A SELF-CONSISTENT CALCULATION OF THE K RESONANCE PARAMETERS*

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The mass and width of the K* meson are calculated by the Balázs method. The solution depends on the following parameters; the slope ϵ of the Regge trajectory of the K* meson, the reference point s_0, the function R(s) which describes the contribution from inelastic processes, and the point t_d which divides the physical energy region into a part in which the contribution from inelastic processes is small ($R(s) \approx 1$) and another in which it is appreciable ($R(s) \gg 1$). The value of the ρ meson mass, t_r, and of the width Γ_1^1 are taken from experiment. The following solutions have been obtained for various values of the parameters ϵ, s_0, t_d, and R(s): (1) $M_{K^*} = 832$ MeV and $\Gamma_{K^*} = 82.5$ MeV (the experimental values are 888 and 50 MeV respectively); (2) $M_{K^*} = 815$ MeV and $\Gamma_{K^*} = 51.4$ MeV.

1. INTRODUCTION

THE bootstrap method proposed by Chew and Frautschi about four years ago[1] is by now rather widely known. Its main content, its positive features, and disadvantages have been presented in the review paper by Shirkov[2]. The bootstrap method has been used to describe almost all the meson and baryon resonances known to date.

In the present paper the bootstrap method is used to find the parameters of the K* resonance ($T = 1/2$; $J = 1$; the mass $M_{K^*} = 888$ MeV, the width $\Gamma_{K^*}^{1/2} = 50$ MeV). This question has already been considered in a number of papers[3-5]. The paper by Capps[3] considered a three-channel problem (πK, $\pi\eta$, ηK). The author failed to obtain good agreement with the experimental data, particularly as regards the width of the resonance. In his opinion it is possible to improve the bootstrap method by using a Regge representation for the asymptotic behavior of the amplitude. The papers by Diu et al.[4] discussed both one-channel and two-channel problems. Neither of these cases gave a convincing proof of the existence of a bootstrap solution. In this discussion two important points came up (a) the results depended on the cut-off and consequently the bootstrap solution was sensitive to the assumption about the asymptotic behavior of the solution; (b) in the two-channel approach there was a problem about the stability of the resulting solutions. Fulco et al.[5] considered the effect of the nearby inelastic channels on the width of the resonance. These authors obtained interesting results which indicated a narrowing of the resonance by the presence of nearby channels. However, their results also depend substantially on the cut-off. In the opinion of these authors an improvement of the model would consist of a more accurate investigation of the asymptotic behavior of the amplitudes and allowance for inelastic processes.

In the present paper the parameters of the K* resonance are determined by the method of Balázs[6] in which the asymptotic behavior of the amplitudes is described by Regge poles in the crossed channels, and the effect of inelastic processes is included by introducing a certain function in the two-channel unitarity condition.

Besides obtaining solutions which are close to the experimental data the paper also discusses the sensitivity of these solutions to the choice of parameters. It turns out in particular that the solutions found depend substantially on the choice of the reference point. Similar results were also found in other papers (see for example [7]).

2. KINEMATICS, STATEMENT OF THE PROBLEM

The amplitude of the scattering process

$$\pi(q_1) + K(p_1) \rightarrow \pi(q_2) + K(p_2)$$

is considered as a function of the three variables s, u, and t. We use the following notation: M and μ are the masses of the K and π mesons, k_s and z_s are the momentum and the cosine of the scattering angle in the c.m.s. of the s channel. In the crossed u channel ($\pi' + K \rightarrow \pi' + K$) s and u change places. The connection between the am-

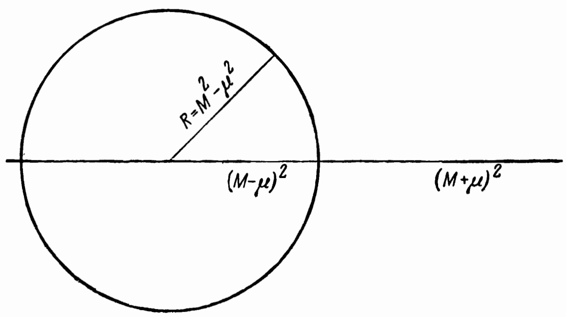

FIG. 1. Location of the cuts in the partial-wave amplitude of the scattering process $\pi K \to \pi K$ in the s plane.

