"FOCUSING" SINGULARITY IN *p*-SPACE OF CONSTANT CURVATURE

R. M. MIR-KASIMOV

Joint Institute for Nuclear Research

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It is shown that the "focusing" singularity [1] arising in field theory in a p-space of constant curvature can be removed if the S-matrix is constructed in accordance with the group of motions of this space.

1. INTRODUCTION

RECENTLY several authors [1-3] have been developing a field theory in which instead of the usual "flat" momentum space they have introduced a p-space of constant curvature. This generalization of field theory is based on the concept of a displacement of a p-space of constant curvature. In contrast to the usual Minkowski space where the displacements form a commutative group, in this case the displacements are noncommutative and do not form a group. It is natural that such a violent change of the geometry will introduce considerable difficulties concurrently with obvious progress (removal of divergences). One of such complications - the so-called "focusing" singularity - has been discussed by Gol'fand [1] who has proposed a method of regularizing this singularity (normalized displacement).

In order to elucidate this phenomenon we consider the simplest second order loop-the polarization operator (cf., diagram). We note that the "focusing" singularity arises only in diagrams containing closed fermion loops. In place of the δ-function $\delta(l + t)$ which expresses the law of conservation of energy and momentum in the usual theory, here we encounter the so-called matrix element of the displacement operators

$$\langle p|\hat{d}_0(l)\hat{d}_0(-t)|p\rangle \tag{1.1}$$

(cf. [2]) which is a generalization of the δ-function to the case of p-space of constant curvature.

Utilizing the explicit form of the displacement operator [1] it can be easily shown that this matrix element must give a contribution not only in the obvious case when $l = t$, but also for arbitrary fixed l and t at the point p satisfying the condition

$$pl = pt = -1. \tag{1.2}$$

The last two equations define a two-dimensional surface in four-dimensional p-space. On the basis of the classical Dirac definition of the δ-function this situation could be expressed by the words: "the argument of the δ-function vanishes on the two-dimensional surface." It is clear that the construction of a consistent field theory on the basis of such a quantity will be very difficult.

In the present note we shall show that it would be more correct not to specify the δ-function in terms of the properties which manifest themselves in the integration of its product with other functions, but to start with expansion in terms of certain complete systems closely related to the geometry of p-space of constant curvature. If such a procedure is adopted the "focusing" singularity gives no contribution to the matrix elements of the displacement operators.

2. A PROJECTIVE TWO-DIMENSIONAL MODEL OF FIELD THEORY

For simplicity we shall utilize a two-dimensional model of the version of the generalized field theory under consideration. Specifically, all the momentum vectors will be assumed to belong to a two-dimensional space of constant curvature. Accordingly the coordinates in this space are determined by a projective method with the aid of a three-dimensional sphere. If we denote by p_α ($\alpha = 1, 2$) vectors in a two-dimensional space of constant curvature, and by P_i ($i = 1, 2, 3$) vectors in three-dimensional Euclidean space, then the following relations hold

$$p_\alpha = P_\alpha / P_3, \qquad P_1^2 + P_2^2 + P_3^2 = 1. \tag{2.1}$$

The formula for the distance $s(p, q)$ in projective coordinates has an exceedingly simple form:

$$\cos s(p, q) = PQ. \tag{2.2}$$

Here Q_i are the projective coordinates corresponding to the vector q. Thus, the distance between points of p-space of constant curvature s(p, q) is simply the angle formed by two vectors in three-dimensional space. Consequently, the group of motions of a space of constant curvature is isomorphic with the group of rotations of a three-dimensional sphere.

The whole construction of a field theory can be carried out in terms of the projective coordinates. We shall state here only the displacement formula which we require (cf., formula (2.1) of [1]):

$$\hat{D}_0(L) = \hat{T}_A\hat{T}_L. \tag{2.3}$$

The operation $\hat{T}_M$ is a three-dimensional reflection in the hyperplane perpendicular to the unit vector M and has the form

$$\hat{T}_M P = P - 2P(PM). \tag{2.4}$$

A is the vacuum vector with the coordinates {0, 0, 1}.

