ON STRONG FLUCTUATIONS OF LIGHT WAVE PARAMETERS IN A TURBULENT MEDIUM

V. I. TATARSKIĬ

Institute of Physics of the Atmosphere, Academy of Sciences, U.S.S.R.

Submitted to JETP editor June 8, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **49**, 1581-1590 (November, 1965)

Amplitude and phase fluctuations of a plane monochromatic wave propagating in a turbulent medium are calculated. The region of strong amplitude fluctuations for which perturbation theory is not valid is investigated. A solution applicable to strong fluctuations is obtained by taking cumulative distortions of an incident wave into account. This accounts for the "saturation" of amplitude fluctuations observed experimentally with increasing distance traversed by a wave in a turbulent medium.

1. Turbulent mixing in a thermally nonuniform gaseous medium induces fluctuations of its dielectric constant which give rise to fluctuations in the amplitude, phase, and other parameters of electromagnetic waves propagating through the medium. The fluctuations are especially strong in the optical region. For example, intensity fluctuations of light propagating in the atmosphere sometimes exceed the mean intensity, although the fluctuations of the dielectric constant are only of the order 10^{-6}—10^{-5}.

Amplitude and phase fluctuations have been calculated by some authors using Rytov's method of smooth disturbances (MSD).[1-4] Gracheva and Gurvich[5] measured fluctuations of light intensity in the surface layer of the atmosphere and compared their results with a theory based on the MSD. They found satisfactory agreement so long as the rms fluctuation σ_1 of the logarithm of the amplitude as calculated by the MSD does not exceed 0.8. In contrast with any higher theoretical values the experimentally measured value ceases to increase or increases much more slowly (Fig. 2).

The discrepancy between the calculation and experiment results from breakdown of the conditions for applying the MSD, which is based on linearizing the nonlinear equation for the logarithm of the field. An attempt to take into account the nonlinear term in the equation by summing a perturbation-theoretical series has not yielded a positive result, since in the given order of perturbation all terms are comparable, differing only in their numerical coefficients. A different procedure is therefore required for the solution of the problem.

2. Let us consider a medium with random fluctuations of the dielectric constant $\epsilon(\mathbf{r})$. We shall assume that the fluctuation frequencies of ϵ are small compared with the wave frequency ω and that the wavelength λ is small compared with the minimum dimensions of the inhomogeneities (the internal scale of turbulence). We can then neglect radiation depolarization effects and can describe wave propagation by means of a scalar wave equation for the complex amplitude of the field:

$$\Delta E + k^2[1+\varepsilon_1(\mathbf{r})]E = 0,$$
$$\varepsilon(\mathbf{r}) = \langle\varepsilon\rangle[1+\varepsilon_1(\mathbf{r})]. \tag{1}$$

Here k is the wave number, and E can represent any field component.

Let the inhomogeneous medium be located in a region $x > 0$ upon which a plane wave A_0e^{ikx} impinges. We shall be interested in the amplitude and phase for $x > 0$. We write the wave as

$$E(\mathbf{r}) = A_0 e^{i\mathbf{kr}+\varphi(\mathbf{r})}, \quad \mathbf{k} = (k, 0, 0). \tag{2}$$

Then $\mathrm{Re}\,\varphi = \ln(A/A_0)$ is the fluctuation of the logarithm of the amplitude and $\mathrm{Im}\,\varphi = S - \mathbf{k}\cdot\mathbf{r}$ is the phase fluctuation. Substituting (2) into (1), we obtain

$$\Delta\varphi + 2i\mathbf{k}\nabla\varphi + (\nabla\varphi)^2 + k^2\varepsilon_1(\mathbf{r}) = 0. \tag{3}$$

In the MSD Eq. (3) is linearized; this means that we drop the term $(\nabla\varphi)^2$. This is equivalent to solving the equation in the first order of perturbation theory.