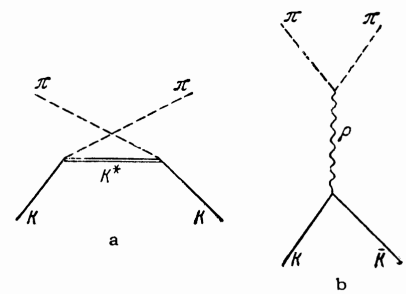

FIG. 2. Diagrams contributing to the low-energy potentials on the left-hand cut.

plitudes for πK, $\pi' K$ scattering and for $\pi\pi \to KK$ is given by the rules

$$A^{I}(s,t) = \sum_{I'} \alpha_{II'} \bar{A}^{I'}(u,t) = \sum_{I'} \lambda_{II'} \bar{A}^{I'}(t,s),$$

$$\alpha_{II'} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}; \quad \lambda_{II'} = \begin{pmatrix} 1 & 6^{-1/2} \\ -1/2 & 6^{-1/2} \end{pmatrix}.$$

It follows from the Mandelstam representation that the partial amplitudes in the s channel have cuts in the s plane, as shown in Fig. 1. In the Balázs method these three cuts are approximated by two cuts along the real axis: (1) the right (physical) cut from $M + \mu$ to $+\infty$; the left (unphysical) cut from $-\infty$ to $M^2 - \mu^2$. The problem is to find the amplitude $A_{J=1}^{T=1/2}$ from which the parameters of the K* resonance can be determined. The weight on the left-hand cut is at low-energies given by the diagrams shown in Fig. 2; at high energies it is determined by the Regge pole in the s channel. The position of the ρ meson and the coupling constants $g_{\rho\pi\pi}$ and $g_{\rho K\bar{K}}$ are known parameters. The mass and width of the K* meson are determined from the bootstrap equations.

3. DERIVATION OF THE BOOTSTRAP EQUATIONS

We seek the amplitude $A_1^{1/2}(s)$ in the form

$$H_1^{1/2}(s) = \frac{1}{s-(M+\mu)^2} A_1^{1/2}(s) = \frac{N(s)}{D(s)}, \tag{1}$$

where N(s) has only a left-hand cut and D(s) only a right-hand cut. We write the unitarity condition for the function $H_1^{1/2}(s)$;

$$\operatorname{Im}[H_1^{1/2}(s)^{-1}] = -(k_s/s^{1/2})[s-(M+\mu)^2]R(s), \tag{2}$$

where R(s) is the ratio of total to elastic cross section. The function R(s) allows for the contribution from inelastic processes. From (1) we find Im N(s) (on the left-hand cut) and Im D(s) (on the right-hand cut), which allows us to write down equations for N(s) and D(s):

$$N(s) = \frac{1}{\pi}\int_{-\infty}^{M^2-\mu^2} ds' \frac{D(s')\operatorname{Im} H_1^{1/2}(s')}{s'-s},$$

$$D(s) = 1 - \frac{s-s_0}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{k_s'}{\sqrt{s'}} \times \frac{N(s')R(s')[s'-(M+\mu)^2]}{(s'-s)(s'-s_0)}. \tag{3}$$

The dispersion relation for D(s) is written with one subtraction at the reference point s_0: $D(s_0) = 1$. The contribution from the left-hand cut is written as a sum of two pole terms:

$$N(s) = \sum_{i=1,2} \frac{a_i}{s-s_i}, \tag{4}$$

where a_i are the residues at suitably chosen poles. The positions of the poles are chosen by the method given in the paper by Balázs [6]; they are $s_1 = -57$ and $s_2 = 10.5$ (if the cut in the t channel is allowed for) or $s_1 = -57$ and $s_2 = 4.5$ (if the cut of the t channel is neglected). The second pair of poles correspond to the case in which the influence of ρ-meson forces is practically zero.

The parameters a_1 and a_2 are determined by comparing the amplitude (1) and its first derivative $\partial H_1^{1/2}(s)/\partial s$ with the calculated function $H_1^{1/2}(s)$ and its first derivative, respectively, at a certain comparison point s_{comp}. We choose $s_{comp} = s_0$ to reduce the number of parameters.