3. THE GEOMETRICAL MEANING OF THE "FOCUSING" SINGULARITY

We shall show that the "focusing" singularity is related to the existence of points which are stationary with respect to motions in p-space of constant curvature. We consider the matrix element (1.1). Writing it in the form

$$\langle p|\hat{d}_0(l)\,\hat{d}_0(-t)\,|\,p\rangle = \langle\hat{d}_0(t)\,\hat{d}_0(-l)\,p\,|\,p\rangle \tag{3.1}$$

and evaluating explicitly the effect of the operator $\hat{d}_0(t)\hat{d}_0(-l)$ on the vector p satisfying equation (1.2), we see that the vector p is stationary with respect to this transformation:

$$\hat{d}_0(t)\,\hat{d}_0(-l)\,p = p. \tag{3.2}$$

In projective notation the condition of stationarity (1.2) has the form

$$(PL) = (PT) = 0. \tag{3.3}$$

The case of a diagonal matrix element (1.1) has been discussed by Gol'fand [1]. It can be easily seen that a similar situation will occur in diagonal matrix elements of products of an arbitrary number of displacement operators. In the case of three displacement operators $\langle p|\hat{d}_0(-l)\hat{d}_0(s)\hat{d}_0(l)|p\rangle$ we have the relation

$$\hat{D}_0(L)\,\hat{D}_0(S)\,\hat{D}_0^{-1}(L)\,P = \hat{T}_{\hat{T}_A\hat{T}_LA}\,\hat{T}_{\hat{T}_A\hat{T}_LS}P = P, \tag{3.4}$$

which is obtained if we use the obvious formulas

$$\hat{T}_M\hat{T}_M = 1, \qquad \hat{T}_M\hat{T}_N\hat{T}_M = \hat{T}_{\hat{T}_MN}. \tag{3.5}$$

The condition of stationarity for the vector P has here the form

$$(\hat{T}_A\hat{T}_LA,\,P) = (\hat{T}_A\hat{T}_LS,\,P) = 0. \tag{3.3'}$$

In the case of a product of an arbitrary number of displacement operators

$$\langle p\,|\,\hat{d}_0(l_1)\,\hat{d}_0(l_2)\,\ldots\,\hat{d}_0(l_n)\,|\,p\rangle \tag{3.6}$$

we can always represent the complex operator appearing above in the form of a product of two reflections

$$\hat{D}_0(L_1)\,\hat{D}_0(L_2)\ldots\hat{D}_0(L_n)\,P = \hat{T}_M\hat{T}_NP, \tag{3.7}$$

where M and N, generally speaking, depend on all the parameters $L_1, \ldots L_n$:

$$M = M(L_1, \ldots, L_n), \qquad N = N(L_1, \ldots, L_n).$$

The conditions of stationarity in this case are the following:

$$(PM) = (PN) = 0. \tag{3.3''}$$

We see that although a displacement is a unique operation and that different parameters correspond to different transformations of the space, for each operator constructed from displacements there exist points which are stationary with respect to the effect of this operator. Knowing the result of the action of the operator on such a point we cannot obtain the parameters for this transformation. It is just because of this that the "focusing" singularity arises. From this the method is clear by means of which the "focusing" singularity can be eliminated. One must specify displacements not by the results of their action on some vector, but as transformations of the space. This is achieved by means of going over to integration over the group of motions of p-space in the formalism of the S-matrix.

It is useful to compare the matrix element (3.6) with the corresponding expression in the usual theory[1)]:

$$\langle p|\hat{d}_0(l_1)\,\ldots\,\hat{d}_0(l_n)\,|p\rangle = \delta(l_1 + l_2 + \ldots + l_n). \tag{3.8}$$

We see that in the usual theory due to the commutativity of the displacements there is no dependence on p, and as a result of this a singularity of the type discussed above cannot arise.

[1)] In the usual theory the effect of the displacement operator $d_0(l)$ reduces to the addition of momenta: $\hat{d}_0(l)\,|\,p\rangle = |l+p\rangle$.