We make the substitution $\varphi = uw$ and impose an additional condition on the functions u and w:

$$2\nabla u + 2i\mathbf{k}u + u^2\nabla w + uw\nabla u = 0, \tag{4}$$

after which the coefficient of ∇w vanishes in the equation for w. The integration of (4) leads to the relation

$$2 \ln u + uw = -2i\mathbf{kr}. \quad (4a)$$

Using (4) and (4a), the equation for w becomes

$$\Delta w + [\Delta \ln u - (\nabla \ln u)^2] w + k^2 \varepsilon_1(\mathbf{r}) / u(\mathbf{r}) = 0. \quad (5)$$

The system (4a) and (5) is equivalent to the initial equation (1).

The linearization of (3) is equivalent to neglecting the term uw in (4a). In this case, denoting the approximation by means of the subscript "1" we obtain

$$u_1 = e^{-i\mathbf{kr}},$$
$$\Delta w_1 + k^2 w_1 = -k^2 \varepsilon_1(\mathbf{r}) e^{i\mathbf{kr}}. \quad (6)$$

These equations are equivalent to the MSD equations; they lead to the solution

$$\varphi_1 = u_1 w_1 \quad (7)$$

which coincides with the MSD solution. We calculate the term uw in (4a) by iteration, making the substitution $u_1 w_1 = \varphi_1$ in first approximation. Successive iterations will be performed approximately by reduction to a nonlinear integral equation that will be solved numerically.

We substitute $uw \approx \varphi_1$ in (4), and

$$u_2 = \exp[-i\mathbf{kr} - {}^1\!/_2 \varphi_1(\mathbf{r})] \quad (8)$$

in (5), thus obtaining

$$\Delta w + [k^2(1 - {}^1\!/_2 \varepsilon_1) - {}^1\!/_4 (\nabla \varphi_1)^2] w$$
$$= -k^2 \varepsilon_1(\mathbf{r}) \exp[i\mathbf{kr} + {}^1\!/_2 \varphi_1(\mathbf{r})]. \quad (9)$$

We here neglect the small quantities $k^2 \epsilon_1$, and $(\nabla \varphi_1)^2$ in the coefficient of w. When these terms are taken into account the mean value $\langle w \rangle$ decreases with distance; we shall neglect this effect. The equation for w then becomes

$$\Delta w + k^2 w = -k^2 \varepsilon_1(\mathbf{r}) \exp[i\mathbf{kr} + {}^1\!/_2 \varphi_1(\mathbf{r})]. \quad (9a)$$

By contrast with the corresponding MSD equation, the scattered wave is here

$$\exp[i\mathbf{kr} + {}^1\!/_2 \varphi_1(\mathbf{r})],$$

i.e., phase and amplitude distortions are here included in first approximation.

It is easily understood that in the additional factor

$$\exp[{}^1\!/_2 \varphi_1] = \exp[{}^1\!/_2 \chi_1 + {}^1\!/_2 i S_1],$$

where $\chi_1 = \ln(A/A_0)$ is the fluctuation in the logarithm of the amplitude and S_1 is the phase fluctuation in first approximation, the phase factor $e^{iS_1/2}$, which can undergo a sign reversal, plays a decisive part. Moreover, in a turbulent medium we have $\langle \chi_1^2 \rangle \ll \langle S_1^2 \rangle$. Therefore we shall use $\varphi_1 \approx iS_1$ in (8) and (9a).[1)] Solving (9a) and multiplying the result by u_2, we obtain

$$\varphi_2(\mathbf{r}) = \frac{k^2}{4\pi} \int \frac{\varepsilon_1(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \exp\Big[ik|\mathbf{r} - \mathbf{r}'| - ik(\mathbf{r} - \mathbf{r}') + \frac{i}{2} S_1(\mathbf{r}') - \frac{i}{2} S_1(\mathbf{r}) \Big] d^3 r'. \quad (10)$$

Equation (10) can be considerably simplified, as in the MSD, for a wavelength that is much smaller than the internal scale of the inhomogeneities. In this case the exact Green's function in (10) can be replaced by an approximation that corresponds to the Fresnel approximation for diffraction. The integration region is then confined to $0 < x' < x$. We thus obtain

$$\varphi_2(x, y, z) = \frac{k^2}{4\pi} \int_0^x \frac{dx'}{x - x'} \iint_{-\infty}^{\infty} dy'\, dz'\, \varepsilon_1(\mathbf{r}')$$
$$\times \exp\left[\frac{ik\rho^2}{2(x - x')} + \frac{i}{2} S_1(\mathbf{r}') - \frac{i}{2} S_1(\mathbf{r}) \right]. \quad (11)$$

This equation differs from the corresponding MSD expression through the presence of the factor

$$\exp\left[\frac{i}{2} S_1(\mathbf{r}') - \frac{i}{2} S_1(\mathbf{r}) \right].$$

We note that (11) contains terms of a perturbation series corresponding to (3) in any order. This can be shown easily by a series expansion of the additional factor.

