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# ON THE UNIQUENESS OF THE WIGHTMAN FUNCTIONAL

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It is shown that an irreducible Wightman functional (defined by a sequence of functions $W_0$; $W_1(x_1); \ldots; W_k(x_1 \ldots x_k); \ldots$), satisfying the usual requirements and the requirement of the existence of a minimal mass, is uniquely determined by its $\Omega$, i.e., the set of all

$$g \equiv (g_0;\ g_1(x_1);\ \ldots;\ g_n(x_1 \ldots x_n);\ 0;\ 0;\ \ldots)$$

(The $g_k(x_1 \ldots x_k)$ are infinitely differentiable functions, decreasing faster than any inverse power as $x_1 \to \infty$, for which

$$W(g^+g) \equiv \sum_{l,\, m} \int \ldots \int dx_1 \ldots dx_{l+m} W_{l+m}(x_1 \ldots x_{l+m}) \bar{g}_l(x_l \ldots x_1) g_m(x_{l+1} \ldots x_{l+m}) = 0.$$

If a functional consists of a finite number of irreducible ones, its $\Omega$ allows a unique reconstruction of all the irreducible components. Several consequences of these assertions are discussed.

## 1. INTRODUCTION

WE briefly recall the formalism to be used. A detailed account can be found in Borchers' article [1]. The necessary mathematical concepts are treated in sufficient detail in the book by Gel'fand and Vilenkin [2]. In this formulation the theory is defined by a functional W over the space $\Sigma$. The elements of $\Sigma$ are the truncated sequences

$$g \equiv \{g_0; g_1(x_1); g_2(x_1x_2); \ldots;\ g_n(x_1 \ldots x_n); 0; 0; \ldots\},$$

where $g_0$ is a complex number and $g_n \in S_{4n}$, the space of infinitely differentiable rapidly decreasing functions of 4n variables. The spaces $S_{4n}$ are nuclear topological vector spaces and $\Sigma$ which is constructed as an exact inductive limit of such spaces, is itself a nuclear space. The introduction of the following operations:

a) multiplications

$$(gh)_n(x_1 \ldots x_n) \equiv \sum_{k=0}^{n} g_k(x_1 \ldots x_k) h_{n-k}(x_{k+1} \ldots x_n),$$

b) involution

$$(g^+)_n(x_1 \ldots x_n) \equiv g_n(x_n \ldots x_1)$$

transforms $\Sigma$ into a *-algebra. In addition, the operation $(a, \Lambda)$ (a is a four-dimensional translation, $\Lambda$ is a Lorentz transformation matrix) is defined on $\Sigma$ as follows:

$$[(a, \Lambda)g]_n(x_1 \ldots x_n) \equiv g_n(\Lambda^{-1}(x_1 - a), \ldots, \Lambda^{-1}(x_n - a)).$$

The following requirements are imposed on the functional W:

1) W must be a linear functional, continuous in the topology of $\Sigma$.

2) W must be a relativistic invariant, i.e., for an arbitrary $(a, \Lambda)$ it must satisfy the condition $W((a, \Lambda)g) = W(g)$.

3) The support of $W_n(p_1 \ldots p_n)$ considered as a support of a generalized function) must be localized in the domain

$$p_n,\ p_{n-1} + p_n;\ \ldots,\ p_2 + \ldots + p_n \in \bar{L}_+ \quad p_1 + \ldots + p_n = 0,$$

where $\bar{L}_+$ is the closed forward light cone (this condition is necessary in order that the spectrum of the energy momentum operator of the theory be localized in $\bar{L}_+$);

4) Local commutativity:

$$W_n(x_1 \ldots x_{i-1}x_ix_{i+1}x_{i+2} \ldots x_n) = W_n(x_1 \ldots x_{i-1}x_{i+1}x_ix_{i+2} \ldots x_n)$$

for arbitrary n and i if

$$(x_i^0 - x_{i+1}^0)^2 - (\mathbf{x}_i - \mathbf{x}_{i+1})^2 \equiv (x_i - x_{i+1})^2 < 0.$$

5) Multiplicative positive semi-definiteness of W: $W(g^+g) \geq 0$.

In order to construct the theory from a given W we introduce the space $\Omega$:

$$\Omega \equiv \{g : W(g^+g) = 0\}.$$

$\Omega$ is closed, since W is continuous. We then construct the quotient space $L = \Sigma/\Omega$ consisting of equivalence classes (cosets) $\psi(g)$ of elements of $\Sigma$ (two elements $g_1$ and $g_2$ of $\Sigma$ are equivalent if $g_1 - g_2 \in \Omega$). The functional W defines on the space L an inner product $\langle \psi(g), \psi(h) \rangle \equiv W(g^+h)$, which does not depend on the particular choice of the representatives g and h in the respective equivalence classes, as a consequence of requirement 5). By completing L with respect to this inner product, we obtain a Hilbert space H. At the same time, since L is a quotient of two nuclear spaces, it is itself a nuclear space. Introducing the dual space $L^+$ of linear continuous functionals over L we arrive at the rigged Hilbert space (Gel'fand triple of spaces)* $L \subset H \subset L^+$. Obviously $W \in L^+$.