4. DEFINITION OF THE δ-FUNCTION IN A SPACE OF CONSTANT CURVATURE

It is well known that in the case of a Euclidean space the δ-function with all its properties is given by the expansion

$$\delta(p-q)=\frac{1}{(2\pi)^2}\int e^{ix(p-q)}\,d^2x. \qquad (4.1)$$

The quantity e^{ipx} is closely connected to the group of displacements of the Euclidean space. Specifically, to every vector of the two-dimensional Euclidean x-space ($x = (x_1, x_2)$) there corresponds uniquely an irreducible one-dimensional representation of the group of displacements of the p-space. The exponential e^{ipx} can be taken as the basis for this representation. The effect of a displacement on the exponential is as follows:

$$d(k)\,e^{ipx}=e^{i(p+k)x}. \qquad (4.2)$$

In other words, $\hat{d}(k) = e^{ikx}$ in the given irreducible representation is also the matrix performing the operation of displacement. Since the group of displacements of a Euclidean space is Abelian all its representations are one-dimensional and the effect of the matrix is simply multiplication by a number. Moreover, in no representation is it possible to distinguish between a vector in the space of the irreducible representation and the displacement operator operating in this space. In other words, it is not possible to distinguish between the points of the space and the elements of the group.

In the case of a field theory in a p-space of constant curvature the symmetry between the coordinates of a point and the vectors parametrizing the displacements of the space of constant curvature is explicitly violated (cf., Sec. 2 of [2]). We shall now show that this asymmetry in the field theory under discussion is not a matter of principle and can be eliminated if the generalization of formula (4.1) to the case of a space of constant curvature is carried out taking into account the properties of the group of motions of this space.

As has been indicated in [1] we can introduce spherical polar coordinates in a space of constant curvature, and as a result of this the element of volume will have the form

$$d\Omega=\sin\vartheta\, d\vartheta\, d\varphi. \qquad (4.3)$$

It is well known that the element of volume of a space of constant curvature or, what is the same thing, the element of area of a three dimensional sphere can be extended to an invariant element of volume of the group of three-dimensional rotations, viz.,

$$dg=\frac{1}{(8\pi)^2}\sin\vartheta\, d\psi\, d\vartheta\, d\varphi, \qquad (4.4)$$

where φ, ϑ, and ψ are the Eulerian angles. All the integrals over the space of constant curvature can be converted in a trivial manner into integrals over the rotation group in the following manner:

$$\int f(\vartheta,\varphi)\sin\vartheta\, d\vartheta\, d\varphi=4\pi\int f(\vartheta,\varphi)\,dg. \qquad (4.5)$$

Utilizing this relation we can in the formulation of the field theory replace the integrals over the space of constant curvature by integrals over the group of motions. This replacement is of a purely formal nature as long as we do not consider the matrix elements of the displacement operators.

In integrating over the space of constant curvature it is natural to utilize the spherical harmonics $Y_m^l(\vartheta, \varphi)$ which constitute a complete system. Any arbitrary function on the sphere $f(\vartheta, \varphi)$ can be expanded in terms of them into a Fourier series:

$$f(\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_m^l(f)\,Y_m^l(\vartheta,\varphi), \qquad (4.6)$$

where

$$c_m^l(f)=\int \overset{*}{Y}{}_m^l(\vartheta',\varphi')f(\vartheta',\varphi')\sin\vartheta'\,d\vartheta'\,d\varphi'. \qquad (4.7)$$

Substituting (4.7) into (4.6) we obtain

$$f(\vartheta,\varphi)$$

$$=\int\sin\vartheta'\,d\vartheta'\,d\varphi'\,f(\vartheta',\varphi')\sum_{l,m}\overset{*}{Y}{}_m^l(\vartheta',\varphi')\,Y_m^l(\vartheta,\varphi), \qquad (4.8)$$

i.e., in the integration in question the quantity

$$\sum_{l,m}\overset{*}{Y}{}_m^l(\vartheta,\varphi)\,Y_m^l(\vartheta',\varphi')$$

acts like a δ-function:

$$\sum_{l,m}\overset{*}{Y}{}_m^l(\vartheta',\varphi')\,Y_m^l(\vartheta,\varphi)=\delta(\vartheta-\vartheta')\,\delta(\varphi-\varphi'). \qquad (4.9)$$

(In future for the sake of brevity we shall not write out the limits for the summations over l and m.)

Since the values of the angles ϑ and φ completely specify the vectors in a space of constant curvature we can write (4.9) in the form

$$\langle p|q\rangle=\delta(p,q)=\sum_{l,m}\overset{*}{Y}{}_m^l(p)\,Y_m^l(q), \qquad (4.9')$$

where p and q are vectors with spherical coordinates respectively given by (φ, ϑ) and (φ', ϑ').