The real and imaginary parts of (11) give expressions for the logarithm of the amplitude and the phase. The mean value $\langle \varphi_2 \rangle$ does not vanish, because ϵ_1 and S_1 are correlated. It can be stated, however, that this mean value $\langle \varphi_2 \rangle$ will be of at least the second order of smallness in ϵ_1. Before determining the mean value $\langle \varphi_2 \rangle$ we shall consider the mean squares of the real and imaginary parts of φ_2. For the sake of brevity we introduce the notation

$$\rho_{1,2} = \{0,\, y - y_{1,2},\, z - z_{1,2}\}, \quad a_{1,2} = k\rho_{1,2}^2 / 2(x - x_{1,2}),$$
$$\alpha_{1,2} = {}^1\!/_2 [S_1(\mathbf{r}_{1,2}) - S_1(\mathbf{r})].$$

For the mean square of the real part of φ_2 we then obtain

$$\langle \chi^2 \rangle = \left(\frac{k^2}{4\pi} \right)^2 \int_0^x \frac{dx_1}{x - x_1} \int_0^x \frac{dx_2}{x - x_2} \iiiint_{-\infty}^{\infty} dy_1\, dz_1\, dy_2\, dz_2$$
$$\times \langle \varepsilon_1(\mathbf{r}_1) \varepsilon_1(\mathbf{r}_2) \cos(a_1 + \alpha_1) \cos(a_2 + \alpha_2) \rangle. \quad (12)$$

[1)]A simple calculation shows that when the term χ_1 is taken into account along with attenuation (extinction) using the formula $\varphi = \chi_1 + iS_1 + \langle \varphi \rangle$, where $\langle \varphi \rangle$ is calculated in the second MSD approximation,[6] the results differ only very slightly from those that will be obtained by us here.

We now note that $S_1(\mathbf{r}_1)$ is given by the integral of ϵ_1 over a large volume enclosing a beam reaching the point $\mathbf{r}_1$. It follows that $S_1(\mathbf{r}_1)$ will be weakly correlated with $\epsilon_1(\mathbf{r}_1)$, and that it will be a Gaussian distribution in virtue of the central limit theorem of probability theory. Therefore, when averaging in (12) we can assume

$$\langle \varepsilon_1(\mathbf{r}_1)\varepsilon_1(\mathbf{r}_2)\cos(a_1+\alpha_1)\cos(a_2+\alpha_2)\rangle \approx \langle \varepsilon_1(\mathbf{r}_1)\varepsilon_1(\mathbf{r}_2)\rangle\langle \cos(a_1+\alpha_1)\cos(a_2+\alpha_2)\rangle,$$

and to calculate the mean product of the cosines we use formulas that are valid for a Gaussian random variable γ:

$$\langle \sin\gamma\rangle = 0, \quad \langle \cos\gamma\rangle = \exp(-{}^1/_2\langle\gamma^2\rangle).$$

The calculations lead to the formula

$$\langle\chi^2\rangle = \frac{k^4}{32\pi^2}\int_0^x \frac{dx_1}{x-x_1}\int_0^x \frac{dx_2}{x-x_2}\int\!\!\int\!\!\int_{-\infty}^{\infty}\!\!\int dy_1\,dz_1\,dy_2\,dz_2$$
$$\times B_\varepsilon(\mathbf{r}_1,\mathbf{r}_2)\{\exp[-{}^1/_8 D_{S_1}(\mathbf{r}_1,\mathbf{r}_2)]\cos(a_1-a_2)$$
$$+\exp[{}^1/_8 D_{S_1}(\mathbf{r}_1,\mathbf{r}_2) - {}^1/_4 D_{S_1}(\mathbf{r},\mathbf{r}_1)$$
$$-{}^1/_4 D_{S_1}(\mathbf{r},\mathbf{r}_2)]\cos(a_1+a_2)\}, \quad (13)$$

where $B_\epsilon(\mathbf{r}_1, \mathbf{r}_2)$ is the correlation function of the fluctuations of ϵ, and $D_{S_1}(\mathbf{r}_1, \mathbf{r}_2)$ is the structure function of the phase fluctuations:

$$D_{S_1}(\mathbf{r}_1,\mathbf{r}_2) = \langle[S_1(\mathbf{r}_1) - S_1(\mathbf{r}_2)]^2\rangle.$$

