RESTRICTIONS OF THE ELASTIC SCATTERING CROSS SECTION IN THE DIFFRACTION PEAK REGION

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It is shown that if the real part of the elastic scattering amplitude is smaller than the imaginary part in the diffraction peak region, then the unitarity condition imposes an upper limit on the elastic scattering cross section (inequality (20)).

1. It has been shown by several authors that the unitarity condition ($|a_l| \leq 1$) together with the conditions that the scattering amplitude be analytic in the momentum transfer and that the energy growth be moderate ($|A| < cs^N$) impose limitations on the behavior of the amplitude at high energies. In particular, Froissart et al.[1-2] obtained, assuming analyticity of the amplitude in the Mandelstam ellipse, the following estimate:

$$|A| < \text{const}\, s \ln^2 s, \qquad |A| < \text{const}\, \vartheta^{-1/2} s \ln^{3/2} s. \tag{1}$$

Kinoshita and Martin[3], using analyticity in a somewhat larger region of the momentum transfer, obtained the estimate

$$|A| < \text{const}\, \vartheta^{-2} \ln^{3/2} s. \tag{2}$$

Since the condition for moderate growth of the energy contains the constants c and N, concerning the magnitude of which we can say nothing, the estimates obtained include an undetermined constant.

In this paper we obtain a limitation on the elastic scattering amplitude in the region of the diffraction peak; this limitation does not contain unknown constants. We make use here of the condition that in the diffraction-peak region the real part of the amplitude is smaller than the imaginary part:

$$|\text{Re}\, A| \leqslant \text{Im}\, A. \tag{3}$$

The limitation obtained contains no unknown constants and is more stringent in the sense of the energy dependence than the estimate of Kinoshita (it does not have a logarithmic factor), but it is more trivial, since the employed condition (3) has a phenomenological character, and is not derived, at least at the given stage, from more general requirements.

In justification of this condition, which in general has always been assumed as natural, we can state that the imaginary part of the amplitude is singled out by the unitarity condition for elastic scattering, and also that this inequality has been experimentally confirmed.[4]

2. Since the cross section decreases with increasing angle in the region of the peak, the cross section for the scattering through a given angle ϑ_0 is smaller than the average value in the angle interval from 0 to ϑ_0

$$\frac{d\sigma}{do}(\vartheta_0) < \left[\int_{z_0}^{1} \left(\frac{d\sigma}{do}\right)^{1/2} w(z)\, dz\right]^2, \tag{4}$$

where $z = \cos\vartheta$, $z_0 = \cos\vartheta_0$, and w is a positive wave function, normalized by the condition

$$\int_{z_0}^{1} w\, dz = 1. \tag{5}$$

We choose it in such a way that the right side in (4) is a minimum. On the other hand, by virtue of (3) we have

$$\left(\frac{d\sigma}{do}\right)^{1/2} = \frac{2}{\sqrt{s}}|A| = \frac{2}{\sqrt{s}}(\text{Re}^2 A + \text{Im}^2 A)^{1/2} \leqslant \frac{2\sqrt{2}}{\sqrt{s}}\text{Im}\, A, \tag{6}$$

so that

$$\frac{d\sigma}{do} < \frac{8}{s}\left[\int_{z_0}^{1} w\, \text{Im}\, A\, dz\right]^2. \tag{7}$$

We substitute in (7) the expansion of the imaginary part of the amplitude in partial waves, and also multiply and divide the corresponding terms by a factor $[(l + 1/2)^2 + \lambda^2]^{1/2}$, where λ is a parameter, for the time being arbitrary, which we shall choose later from the condition of the optimality of the limitation

$$\frac{d\sigma}{do} < \frac{2}{q_s^2}\left\{\sum_{l=0}^{\infty}(2l+1)\,\text{Im}\, a_l \int_{z_0}^{1} wP_l(z)\, dz \times [(l+1/2)^2+\lambda^2]^{1/2}[(l+1/2)^2+\lambda^2]^{-1/2}\right\}^2 \tag{8}$$

We use further the Cauchy inequality $[(xy)^2 \leq x^2y^2]$ and replaced Im a_l by unity

$$\frac{d\sigma}{do} < \frac{2}{q_s^2} \sum_{l'=0}^{\infty} \frac{1}{(l'+1/2)^2+\lambda^2} \times \sum_{l=0}^{\infty} [(l+1/2)^2+\lambda^2](2l+1)^2 \left[\int_{z_0}^{1} wP_l dz\right]^2. \quad (9)$$

The first sum in (9) can be calculated in explicit form

$$\sum_{l=0}^{\infty} [(l+1/2)^2+\lambda^2]^{-1} = \frac{\pi \operatorname{th} \lambda}{2\lambda} < \frac{\pi}{2\lambda}. \quad (10)^*$$