We next calculate the function $H_1^{1/2}(s)$. At a fixed value of s we write the following partial-wave dispersion relation:

$$A_1^{1/2}(s) = \frac{1}{2\pi k_s^2}\left\{\int_{4\mu^2}^{\infty} A_t^{1/2}(s,t')Q_1\left(1+\frac{t'}{2k_s^2}\right)dt' - \int_{(M+\mu)^2} A_u^{1/2}(s,u')Q_1\left(-1-\frac{u'-(M^2-\mu^2)^2/s}{2k_s^2}\right)du'\right\}, \tag{5}$$

where $A_u^{1/2}(s, u')$ and $A_t^{1/2}(s, t')$ are the imaginary parts of the amplitudes in the u and t channels, which can be expanded, as usually in physical channels, in powers of the cosines of the angles z_u and z_t:

$$z_u = 1 + \frac{2(M^2+\mu^2) - u - s}{2k_u^2}; \quad z_t = \frac{s + p^2 + q^2}{2pq},$$

Here p and q are the momenta of the k and π mesons in the t channel. One limitation on the choice of the comparison point $s_{comp} = s_0$ results from a consideration of the region of analyticity of $A_u^{1/2}$ and $A_t^{1/2}$. The comparison point must be chosen at a value of s for which their expansions are valid.

In Eq. (5) each integral consists of two parts, a low-energy part $A_1^{1/2(H)}(s)$ and a high-energy part $A_1^{1/2(H)}(s)$. It is assumed that the dominant part of the low-energy contribution in the u channel comes from the diagram giving an exchange of K* mesons (Fig. 2a) and therefore the expansion of $A_u^{1/2}(s, u')$ contains only one term with $l = 1$. The partial wave $A_{1u}^{I'}(u')$ is approximated by a Breit-Wigner formula. In the approximation of zero width this becomes

$$A_{1u}{}^{I'}(u') = -[u' - (M+\mu)^2]\pi\Gamma_1{}^{1/2}\delta(u' - u_r), \quad (6)$$

where u_r is the position of the K* resonance and $\Gamma_1^{1/2}$ is related to the width of the K* resonance by

$$\Gamma_{K^*}{}^{1/2} = \frac{k_r[s_r - (M+\mu)^2]}{s_r}\Gamma_1{}^{1/2}.$$

The expression for $A_{lt}^{I'}(t')$ is found from the appropriate diagram (Fig. 2b) by perturbation theory:

$$A_{1t}{}^{I'}(t') = 3(pq)\pi\Gamma_1{}^1\,\delta(t - t_r), \quad (7)$$

where $\Gamma_1^1 = {}^8\!/_3\, g_{\pi\pi\rho} g_{K\bar{K}\rho}$, and $t_r = M_\rho^2$ (M_ρ is the mass of the ρ meson). Using (6) and (7) we find for the low-energy part $A_1^{1/2(L)}(s)$ of the amplitude (5) the expression

$$A_1{}^{1/2(L)}(s) = \frac{3\Gamma_1{}^1}{4k_s^2}(s + p_r^2 + q_r^2)\,Q_1\left(1 + \frac{t_r}{2k_s^2}\right) + \frac{\Gamma_1{}^{1/2}}{2k_s^2}$$
$$\times\left\{u_r - (M+\mu)^2 + \frac{2u_r[2(M^2+\mu^2) - u_r - s]}{u_r - (M-\mu)^2}\right\}Q_1$$
$$\times\left(1 + \frac{u_r - (M^2-\mu^2)^2/s}{2k_s^2}\right). \quad (8)$$

The high-energy part $A_1^{1/2(H)}(s)$ will, by assumption, be determined by the K* meson in the s channel, i.e.,

$$A_\alpha{}^{1/2}(s,t) = -\frac{\pi[2\alpha(s)+1]}{2\sin\pi\alpha(s)}\beta(s)\,[P_\alpha(-z) - P_\alpha(z)].$$

A similar expression can be obtained for $A_\alpha^{1/2}(s, u)$. The cuts of the function $P_\alpha(-z)$ and $P_\alpha(z)$ in the u and t channels are known.

Inserting in (5) the asymptotic expression for the imaginary parts of $A_\alpha^{1/2}(s, t)$ and $A_\alpha^{1/2}(s, u)$ and of the functions $Q_1(z_u)$ and $Q_i(z_t)$ we find

$$A_1{}^{1/2(H)}(s) = -\frac{1}{3}\frac{[2\alpha(s)+1]\,\beta(s)C_1(\alpha)}{\alpha - 1}\left(\frac{t_d}{2k_s^2}\right)^{\alpha-1}, \quad (9)$$

where t_d is the lower limit in both integrals in (5), which is determined from two considerations: (a) first, this limit must be far enough to relate to the asymptotic behavior of the amplitude; (b) second, it cannot lie closer than the singularities of the functions $P_\alpha(z_t)$ and $P_\alpha(-z_u)$. These requirements are satisfied if $t \gtrsim 130$. In our work we took $t_d = 130$.