Equation (13) differs from the corresponding MSD expression by the presence of the additional exponential factors. An analogous formula for the mean square phase fluctuation differs from (13) in the sign of the second term within curly brackets.

3. It is difficult to perform a calculation using (13). We shall therefore use several additional simplifications that have clear physical meanings but require further mathematical justification. First of all, we shall make use of the fact that the longitudinal radius of correlation of phase fluctuations is very large, being of the order of the length of the entire trace. We can therefore use the approximation

$$D_{S_1}(\mathbf{r}_1,\mathbf{r}_2) \approx D_{S_1}\left(\frac{x_1+x_2}{2},\ \boldsymbol{\rho}_1-\boldsymbol{\rho}_2\right),$$

where $D_{S_1}(x, \rho)$ is the structure function of the phase in the plane x = const. We now consider the expression

$$A = 2D_{S_1}(\mathbf{r},\mathbf{r}_1) + 2D_{S_1}(\mathbf{r},\mathbf{r}_2) - D_{S_1}(\mathbf{r}_1,\mathbf{r}_2).$$

Identical considerations lead to the approximation

$$A \approx 2D_{S_1}\left(\frac{x_1+x_2}{2},\ \boldsymbol{\rho}_1\right) + 2D_{S_1}\left(\frac{x_1+x_2}{2},\ \boldsymbol{\rho}_2\right) - D_{S_1}\left(\frac{x_1+x_2}{2},\ \boldsymbol{\rho}_1-\boldsymbol{\rho}_2\right)$$

(in all three terms, for the plane x = const we have selected $x_1 + x_2$ = const, which is valid to about the same degree of accuracy). The structure function $D_{S_1}(x, \rho)$ in a turbulent medium has the form $\sim\rho^{5/3}$. For a structure function of the form $\sim\rho^2$ we have

$$A = D_{S_1}\left(\frac{x_1+x_2}{2},\ \boldsymbol{\rho}_1+\boldsymbol{\rho}_2\right).$$

Approximately the same relations hold true for $D_{S_1}(x, \rho) \sim \rho^{5/3}$ because of the small difference between the exponents $5/3$ and 2.[2)]

As a result of all these approximations (13) becomes

$$\langle\chi^2\rangle = \frac{k^4}{32\pi^2}\int_0^x \frac{dx_1}{x-x_1}\int_0^x \frac{dx_2}{x-x_2}\int\!\!\int\!\!\int_{-\infty}^{\infty}\!\!\int dy_1\,dz_1\,dy_2\,dz_2$$
$$\times B_\varepsilon(\mathbf{r}_1,\mathbf{r}_2)\left\{\exp\left[-\frac{1}{8}D_{S_1}\left(\frac{x_1+x_2}{2},\ \rho_1-\rho_2\right)\right]\right.$$
$$\times\cos\left[\frac{k\rho_1^2}{2(x-x_1)} - \frac{k\rho_2^2}{2(x-x_2)}\right]$$
$$+\exp\left[-\frac{1}{8}D_{S_1}\left(\frac{x_1+x_2}{2},\ \rho_1+\rho_2\right)\right]$$
$$\left.\times\cos\left[\frac{k\rho_1^2}{2(x-x_1)} + \frac{k\rho_2^2}{2(x-x_2)}\right]\right\}. \quad (14)$$