Equation (3.9) can be regarded as the definition of the matrix element $\langle p|q\rangle$.

We now consider the matrix element $\langle p|\hat{d}_0(l)|q\rangle$, which can also be written in the form $\langle p|\hat{d}_0(l)q\rangle$. Utilizing expansion (4.9) we obtain

$$\langle p|\hat{d}_0(l)q\rangle = \sum_{l,m} \overset{*}{Y}{}^l_m(p) Y^l_m(\hat{d}_0(l)q) = \sum_{l,m,n} \overset{*}{Y}{}_m{}^l(p) T_{mn}{}^l(\hat{d}_0(l)) Y_n{}^l(q), \tag{4.10}$$

where $T^l_{mn}(\hat{d}_0(l))$ is the matrix of the transformation $\hat{d}_0(l)$ acting in the space of the representation of the group of rotations of weight l. The expression $T^l_{mn}(\hat{d}_0(l))$ denotes that this matrix depends on the parameters of the transformation $\hat{d}_0(l)$, for example on the corresponding Eulerian angles.

In formula (4.10) once again the asymmetry between the vectors of the space and the transformation parameters is manifested. However, now the method is clear whereby we can alter our definitions and make the theory completely symmetric with respect to the momenta of all the particles independently of the manner in which a Lagrangian is constructed from them (cf. [2]). Specifically, we generalize the expansion (4.6) to the case of integration over the group. The Fourier series will have the form

$$f(\varphi, \vartheta, \psi) = \sum_{l=0}^{\infty} \sum_{m,n=-l}^{l} c_{mn}{}^l T_{mn}{}^l(\varphi, \vartheta, \psi). \tag{4.11}$$

We recall that the matrix elements $T^l_{mn}(\varphi, \vartheta, \psi)$ which depend on the continuous parameters φ, ϑ, ψ, at the same time constitute a complete functions over the group. The coefficients c^l_{mn} have the form[2)]

$$c_{mn}{}^l(f) = \int \overset{*}{T}_{mn}{}^l(\varphi', \vartheta', \psi') f(\varphi', \vartheta', \psi') \sin\vartheta'\, d\varphi'\, d\vartheta'\, d\psi'. \tag{4.12}$$

The matrix element $\langle p|q\rangle$ can be expanded by means of the formula

$$\langle p|q\rangle = \sum_{l,m,n} \overset{*}{T}_{mn}{}^l(g_p) T_{mn}{}^l(g_q) = \langle g_p|g_q\rangle. \tag{4.13}$$

We interpret the symbol $T^l_{mn}(g_p)$ in the following manner. We take a fixed vector B (for example, the vacuum vector B = A = (0, 0, 1)), and we obtain all the other vectors P by acting on B by an appropriate transformation of displacement:

$$P = \hat{d}_0(k_p)B. \tag{4.14}$$

Thus, we make each vector P correspond to a certain displacement $\hat{d}_0(k_p)$. If the displacement operator is written in k-parametrization (cf., formula (1.1) of [1]), then the vector k_p coincides with p. It is clear that this relationship is mutually unique. Therefore the following formula always holds

$$\langle p|q\rangle = \langle \hat{d}_0(k_p)B|\hat{d}_0(k_q)B\rangle = \langle B|\hat{d}_0^{-1}(k_p)\hat{d}_0(k_q)|B\rangle. \tag{4.15}$$

Since only the fact of the coincidence of the transformations $\hat{d}_0(k_p)$ and $\hat{d}_0(k_q)$ is significant, while the vector B is arbitrary, we can take another step and write

$$\langle p|q\rangle = \langle \hat{d}_0(k_p)|\hat{d}_0(k_q)\rangle. \tag{4.16}$$

Equation (4.16) justifies the replacement of (4.9a) by (4.13).