The operation $(a, \Lambda)$ generates unitary representations on H of the inhomogeneous Lorentz group: $U(a, \Lambda)\psi(g) \equiv \psi((a, \Lambda)g)$, and the algebraic operations in $\Sigma$ induce on H the algebra a of operators $a(g)$: $a(g)\psi(h) \equiv \psi(gh)$. If W admits a representation of the form

$$W(g) = \sum_\alpha |\rho_\alpha|^2 W_\alpha(g),$$

where the $W_\alpha$ satisfy all the above requirements, the functional W is called reducible. We shall consider below only functionals consisting of a finite number of irreducible ones. To each irreducible functional $W_\alpha$ corresponds a vector $\psi_{0\alpha} \in H$, such that $U(a, \Lambda)\psi_{0\alpha} = \psi_{0\alpha}$, and

$$\langle \psi_{0\alpha}, \psi_{0\beta} \rangle = \delta_{\alpha\beta}, \quad \psi(1) = \sum_\alpha \rho_\alpha \psi_{0\alpha},$$

$$W_\alpha(g) = \rho_\alpha^{-1} \langle \psi_{0\alpha}, \psi(g) \rangle.$$

We impose yet another condition on W:

6) The energy-momentum operator must have zero as an isolated point of the spectrum.

Owing to the relativistic invariance of the theory, the condition 6) is equivalent to the existence of a $\mu > 0$ such that the whole continuous spectrum of the energy-momentum operator is localized above the hyperboloid $p^2 = \mu^2$. We note that this condition is necessary also for the proof of the possibility of constructing the asymptotic states $\psi_{in,out}$ [3].

We will show for a theory which satisfies the requirements 1) – 6) that an irreducible W is uniquely determined by its $\Omega$ and that the intersection of a finite number of $\Omega_\alpha$, corresponding to irreducible functionals, allows to reconstruct all $\Omega_\alpha$ out of which this intersection is formed. This is the content of Sec. 2. In Sec. 3 we consider the possibility of extending this result to theories which do not satisfy the requirement 6). Several consequences of these propositions are discussed in Section 4.

We note that all results which are obtained remain valid also for an arbitrary number of fields with arbitrary spins, and that only the requirements 1), 2), 6) are important for the derivation of our results.

## 2. PROOF THAT W IS UNIQUELY DETERMINED BY $\Omega$.

Let $W_1$ be a functional satisfying the requirements 1)—6) and consisting of a finite number of irreducible functionals. Let $W_2$ be a functional satisfying the conditions 1)—5). Assume that in addition $\Omega_1 = \Omega_2$. It is easy to see that in this case $W_2$ satisfies also the condition 6). By definition $W_1$ admits the representation

$$W_1 = \sum_{\alpha=1}^{N} |\rho_\alpha|^2 W_\alpha.$$

We want to show that the functional $W_2$ consists of the same $W_\alpha$.

One can construct the Hilbert spaces $H_{1,2}$ from the functionals $W_{1,2}$. Consider the mapping V from $H_1$ into $H_2$:

$$V\psi_1(g) \equiv \psi_2(g).$$

If $\psi_1(g) = 0$, $g \in \Omega_1 = \Omega_2$ and consequently $V\psi_1(g) = 0$. Conversely, if $V\psi_1(g) = 0$, $g \in \Omega_2 = \Omega_1$ and hence $\psi_1(g) = 0$. It follows that V is a linear and one-to-one mapping of $L_1 \subset H_1$ onto $L_2 \subset H_2$. Then

$$W_2(g^+h) = \langle V\psi_1(g), V\psi_1(h) \rangle.$$

For a fixed g, this is a linear functional of $\psi_1(h)$, continuous in the topology of $L_1$. This is a consequence of the continuity of $W_2$. Therefore the expression $V^+V\psi_1(g)$ is meaningful, namely: $V^+V\psi_1(g)$ belongs to $L_1^+$ for any $\psi_1(g)$, i.e., $V^+V$ is a linear continuous mapping of $L_1$ into $L_1^+$. We denote $V^+V\psi(1) \equiv f \in L_1^+$. Since $W_2$ is relativistically invariant,

$$\langle f, \psi_1(g) \rangle = W_2(g) = W_2((a, \Lambda)g) = \langle f, U_1(a, \Lambda)\psi_1(g) \rangle.$$

It follows that f is a generalized eigenvector of the energy-momentum operator, corresponding to the eigenvalue 0.