Let us now consider the integrals with respect to the variables ρ_1 and ρ_2. The important integration region is determined for the first term by the factor

$$\cos\left[\frac{k\rho_1^2}{2(x-x_1)} - \frac{k\rho_2^2}{2(x-x_2)}\right].$$

The argument of the cosine is of the order $k(\rho_1^2 - \rho_2^2)/(2x - x_1 - x_2)$ where the mean value of the denominator has been taken. The integral over the region

$$q \equiv \left|\frac{k(\rho_1^2-\rho_2^2)}{2x-x_1-x_2}\right| \geqslant 1$$

is small because of the oscillations of the cosine. Therefore integration over the region q < 1 is most important, although in this region the argu-

2)A simple calculation shows that the corresponding error in the exponential index does not exceed 0.25 in the important integration region given by (15a), and is even smaller in most of the integration region.

ment $\rho_1 - \rho_2$ of the function

$$D_{S_1}\left(\frac{x_1+x_2}{2},\ \rho_1-\rho_2\right)$$

is of the order

$$|\rho_1-\rho_2| = \beta[(2x-x_1-x_2)/k]^{1/2}, \qquad (15)$$

where β is a numerical constant of the order of unity. Applying the theorem of the mean to the integrals over ρ_1 and ρ_2, we can place the function D_{S_1} outside of the integrand and use the substitution (15) for $|\rho_1 - \rho_2|$. The same considerations apply to the second term in (14), which leads to

$$|\rho_1+\rho_2| = \beta[(2x-x_1-x_2)/k]^{1/2}. \qquad (15a)$$

The constants β in (15) and (15a) can differ in general, but will be taken as identical for the sake of simplicity.

Using (15) and (15a), we obtain

$$\langle\chi^2\rangle = \left(\frac{k^2}{4\pi}\right)^2 \int_0^x \frac{dx_1}{x-x_1}\int_0^x \frac{dx_2}{x-x_2}\iiiint_{-\infty}^{\infty} dy_1\,dz_1\,dy_2\,dz_2$$
$$\times B_\varepsilon(\mathbf{r}_1,\mathbf{r}_2)$$
$$\times \exp\left\{-\frac{1}{8}D_{S_1}\left(\frac{x_1+x_2}{2},\ \beta\left[\frac{2}{k}\left(x-\frac{x_1+x_2}{2}\right)\right]^{1/2}\right)\right\}$$
$$\times \cos\frac{k\rho_1^2}{2(x-x_1)}\cos\frac{k\rho_2^2}{2(x-x_2)}. \qquad (16)$$

The analogous formula for $\langle S^2\rangle$ differs from (16) only by the replacement of cosines with sines.

Equation (16) reduces our present problem to the problem that was solved in [4] for amplitude and phase fluctuations in a medium with smoothly varying parameters of dielectric-constant fluctuations. Indeed, Eq. (16) can be treated as the (MSD) solution for amplitude fluctuations in a medium with the "effective" correlation function

$$B_э(\mathbf{r}_1,\mathbf{r}_2) = B_\varepsilon(\mathbf{r}_1,\mathbf{r}_2)$$
$$\times \exp\left\{-\frac{1}{8}D_{S_1}\left(\frac{x_1+x_2}{2},\ \beta\left[\frac{2}{k}\left(x-\frac{x_1+x_2}{2}\right)\right]^{1/2}\right)\right\}, \qquad (17)$$

depending on the longitudinal coordinate $(x_1 + x_2)/2$. When the fluctuations of the dielectric constant are described by a $2/3$ law[7] we have

$$\langle[\varepsilon_1(\mathbf{r}_1)-\varepsilon_1(\mathbf{r}_2)]^2\rangle = C_\varepsilon^2|\mathbf{r}_1-\mathbf{r}_2|^{2/3}.$$

Equation (17) gives an expression for the "effective" structural characteristic

$$C_э^2(\xi) = C_\varepsilon^2\exp\{-{}^1/_8 D_{S_1}(\xi,\ \beta[2(x-\xi)/k]^{1/2})\}$$
$$= C_\varepsilon^2\exp\{-0.16\beta^{5/3}C_\varepsilon^2k^{7/6}\xi(x-\xi)^{5/6}\}, \qquad (17a)$$

where the expression

$$D_{S_1}(\xi,\ \rho) = 0.73C_\varepsilon^2k^2\xi\rho^{5/3}, \quad \xi = (x_1+x_2)/2$$

has been used for the structure function $D_{S_1}(\xi, \rho)$ derived in the MSD approximation.