The matrix element (4.10) can now be written in the completely symmetric form:

$$\langle p|\hat{d}_0(l)|q\rangle = \sum_{l,m} \overset{*}{T}{}^l_{mn}(\hat{d}_0(k_p)) T^l_{ms}(\hat{d}_0(l)) T^l_{sn}(\hat{d}_0(k_q)). \tag{4.17}$$

We now return to the "focusing" singularity and consider the matrix element (1.1)

$$\begin{aligned}\langle p|\hat{d}_0(l)\hat{d}_0^{-1}(t)|p\rangle &= \sum_{l,m,n,r,s} \overset{*}{T}{}^l_{mn}(\hat{d}_0(k_p)) T^l_{ms}(\hat{d}_0(l)) T^l_{sr}(\hat{d}_0^{-1}(t)) T^l_{rn}(\hat{d}_0(k_p)) \\ &= \sum_{l,m,n,r,s} T^l_{rn}(\hat{d}_0(k_p)) T^l_{nm}(\hat{d}_0^{-1}(k_p)) T^l_{ms}(\hat{d}_0(l)) T^l_{sr}(\hat{d}_0^{-1}(t)) \\ &= \sum_{l,m,r,s} T^l_{rm}(\hat{d}_0(k_p)\hat{d}_0^{-1}(k_p)) T^l_{ms}(\hat{d}_0(l)) T^l_{sr}(\hat{d}_0^{-1}(t)) \\ &= \sum \overset{*}{T}{}^l_{rs}(\hat{d}_0(t)) T^l_{rs}(\hat{d}_0(l)) = \langle t|l\rangle = \delta(l, t).\end{aligned} \tag{4.18}$$

We have utilized here the basic property of the representation:

$$T(\hat{d}_0(l)) T(\hat{d}_0(t)) = T(\hat{d}_0(l)\hat{d}_0(t)) \tag{4.19}$$

and the unitarity property of the matrices T:

$$T^l_{mn}(\hat{d}_0(l)) = \overset{*}{T}{}^l_{nm}(\hat{d}_0^{-1}(l)). \tag{4.20}$$

The chain of equations (4.18) shows that there is actually no dependence on p in complete agreement with formula (3.8). This proves the absence of the "focusing" singularity.

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2)For the sake of simplicity we omit here the normalizing factors which, generally speaking, are different for different values of l.

5. A MORE PRECISE FORMULATION OF THE FORMALISM OF FIELD THEORY

In this section we shall show that the whole formalism of field theory in the case of p-space of constant curvature can be altered in accordance with the generalized law of energy-momentum conservation (4.13). First of all we note that in going over from integration over p-space to integration over the group of motions of a p-space of constant curvature it appears that the special role of displacements of p-space is suppressed[3)] (cf., formula (2.16) of [2]). We shall show that in the case of integrating over the group the special role played by the displacements can be preserved.

We point out first of all that if the vector K which parametrizes the displacement has the following coordinates:

$$K_1 = \sin\vartheta\sin\varphi, \quad K_2 = \sin\vartheta\cos\varphi, \quad K_3 = \cos\varphi, \qquad (5.1)$$

then the matrix of the displacement $\hat{D}_0(K)$ will have the form

$$\hat{D}_0(K) = g(-\varphi)g(\vartheta)g(\varphi) = g^{-1}(\varphi)g(\vartheta)g(\varphi), \qquad (5.2)$$

$$g(\varphi) = \begin{Vmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad g(\vartheta) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{Vmatrix} \qquad (5.3)$$

(cf. [4,6]). In other words, the displacements $\hat{D}_0(K) = \hat{d}_0(\vartheta, \varphi)$ are picked out from the arbitrary rotation $g(\varphi, \vartheta, \psi)$ which depends on the three Eulerian angles φ, ϑ, ψ, by the condition $\psi = -\varphi$.

We can now proceed to generalize the formalism of quasifields [2]. As in [1,2], we restrict ourselves to the case of pseudoscalar mesodynamics. We assume that the quasifields of the nucleons ψ and of the mesons φ are functions over the group of motions of p-space:

$$\psi = \psi(g_p), \qquad \varphi = \varphi(g_k). \qquad (5.4)$$

The only restriction imposed on the operators of the quasifields ψ and φ is the form of the pairings:

$$\langle\bar{\psi}(g_p)\psi(g_q)\rangle_0 = (m + \hat{p})^{-1}\delta(g_p, g_q),$$

$$\langle\varphi(g_k)\varphi(g_s)\rangle_0 = (\mu^2 + k^2)^{-1}\delta(g_k, g_s)\delta(\varphi_k + \psi_k). \qquad (5.5)$$

The remaining pair products of quasifields yield zero when averaged over the vacuum state.