We transform the second sum in (9) by representing w in the form

$$w = \int_{\vartheta}^{\vartheta_0} \frac{v(\varphi)\, d\varphi}{[2(z-\cos\varphi)]^{1/2}}. \quad (11)$$

Then we have for the average value of the Legendre polynomial

$$\int_{z_0}^{1} wP_l dz = \int_{0}^{\vartheta_0} v(\varphi)\, d\varphi \int_{\cos\varphi}^{1} \frac{P_l(z)\, dz}{[2(z-\cos\varphi)]^{1/2}} = \frac{1}{l+1/2} \int_{0}^{\vartheta_0} v(\varphi) \sin(l+1/2)\varphi\, d\varphi. \quad (12)$$

Substituting this expression in (9), we obtain

$$\frac{d\sigma}{do} < \frac{2}{q_s^2}\, \frac{\pi}{2\lambda}\, 4 \sum_{l=0}^{\infty} [(l+1/2)^2+\lambda^2] \left[\int_{0}^{\vartheta_0} v(\varphi)\sin(l+1/2)\varphi\, d\varphi\right]^2$$

$$= \frac{4\pi}{q_s^2\lambda} \sum_{l=0}^{\infty} \left\{\left[\int_{0}^{\vartheta_0} v(\varphi) \frac{d}{d\varphi}\cos(l+1/2)\varphi\, d\varphi\right]^2 + \lambda^2 \left[\int_{0}^{\vartheta_0} v(\varphi)\sin(l+1/2)\varphi\, d\varphi\right]^2\right\}. \quad (13)$$

We evaluate the first integral by parts, putting $v(\varphi_0) = v(0) = 0$ (this is necessary for the sum over l to converge), and use the completeness property of the systems of functions $\cos(l+1/2)\varphi$ and $\sin(l+1/2)\varphi$:

$$\sum_{l=0}^{\infty} \cos(l+1/2)\varphi \cos(l+1/2)\varphi' = \sum_{l=0}^{\infty} \sin(l+1/2)\varphi \sin(l+1/2)\varphi' = \frac{\pi}{2}\delta(\varphi-\varphi'). \quad (14)$$

We then obtain

$$\frac{d\sigma}{do} < \frac{2\pi^2}{q_s^2\lambda} \int_{0}^{\vartheta_0} [v'^2+\lambda^2 v^2]\, d\varphi. \quad (15)$$

The condition (5) for the normalization of w is expressed in terms of v in the form

$$\int_{z_0}^{1} w\, dz = 2\int_{0}^{\vartheta_0} v(\varphi)\sin\frac{\varphi}{2}\, d\varphi = 1. \quad (16)$$

*th = tanh.

(It is obtained from (12) with $l = 0$.)

We have thus arrived at a standard isoperimetric problem: find the minimum of the functional (15) under boundary conditions $v(\varphi_0) = v(0) = 0$ and the additional condition (16). Solving it by standard means,[5] we find that the minimum is realized by the function

$$v(\varphi) = c\left[\sin\frac{\varphi}{2} - \operatorname{sh}\lambda\varphi \sin\frac{\vartheta_0}{2} \Big/ \operatorname{sh}\lambda\vartheta_0\right], \quad (17)^*$$

$$c = (\lambda^2+1/4)\left[(\lambda^2+1/4)(\vartheta_0-\sin\vartheta_0) + 1/2\sin\vartheta_0 - 2\lambda \operatorname{cth}\lambda\vartheta_0 \sin^2\frac{\vartheta_0}{2}\right]^{-1} \quad (18)$$

and is equal to

$$\frac{\pi^2(\lambda^2+1/4)}{q_s^2\lambda}c.$$

In the case of small angles, which are of principal interest to us, we obtain, discarding higher terms in ϑ_0^2

$$\frac{d\sigma}{do} < \frac{2\pi^2}{q_s^2\vartheta_0^4} \frac{\gamma^3}{[1/3\gamma^2+1-\gamma\operatorname{cth}\gamma]} \quad (19)$$

(γ stands for $\lambda\vartheta_0$). The parameter γ is arbitrary, and we choose its value from the condition that the second factor in (19) be a minimum. This minimum is attained for $\gamma = 3.7$ and equal to 27.25.

As a result we have

$$\frac{d\sigma}{do} < 538\frac{q_s^2}{t^2}. \quad (20)$$

The obtained upper bound has the same angular dependence as the lower bound derived in the paper of Kinoshita and Martin (ϑ^{-2}), this being connected with the analogous mechanism for the occurrence of the upper bound, namely: high partial waves, starting with $l \sim 1/\vartheta$, oscillate and suppress each other, while the first const. ϑ^{-1} of the partial waves make a contribution $\sim \vartheta$.

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*sh = sinh; cth = coth.

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