We now use the formulas given in [4] for the mean square fluctuations of the logarithm of the amplitude and the phase difference in the case of variable C_ϵ:[4]

$$\langle\chi^2\rangle = {}^{11}/_6\cdot 0.077k^{7/6}\int_0^x C_э^2(\xi)(x-\xi)^{5/6}\,d\xi, \qquad (18)$$

$$D_S(x,\rho) = 0.73k^2\rho^{5/3}\int_0^x C_э^2(\xi)\,d\xi. \qquad (19)$$

Substituting (17a), we obtain $\langle\chi^2\rangle$ and $D_S(x, \rho)$ corresponding approximately to $\varphi_2(\mathbf{r})$.

4. The foregoing calculation of the amplitude and phase fluctuations was based on the MSD phase structure function $D_{S_1}(x, \rho)$. When (19) is used to calculate the phase structure function in the next approximation, it is found to differ greatly from the initially used function $D_{S_1}(x, \rho)$ if

$$\sigma_1^2 = 0.077C_\varepsilon^2k^{7/6}x^{11/6},$$

which is the calculated MSD mean square fluctuation of the logarithm of the amplitude, is considerably larger than unity. Calculating a new phase structure function and inserting it into (19), we obtain the next approximation etc.

Obviously, if this iterative process converges, as can easily be proved, the limiting structure function of the phase satisfies the integral equation

$$D_S(x,\rho)$$
$$= 0.73C_\varepsilon^2k^2\rho^{5/3}\int_0^x \exp\left\{-\frac{1}{8}D_S\left(\xi,\beta\left[\frac{2(x-\xi)}{k}\right]^{1/2}\right)\right\}d\xi. \qquad (21)$$

The solution of this equation can be sought in the form

$$D_S(x,\ \rho) = 0.73C_\varepsilon^2k^2\rho^{5/3}xf(x/l), \qquad (22)$$

where the scale l is defined by

$$2.1\beta^{5/3}\sigma_1^2(l) = 2.1\beta^{5/3}\cdot 0.077C_\varepsilon^2k^{7/6}l^{11/6} = 1$$

and $f(t)$ is a dimensionless function of a dimensionless argument. Using a new integration variable $\xi = xp$, Eq. (21) can be transformed into

$$f(t) = \int_0^1 \exp[-t^{11/6}p(1-p)^{5/6}f(pt)]\,dp, \qquad (21a)$$

where $t = x/l$. Assuming $t = 0$ in (21a), we obtain $f(0) = 1$. Differentiating (21a) with respect to the

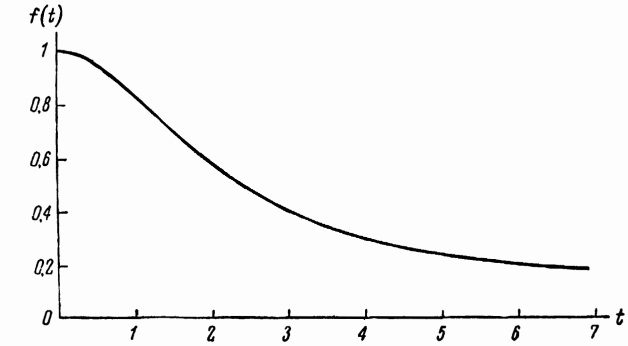

FIG. 1

variable $\tau = t^{11/6}$ and then assuming $t = 0$, we obtain $[df(\tau)/d\tau]_{\tau=0}$ and thus arrive at the series expansion of $f(t)$:

$$f(t) = 1 - {}^6/_{11}\cdot{}^6/_{17}t^{11/6} + \ldots \qquad (23)$$

When $t \gg 1$ the most important region in (21a) is the vicinity of the point $p = 1$. Hence the asymptotic behavior of $f(t)$ for large values of the argument is represented by

$$f(t) = \frac{1}{t}\left[\Gamma\left(\frac{11}{5}\right)\right]^{5/11} + \ldots = \frac{1.045}{t} + \ldots \quad (t \gg 1). \qquad (24)$$