The interaction Lagrangian has the form

$$\hat{\Lambda} = g\langle\bar{\psi}|\hat{\varphi}|\psi\rangle$$
$$= g\int\bar{\psi}(g_p)\langle g_p|g_k|g_q\rangle\psi(g_q)\varphi(g_k)\,dg_p dg_q dg_k. \qquad (5.6)$$

It can be easily seen that in the evaluation of the matrix elements the "extra" integrations introduced in going over from the element of space volume to the element of group volume are eliminated in a trivial manner. In order that the matrix elements obtained in the expansion of the S-matrix

$$S = e^{i\hat{\Lambda}} \qquad (5.7)$$

should completely coincide with those given, for example, in [2] it is necessary to divide the matrix element corresponding to an arbitrary diagram by $(2\pi)^\alpha$, where α is the number of independent loops (consisting both of fermion and of boson lines).

6. REMARKS

We shall make a number of remarks with respect to the limitations of the two-dimensional spherical model of field theory considered above.

The first remark is related to the fact that in the present paper we have considered the case of spherical geometry in p-space. This geometry cannot be utilized for a sensible generalization of field theory since the correspondence principle is not satisfied here. A physical meaning can be ascribed only to the case of elliptical geometry. In going over from spherical geometry to elliptical geometry the formula for the displacement

$$Q = D_0(L)P, \qquad (6.1)$$

if it is regarded as an equation in L has two solutions. (These are the so called "classical" and "nonclassical" displacements, cf. [1]). It can be easily seen that the "focusing" singularity occurs only for "classical" displacements. But the inclusion of "nonclassical" displacements needed for the transition to elliptic geometry does not introduce any changes into our arguments.

The second remark concerns complications which arise when the number of dimensions of the space of constant curvature is increased. In the case of four dimensions conditions (3.3)-(3.3″) which separate out the "focusing" singularity give us two-dimensional surfaces. However, this complication in no way affects the arguments given above.

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3)The transition to the integration over the group of motions of p-space can be regarded as a generalization of the concept of virtual momentum. In this paper this generalization is minimal, since the values of the matrix elements remain as before. However, if we completely do away with the special role played by the displacements, then far-reaching physical consequences will result.

It is worth while to make the third remark in connection with the fact that in some versions of the field theory under consideration investigations were made not of an elliptical, but of a pseudoelliptical p-space (cf., for example, [3]). The spectrum of the representations of the group of motions of this space is much more complicated than in the case of elliptical space. Since the group of motions is noncompact infinite dimensional representations arise in this case. However, it can be shown, that there exists a series of unitary infinite dimensional representations which constitutes a complete system of functions [5]. Consequently, a Fourier expansion in terms of these functions exists, and our arguments utilizing only the possibility of expansion and unitarity can be completely repeated in this case also.

The result of the present paper consists of the fact that the geometrical meaning of the "focusing" singularity has been elucidated and it has been shown that this danger in the generalization of the field theory under consideration turns out to be fictitious and can be easily circumvented if the scattering matrix formalism is constructed taking into account the group properties of the operation of displacement in a p-space of constant curvature.

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[1] Ya A. Gol'fand, JETP 44, 1248 (1963), Soviet Phys. JETP **17**, 842 (1963).

[2] Ya. A. Gol'fand, JETP **43**, 256 (1962), Soviet Phys. JETP **16**, 184 (1963).

[3] V. G. Kadyshevskiĭ, DAN SSSR **147**, 1336 (1962), Soviet Phys. Doklady **7**, 1138 (1963).

[4] Ya. A. Gol'fand, R. A. Minlos and Z. A. Shapiro, Predstavleniya gruppy vrashcheniĭ i gruppy Lorentsa (Representations of the Rotation and the Lorentz Groups), Fizmatgiz, 1958.

[5] M. A. Naĭmark, Lineĭnye predstavleniya gruppy Lorentsa (Linear Representations of the Lorentz Group) Fizmatgiz, 1958.

[6] N. Ya. Vilenkin, Spetsial'nye funktsii svyazannye s predstavleniyami klassa 1 grupp dvizheniĭ prostranstv postoyannoĭ krivizny (Special Functions Connected with the Representations of Class 1 of the Groups of Motions of Spaces of Constant Curvature), Proc. Moscow Math. Soc. **12** (1963).

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