We assume that[1)]

$$C_1(\alpha)\,[2\alpha(s) + 1]\,\beta(s)\,(2k_s^2)^{1-\alpha} \approx \text{const},$$
$$\operatorname{Re}\alpha \approx 1 + \varepsilon(s - s_r),$$

where ϵ is the slope of the Regge trajectory. We determine the residue $\beta(s)$:

$$\beta(s) = [s - (M+\mu)^2]\,\Gamma_1{}^{1/2}\frac{d\alpha}{ds}\bigg|_{s=s_r}.$$

The expression for $A_1^{1/2(H)}(s)$ now takes the form

$$A_1{}^{1/2(H)}(s) = \Gamma_1{}^{1/2}\frac{s - (M+\mu)^2}{s_r - s}\,t_d{}^{\varepsilon(s-s_r)}. \quad (10)$$

Hence we find from (1), (4), (5), (8), and (10) the following equations for the residues a_1 and a_2:

$$A_1{}^{1/2}(s_0) = [s_0 - (M+\mu)^2]\left[\frac{a_1}{s_0 - s_1} + \frac{a_2}{s_0 - s_2}\right],$$

$$\frac{\partial A_1{}^{1/2}(s)}{\partial s}\bigg|_{s=s_0} = \sum_{i=1,2} a_i\left\{\frac{1}{s_0 - s_i} - \frac{s_0 - (M+\mu)^2}{(s_0 - s_i)^2}\right.$$
$$+ \frac{A_1{}^{1/2}(s_0)}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\,\frac{k_s'[s' - (M+\mu)^2]}{\sqrt{s'}(s - s_0)^2(s' - s_i)}[\theta(s_d - s)$$
$$\left. + R(s)\,\theta(s - s_d)]\right\}. \quad (11)$$

In Eq. (11) we treat the quantity $R(s)$ as a constant and take it out of the integral. An approximate expression for it will be given below. For the value of $A_1^{1/2}(s_0)$ at the point $s_0 = s_{comp}$ on the left we insert the sum of (8) and (10), and in place of the derivative $\partial A_1^{1/2}(s)/\partial s|_{s=s_0}$ we use the derivative of the same sum. The function $D(s)$ can now be found by inserting for a_1 and a_2

1)It is shown in [8] that this choice ensures a satisfactory narrowing of the diffraction peak.

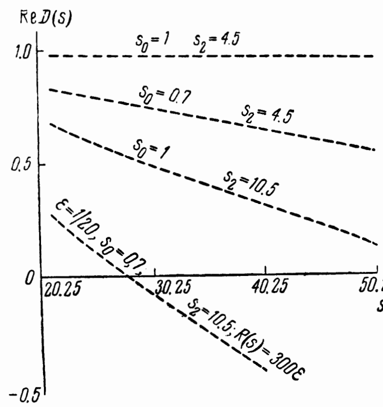

FIG. 3. Behavior of ReD(s) and its dependence on the parameters $s_0 = s_{comp}$ and s_2 ($s_2 = 2.5$ corresponds to the contribution from ρ-exchange forces).

from (11) in (3), and the position and width of the resonance are then found from the conditions

$$\operatorname{Re} D(s_r) = 0, \quad \Gamma_1{}^{1/2} = -N(s_r)\left|\frac{\partial \operatorname{Re} D(s)}{\partial s}\right|_{s=s_r}. \quad (12)$$

The behavior of Re D(s) near the resonance is very well approximated by (see Fig. 3):

$$\operatorname{Re} D(s) \approx \frac{s - s_r}{s_0 - s_r}$$

and therefore

$$\Gamma_1{}^{1/2} \approx (s_r - s_0) N(s_r).$$

The quantity R(s) is calculated exactly as in the paper of Balázs[6] on the assumption that the contribution of inelastic processes to $\pi K \to \pi K$ scattering is taken care of by the Pomeranchuk pole:

$$R(s) \approx \frac{64\pi\varepsilon \ln s}{\sigma_{tot}(1 + \pi^2/4\ln^2 s)},$$