For intermediate values of the argument the solution of (21a) is easily obtained numerically by means of a few iterations; the result is shown in Fig. 1. Using the asymptotic expression (24), we easily find that in the region $x \gg l$ the phase structural function ceases to increase with x, but approaches the constant limit

$$D_S(x, \rho) = 0.76 C_\varepsilon^2 k^2 \rho^{5/3} l \quad (x \gg l).$$

Let us now consider the quantity $\langle \chi^2 \rangle$. Substituting the derived function $D_S(x, \rho)$ into the formula

$$\langle \chi^2 \rangle = \frac{11}{6}\cdot 0.077 C_\varepsilon^2 k^{7/6} \int_0^x (x-\xi)^{5/6} \times \exp\left\{-\frac{1}{8} D_S\left(\xi, \beta\left[\frac{2(x-\xi)}{k}\right]^{1/2}\right)\right\} d\xi$$

and introducing again the variables $p = \xi/x$ and $t = x/l$, we obtain

$$\langle \chi^2 \rangle = \frac{11}{6}\frac{t^{11/6}}{2.1\beta^{5/3}} \int_0^1 (1-p)^{5/6} \exp[-t^{11/6} p (1-p)^{5/6} f(pt)]\, dp. \qquad (25)$$

In the region $\sigma_1^2(x) \ll 1$, i.e., $x \ll l$, we now obtain $\langle \chi^2 \rangle \approx \sigma_1^2(x)$. In the region $\sigma_1^2(x) \gg 1$, i.e., $x \gg l$, the principal contribution to the integral comes from the vicinities of the points $p = 0$ and $p = 1$. As a result of some simple calculations it is found that for $x \gg l$ Eq. (25) approaches the constant limit $121/30 \times 2.1\beta^{5/3}$, depending only on β. Replacing β with the parameter $\langle \chi^2 \rangle_\infty$ in (25), we obtain

$$\frac{\langle \chi^2(x) \rangle}{\langle \chi^2 \rangle_\infty} = \frac{5}{11} t^{11/6} \int_0^1 (1-p)^{5/6} \exp[-t^{11/6} p (1-p)^{5/6} f(pt)]\, dp, \qquad (25a)$$

where t is related to $\sigma_1^2(x)/\langle \chi^2 \rangle_\infty$ by

$$t^{11/6} = \frac{121}{30}\frac{\sigma_1^2(x)}{\langle \chi^2 \rangle_\infty}.$$

It follows that the ratio $\langle \chi^2 \rangle / \langle \chi^2 \rangle_\infty$ is a function of $\sigma_1^2(x)/\langle \chi^2 \rangle_\infty$. Equation (25a) can easily be calculated numerically.

5. The mean square fluctuation of the logarithm of the amplitude is expressed in terms of $\langle \chi^2 \rangle$ and $\langle \chi \rangle$ by

$$\langle [\ln A - \langle \ln A \rangle]^2 \rangle \equiv \sigma^2 = \langle \chi^2(x) \rangle - \langle \chi(x) \rangle^2.$$

The first of these two quantities has already been derived. The second quantity could be determined by averaging Eq. (11). However, $\langle \chi \rangle$ can be obtained considerably more simply on the basis of energy conservation. The energy flux is expressed in terms of χ by means of

$$\mathbf{P} = \mathrm{const}\, EE^*\mathbf{n} = \mathrm{const}\, A_0^2 \exp[2\chi(\mathbf{r})]\mathbf{n},$$

where n is a unit vector in the direction of wave propagation. For an incident plane wave we obviously require

$$\langle \exp[2\chi(\mathbf{r})]\mathbf{n} \rangle = \mathrm{const}.$$

When, as occurs customarily for wave propagation in a real turbulent atmosphere, the fluctuations of the propagation direction are small, the vector $\mathbf{n}$ can be replaced by its undisturbed value $\mathbf{n}_0$. In this case we have

$$\langle \exp[2\chi(\mathbf{r})] \rangle = \mathrm{const}. \qquad (26)$$

For the purpose of explicit averaging in this formula we must know the probability distribution of $\chi(\mathbf{r})$. From (11), in which S_1 has been replaced by S, it follows that φ is a linear functional of the random quantity

$$\varepsilon_1(\mathbf{r}') \exp\{{}^1/_2 i[S(\mathbf{r}') - S(\mathbf{r})]\}.$$