*The term rigged Hilbert space is a literal translation of the Russian term defined by Gel'fand and was used in the English translation of [2]. In the translator's opinion the term "Gel'fand triple" (used in the German translation) is more esthetic. (Translator's note).

We now make use of the assumption that this eigenvalue is an isolated point of the spectrum. This assumption implies that any $\psi_1(g)$ can be represented in the form

$$\psi_1(g) = \psi_1(g^0) + \psi_1(g^1), \quad g_0{}^0 = g_0, \quad g_0{}^1 = 0,$$

$$g_n{}^{0,1}(p_1 \ldots p_n) = g_n(p_1 \ldots p_n)\varphi_{0,1}(p_1 + \ldots + p_n),$$

$$\varphi_0(p) = \begin{cases} 0, & p \in \overline{V}^+_{3/2\mu} \\ 1, & p \overline{\in} \overline{V}^+_{3/2\mu} \end{cases}, \quad \varphi_1(p) = \begin{cases} 1, & p \in \overline{V}^+_{3/2\mu} \\ 0, & p \overline{\in} \overline{V}^+_{3/2\mu} \end{cases}.$$

Here $\overline{V}^+_\mu$ is the part of the forward cone situated above the mass-hyperboloid $p^2 = \mu^2$ together with its boundary. Since f is a generalized eigenvector corresponding to the eigenvalue 0, $\langle f, \psi_1(g^1)\rangle = 0$, and consequently

$$W_2(g) = \langle f, \psi_1(g^0)\rangle.$$

It remains only to note that the ensemble $\{\psi_1(g^0)\}$ $g \in \Sigma$ forms a finite-dimensional space, since

$$\psi_1(g^0) = \sum_{\alpha=1}^{N} \psi_{0\alpha}\langle\psi_{0\alpha}, \psi_1(g^0)\rangle = \sum_{\alpha=1}^{N} \psi_{0\alpha}\langle\psi_{0\alpha}, \psi_1(g)\rangle$$

$$= \sum_{\alpha=1}^{N} \rho_\alpha \psi_{0\alpha} W_\alpha(g).$$

The functional f is densely defined in the space and as a consequence of its linearity we have

$$\langle f, \psi_1(g^0)\rangle = \sum_{\alpha=1}^{N} f_\alpha |\rho_\alpha|^2 W_\alpha(g).$$

Since the functional $W_2$ is positive semidefinite, it follows that $f_\alpha \equiv |\lambda_\alpha|^2 \geq 0$. In addition, $\lambda_\alpha \neq 0$, since otherwise we would arrive at a contradiction by interchanging $W_1$ and $W_2$ in our reasoning.

Thus, $W_2$ consists of the same irreducible functionals as $W_1$, Q.E.D.

## 3. ON THE POSSIBILITY OF RELINQUISHING CONDITION 6)

The question whether the result holds also in theories satisfying only the requirements 1)—5) but not satisfying 6) remains unanswered in the general case. We can however indicate a class of theories for which this result is certainly inapplicable.

We introduce the functional $W_0$:

$$W_0 \equiv (1, 0, 0, 0, \ldots).$$

Obviously $W_0$ is irreducible and its $\Omega_0$ consists of all g for which $g_0 = 0$.

Consider an irreducible theory W, for which the set $\{\psi(\tilde{g})\}$ $\tilde{g}_0 = 0$ does not contain translation-invariant states. The only representative of such theories satisfying requirements 1)—6) is $W_0$. All other theories of this type do not satisfy the condition 6). Assume there exists at least one such theory W, differing from $W_0$. Then $\Omega \subset \Omega_0$, since otherwise $\Omega$ would contain at least one element g with $g_0 \neq 0$, contrary to the assumption. Thus $\Omega = \Omega \cap \Omega_0$. If the result we have proved would remain valid also in this case, it would imply $W = W_0$, which is false.

Thus, if there exist theories with this property which differ from $W_0$ one could not extend the proof to incorporate these theories.

## 4. SOME CONSEQUENCES

We consider the class of theories which can be reconstructed from their $\Omega$ up to the expansion coefficients in terms of their irreducible ingredients. This class of theories will be denoted by A. According to what we have proved the class A contains the theories satisfying 1)—6) and consisting of a finite number of irreducible ones. The following propositions are obvious.

1. If the space $\Omega$ belonging to a theory W which satisfies the conditions 1)—5) contains the space $\Omega_0$ belonging to the theory $W_0 \in A$, then W consists of the same irreducible functionals as $W_0$. The decomposition of W into irreducible functionals contains all the irreducible $W_\alpha$ which occur in $W_0$ if and only if $\Omega = \Omega_0$. In particular, there does not exist a theory W satisfying 1)—5) for which the space $\Omega$ is strictly larger than the space $\Omega$ of an irreducible theory belonging to the class A. This fact has already been used in assumption 3).