where σ_{tot} is the total cross section for the annihilation reaction $\pi\pi \to K\bar{K}$. To simplify the calculations this expression is always taken at the point $s = s_d$. For the particular values $\sigma_{tot} = 50$ mb and $\epsilon = 1/20$, we get

$$R(s) = 300\ \varepsilon = 15.$$

4. CALCULATION OF THE K* RESONANCE PARAMETERS. CONCLUSIONS

Equations (8), (10), and (11) show that Γ_1^1, t_r, ϵ, t_d, s_0, and R(s) are free parameters in our calculation. In reality none of these quantities can take arbitrary values. The width Γ_1^1 and position t_r of the ρ-meson resonance must be taken from experiment. The slope ϵ of the Regge trajectory must lie between the physically reasonable limits $1/50 < \epsilon < 1/10$, and the quantity R(s), which depends on σ_{tot} and ϵ, is bounded by the physical limits 200ϵ, and 350ϵ, if one assumes that σ_{tot} lies between 40 and 80 mb. The quantity t_d is defined as the limit above which the contribution from inelastic processes becomes substantial, and is also dependent on the considerations mentioned in the explanation of Eq. (9). The position of the chosen point s_0 (or the comparison point) is restricted by a number of arguments[4,6]: to the left its value is limited by the condition for the expansion of the functions $A_u^{1/2}(s,u)$ and $A_t^{1/2}(s,t)$, and to the right it cannot lie beyond the threshold for the physical process. In our problem we choose it near the end of the unphysical cut.

The self-consistent calculation of the K* parameters consisted in assuming, for given Γ_1^1, t_r, ϵ, t_d, s_0, and R(s), some input values of the parameters $\Gamma_1^{1/2\,\text{in}}$ and s_r^{in}, chosen close to the experimental data. On the basis of these values we then calculated the residues a_1 and a_2 (see Eq. (11)) and used these to solve Eq. (12). If the resulting output values of the parameters, $\Gamma_1^{1/2\,\text{out}}$ and s_r^{out} agreed with $\Gamma_1^{1/2\,\text{in}}$ and s_r^{in}, the calculation was finished. Since the calculation was done numerically and the output values $\Gamma_1^{1/2\,\text{out}}$, s_r^{out} agreed only approximately with $\Gamma_1^{1/2\,\text{in}}$ and s_r^{in}, they were recalculated from the same equations to check the convergence of the solution.

In this manner several solutions were found. We show the most interesting cases in the table.

All the solutions listed in the table took the ρ meson forces into account. Within the range of values for the parameters which we considered we did not find a single bootstrap solution without a contribution from ρ mesons. To improve the agreement of the resulting value of the width $\Gamma_{K^*}^{1/2}$ with experiment it is necessary either to increase the slope of the Regge trajectory (see solutions 1 and 2) or to decrease the value of s_r^{in}. It is possible that there are two different comparison points, or two different slopes for the

Solution No.	ϵ	s_0	R (s)	t_r for ρ meson	Γ_1'	$s_r^{\text{in}} = u_r^{\text{in}}$, MeV	s_r^{out}, MeV	$\Gamma_{K^*}^{1/2\,\text{in}}$, MeV	$\Gamma_{K^*}^{1/2\,\text{out}}$, MeV
1	1/20	1.3	310	30	0.8	827	832	82.7	82.5
2	1/12	1.75	300	30	0.8	817	815	52.4	51.4
3	1/20	1.62	350	30	10	817	816	76.2	75.2
4	1/12	1.514	310	30	30	803	803	61.0	61.0

Regge trajectory and two different comparison points, which lead to the same solution for s_r and $\Gamma^{1/2}_{K^*}$. It is very important to allow for inelasticity. For reasonable agreement with experiment the quantity R(s) must be taken between 15 and 25. The solution depends strongly on the choice of the point $s_0 = s_{comp}$ (see Fig. 3).

In the range $-10 < s_0 < 20$, 25 and $s_0 < s < 50$ all solutions which were found were unique.

Hence it is possible to obtain even in the simplest single-channel problem, satisfactory agreement with experiment, particularly for the width. It is known that the inclusion of other channels has the effect of reducing the width of the resonance[5]. In the Balázs method the influence of other channels is included through the function R(s). One disadvantage of the method is the dependence of the solution on a large number of parameters. As a result it produces in addition to the physically interesting solutions also others which have no relation to the experimental data.

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