Although $S(\mathbf{r})$ is strongly correlated in the direction of wave propagation, it is considerably more weakly correlated in the transverse direction. The function $\epsilon_1(\mathbf{r})$ has the finite correlation radius L_0. Under the condition $x \gg L_0$, the product $\epsilon_1 \exp\{{}^1/_2 i[S(\mathbf{r}') - S(\mathbf{r})]\}$ also has a characteristic correlation scale of the order L_0, which is small compared with the integration

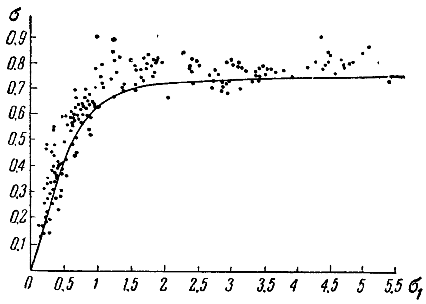

FIG. 2

region. Therefore (11) can be represented as the sum of a large number of uncorrelated terms. It follows that the logarithm of the amplitude and the phase are Gaussian random quantities, as in the case of $\sigma_1^2(x) \ll 1$, when employing the MSD. This is well confirmed by experimental results given in [5].

With the Gaussian form of $\chi(r)$ the averaging in (26) is easily performed and leads to

$$\exp\{2[\langle\chi(x)\rangle + \langle\chi^2(x)\rangle - \langle\chi(x)\rangle^2]\} = \text{const.} \qquad (27)$$

Since for $x = 0$ we have const = 1, it follows that

$$\langle\chi^2\rangle - \langle\chi\rangle^2 + \langle\chi\rangle = 0 \qquad (28)$$

is the relation between $\langle\chi^2\rangle$ and $\langle\chi\rangle$. Solving (28), we obtain the mean square fluctuation of the logarithm of the amplitude in terms of $\langle\chi^2\rangle$:

$$\sigma^2 = \langle\chi^2\rangle - \langle\chi\rangle^2 = (\langle\chi^2\rangle + {}^1/_4)^{1/2} - {}^1/_2. \qquad (29)$$

The function defined by Eq. (29) can then be determined only to within an unknown parameter $\langle\chi^2\rangle_\infty$, which remains undetermined because of the crude method of calculation employed in Sec. 3. When $\sigma_1^2(x) \gg 1$, $\sigma^2(x)$ approaches a constant limit:

$$\sigma_\infty^2 = (\langle\chi^2\rangle_\infty + {}^1/_4)^{1/2} - {}^1/_2. \qquad (30)$$

On the basis of the experimental data in [5] this limit is found to be $(0.8)^2$. The corresponding value of $\langle\chi^2\rangle_\infty$ is close to unity. The value of the parameter β required to account for the observed saturation level is 1.4; this is an entirely reasonable result.

Figure 2 shows the function in (29) for $\langle\chi^2\rangle_\infty = 1$, obtained by numerical integration of (25a) in conjunction with the experimental results in [5].

[1] S. M. Rytov, Izv. AN SSSR, Ser. fiz. 1, 223 (1937).

[2] A. M. Obukhov, Izv. AN SSSR, Ser. geofiz. **17**, 155 (1953).

[3] L. A. Chernov, Rasprostranenie voln v srede so sluchaĭnymi neodnorodnostyami (Wave Propagation in a Medium with Random Inhomogeneities), AN SSSR, 1959.

[4] V. I. Tatarskiĭ, Teoriya fluktuatsionnykh yavleniĭ pri rasprostranenii voln v turbulentnoĭ atmosfere (Fluctuation Theory for Wave Propagation in a Turbulent Atmosphere), AN SSSR, 1959.

[5] M. E. Gracheva and A. S. Gurvich, Izv. Vuzov, Radiofizika (in press).

[6] V. I. Tatarskiĭ, Izv. Vuzov, Radiofizika 5, 490 (1962).

[7] A. M. Obukhov, Izv. AN SSSR, Ser. geogr. i geofiz. **13**, 58 (1959).

Translated by I. Emin
203