2. If $W \in A$ and the theory $W'$ is obtained from W by a one-to-one similarity mapping V

$$\psi'(g) = V\psi(g), \quad \varphi'(x) = V\varphi(x)V^{-1}$$

($\varphi$, $\varphi'$ are the field operators of the theories W, $W'$), then in order for $W'$ to satisfy 1)—5) it is necessary and sufficient that it be composed of exactly the same (all of them) irreducible functionals as W. If W is irreducible, then $W' = W$.

Let us now consider the formulation of the symmetry problem in this formalism. The starting point is the introduction of a group $\mathcal{T}$ of automorphisms on $\Sigma$. We denote the group elements by $\tau$. We have to remember that in addition to the automorphisms associated with the "usual" symmetries (parity, isospin transformations etc.) there exist other automorphisms, e.g.,

$$g_0{}^{(a)} = g_0, \quad g_n{}^{(a)}(p_1 \ldots p_n) = a(p_1{}^2) \ldots a(p_n{}^2) g_n(p_1 \ldots p_n),$$

where $a(p^2)$ are such functions that $a$ and $a^{-1}$ are both infinitely differentiable functions increasing like a power (multipliers on $S_{4n}$). Finding and investigating such "unusual" groups of automorphisms may present a certain interest.

Let W be an irreducible functional belonging to the class A. With each element $\tau \in \mathcal{T}$ we associate the functional $W_\tau$ defined by $W_\tau(g_\tau) \equiv W(g)$. We consider only such groups $\mathcal{T}$, for which $W_\tau \in A$ for all $\tau \in \mathcal{T}$. It is easy to see that $W_\tau$ is also irreducible. A theory is invariant under $\mathcal{T}$ if $W_\tau = W$.

Let us assume that $W_\tau \neq W$. One can introduce a reducible functional $\overline{W}$ which is invariant under the group:

$$\overline{W} = \sum_\tau W_\tau \Big/ \sum_\tau .$$

if $\overline{W} \in A$ (this is certainly true for finite groups) all the $W_\tau$ can be reconstructed from $\overline{W}$. In other words, a part of an irreducible functional (belonging to the class A) which is symmetric under a certain group of automorphism, determines the original functional up to an automorphism of the group.

Consider for example the group $\mathcal{T}$ corresponding to parity. It consists of the two elements 1 and P, with

$$(Pg)_0 = g_0, \quad (Pg)_n(x_1 \ldots x_n) = g_n(x_1^p \ldots x_n^p),$$

$$x_i^p \equiv -\mathbf{x}_i, \; x_i^0.$$

Define $W_0^+ = 1$, $W_0^- = 0$,

$$W_n^\pm(x_1 \ldots x_n) = {}^1\!/_2 [W_n(x_1 \ldots x_n) \pm W_n(x_1^p \ldots x_n^p)].$$

Then, according to our results, if all $W_n^+$ are given, the $W_n^-$ are all determined up to a common sign.

## 5. CONCLUSION

Let us analyze the meaning of the results we have derived. Lt us define the set $M \subset \Omega$. By definition the set M consists of those functions $g \in \Omega$ for which $gh \in \Omega$ for any $h \in \Sigma$. The properties of W imply that M is a closed linear manifold. If $g \in M$, $a(g)\psi(h) = 0$, i.e., $a(g) = 0$. Conversely, if for a $g \in \Sigma$ there exists an $a(g) = 0$, then $g \in M$. Thus M is the set of all $g \in \Sigma$ for which $a(g) = 0$. This representation shows that the set M is invariant under $(a, \Lambda)$, under involution, and that it is a bilateral ideal. Obviously M is a maximal bilateral (and hence a maximal right) ideal in $\Omega$. M can be uniquely determined if $\Omega$ is known. Whether $\Omega$ can be constructed if M is known, remains an open question, even for irreducible functionals.

The following proposition seems very plausible: An irreducible theory is uniquely determined by the ensemble of its dynamical equations, i.e., by M. One might consider the results of the present paper as a first step towards a proof of this proposition. The next step should consist in a proof that it is possible to determine $\Omega$ uniquely in terms of M.

One might look at this problem from another angle. Consider two irreducible theories of the class A. Each of them generates its own operator algebra $a_1 \equiv \{a_1(g)\}$ and $a_2 \equiv \{a_2(g)\}$ over the Hilbert spaces $H_1$ and $H_2$. The algebras $a_1$ and $a_2$ are isomorphic under the correspondence $a_1(g) \leftrightarrows a_2(g)$ if and only if $M_1 = M_2$. If in addition $\Omega_1 = \Omega_2$, this isomorphism is implementable by a similarity mapping $a_2(g) = Va_1(g)V^{-1}$. In this formulation the result we have obtained is related with the results obtained by Haag and Kastler [4].

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