_a^i is Δ^i in which the a-th column is replaced by

$$\begin{pmatrix} \frac{\partial(R_4 - R_i)}{\partial s_{i4}} \\ \vdots \\ \frac{\partial(R_{n-2} - R_i)}{\partial s_{i,\ n-2}} \end{pmatrix};$$

$\Delta_a^i = 0$ if $i \neq 1, 2, 3$.

The supplementary conditions are

$$\frac{\partial(R_j - R_i)}{\partial s_{ij}} = \sum_a\left(\frac{\Delta_a{}^i}{\Delta^i} - \frac{\Delta_a{}^j}{\Delta^j}\right)\frac{\partial s_a}{\partial s_{ij}},$$

$$i = 1, \quad j = 2, 3,$$
$$i = 2, \quad j = 3,$$
$$j = n-1, \quad i = 1, 2, 3.$$

[1] V. Ya. Faĭnberg, Lectures at International School in Dubna, 1964.

[2] Ya. B. Faĭnberg, JETP 47, 2285 (1965), Soviet Phys. JETP 20, 1529 (1966).

[3] N. N. Bogolyubov, B. V. Medvedev, and M. K. Polivanov, Voprosy teorii dispersionnykh sootnosheniĭ (Problems in the Theory of Dispersion Relations), Fizmatgiz 1958. B. V. Medvedev and M. K. Polivanov, op. cit.[1]

[4] O. Steinman, Ann of Phys. 29, 76 (1964).

[5] R. É. Kallosh, Diploma Thesis, Moscow State Univ. Phys. Dept. 1964.

[6] B. L. Voronov, Preprint, Phys. Inst. Acad. Sci. 1965.

[7] D. Hall and A. S. Wightman, Kgl. Danske, videnskab. Selsk. Math. Fys. Medd. 31, No. 5 (1957).

[8] K. Bardakei and E. C. G. Sudashan, Nuovo Cimento 21, 722 (1961).

[9] G. F. Chew and F. E. Low, Phys. Rev. 101, 157 1571-1579 (1956).

[10] Chew, Goldberger, Low, and Nambu, Phys. Rev. 106, 1337 (1957).

[11] D. A. Kirzhnits, Preprint Phys. Inst. Acad. Sci.

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