ION ESCAPE FROM A MAGNETIC MIRROR TRAP DUE TO DEVELOPMENT OF INSTABILITY CONNECTED WITH THE "LOSS CONE"

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We consider the nonlinear stage of development of microinstability in a plasma in magnetic-mirror traps, caused by the "loss cone" in the ion-velocity distribution. The oscillations that arise as a result of the instability lead to fast diffusion of the ions in velocity space to the "loss cone" and to their subsequent escape from the trap.

1. Flute instability of a plasma in magnetic-mirror traps can be suppressed, in accordance with current opinion, by special choice of the magnetic-field geometry (the so-called "minimum-**B** configurations")[1]. In this connection, more careful attention should be paid to another type of plasma instability in devices of this kind—the instability connected with local anisotropy of the ion-velocity distribution function. It has been shown[2,3] that if the average transverse energy of the particles greatly exceeds their average energy of longitudinal motion (along the magnetic field H_0), $T_\perp > T_\parallel$, then short-wave fluctuations of the electric and magnetic fields should become excited in the plasma spontaneously. Instabilities of this kind become amplified if the ion-velocity distribution has sharp maxima[4], as is usually the case when ion beams are injected.

However, the prevalent conviction was that under real conditions there is a certain stability margin with respect to this class of instabilities. Recently Rosenbluth and Post[5] called attention to the fact that in a dense plasma, with $n \gtrsim 10^{13}$ cm^{-3} (the numerical estimate was made for typical magnetic fields $H_0 \approx 10^4$), the presence of the "loss cone" in the velocity distribution of the ions always leads to the development of strong instability. It is therefore of interest to investigate the influence of this instability on plasma confinement in traps.

This raises the question of finding the spectrum of the oscillations in an unstable plasma and the resultant diffusion of ions in the "loss cone," which leads to escape of the ions from the trap through the magnetic mirrors. A strict mathematical description of the turbulent transport processes is possible only in the case of weak instability, when the turbulent state of the plasma can be represented in the form of a set of weakly interacting oscillations. The representation of plasma turbulence in the form of a set of oscillations is reasonable if the instability increment is small compared with the frequency ($\gamma \ll \omega$) and if the oscillation amplitude varies little during the time of a single oscillation. Under the simplifying assumptions made by Rosenbluth and Post[5], the instability develops within a time on the order of one oscillation cycle, so that $\gamma \sim \omega$. Therefore the method of analysis employed in our paper cannot be applied directly to the case considered in[5]. However, as we shall show below, in many real situations the instability is appreciably weakened, and we can use for perturbation theory its rigorous description.

2. The dumping of the ions in the "loss cone" can be described in terms of "quasilinear diffusion" of ions in velocity space under the influence of the oscillations. Since the diffusion coefficient depends essentially on the oscillation energy level and its spectral distribution, we proceed directly to determine these quantities. As in[5], we assume the perturbation scale to be sufficiently small for the plasma to be regarded as homogeneous, so that the electric field potential φ can be expanded in a sum over the fields of the individual harmonic oscillations[1)]:

$$\varphi = \sum_{\mathbf{k},\,\omega} \varphi_{\mathbf{k}\omega} \exp(-i\omega t + ik_z z + i\mathbf{k}_\perp \mathbf{r}), \tag{1}$$

where k_z and $\mathbf{k}_\perp$ are the components of the wave vector respectively along and transverse to the unperturbed magnetic field $\mathbf{H}_0 = \{0, 0, H_z\}$, ω is the frequency, and $\varphi_{\mathbf{k}\omega}$ is the amplitude of the oscillation.

We assume, as usual, that in the interaction between the oscillations themselves and between the

1)For simplicity we set the volume V of the system equal to unity, V = 1.

oscillations and the plasma particles, which determines the energy spectrum distribution of the spectrum, the phases of the amplitude $\varphi_{k\omega}$ are random. This assumption is valid in the case of weak instability. As a result we obtain the following equation for the time variation of the oscillation energy[6,7]:

$$\frac{\partial}{\partial t}\frac{\partial \varepsilon_{\mathbf{k}}^{(1)}(\omega)}{\partial \omega}\frac{k^2|\varphi_{\mathbf{k}}|^2}{8\pi} = -\frac{k^2}{4\pi}\operatorname{Im}\Big\{\varepsilon_{\mathbf{k}}^{(1)}(\omega)$$

$$-\sum_{\mathbf{k}',\omega'}\Big[\frac{\varepsilon^{(2)}_{-\mathbf{k}',\mathbf{k}+\mathbf{k}'}(-\omega',\omega+\omega')\,\varepsilon^{(2)}_{\mathbf{k},\mathbf{k}'}(\omega,\omega')}{\varepsilon^{(1)}_{\mathbf{k}+\mathbf{k}'}(\omega+\omega')}$$

$$-\varepsilon^{(3)'}_{-\mathbf{k},\mathbf{k},\mathbf{k}'}(-\omega',\omega,\omega')\Big]|\varphi_{\mathbf{k}'}|^2\Big\}|\varphi_{\mathbf{k}}|^2$$

$$+\frac{k^2}{4\pi}\operatorname{Im}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\frac{|\varepsilon^{(2)}_{\mathbf{k}',\mathbf{k}''}(\omega',\omega'')|^2}{\varepsilon^{(1)}_{\mathbf{k}}(\omega'+\omega'')}|\varphi_{\mathbf{k}'}|^2|\varphi_{\mathbf{k}''}|^2, \qquad (2)$$

where ω, ω', and ω'' are the frequencies of the natural oscillations with wave vectors $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$ and amplitudes $\varphi_{\mathbf{k}}$, $\varphi_{\mathbf{k}'}$, $\varphi_{\mathbf{k}''}$; $\epsilon^{(1)}_{\mathbf{k}}$, $\epsilon^{(2)}_{\mathbf{k}',\mathbf{k}''}$, and $\epsilon^{(3)}_{\mathbf{k}',\mathbf{k}'',\mathbf{k}'''}$ are the coefficients of expansion of the dielectric constant in powers of the amplitudes $\varphi_{\mathbf{k}}$.

The coefficients $\epsilon^{(n)}$ are simply related to the corresponding expansion coefficients of the particle distribution function with respect to the oscillation amplitudes:

$$f_j(\mathbf{r},\mathbf{v},t) = f_{0j}(\mathbf{r},\mathbf{v},t) + \sum_{\mathbf{k},\omega}\mu^{j(1)}_{\mathbf{k}\omega}(\mathbf{v},t)\varphi_{\mathbf{k}\omega}(\mathbf{r},t)$$

$$+\sum_{\omega'\mathbf{k}',\mathbf{k}''\omega''}\mu^{j(2)}_{\mathbf{k}'\omega',\mathbf{k}''\omega''}(\mathbf{v},t)\varphi_{\mathbf{k}'\omega'}(\mathbf{r},t)\varphi_{\mathbf{k}''\omega''}(\mathbf{r},t)+\ldots$$

In the linear approximation we have

$$\varepsilon_{\mathbf{k}}^{(1)}(\omega) \equiv 1 - \sum_j \frac{4\pi e_j}{k^2}\int \mu_{\mathbf{k}\omega}{}^{j(1)}(\mathbf{v})\,d\mathbf{v}. \qquad (3)$$

The subsequent coefficients $\epsilon^{(n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}(\omega_1,\ldots,\omega_n)$ are defined by

$$\varepsilon^{(n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}(\omega_1\ldots\omega_n)$$

$$\equiv -\sum_j\frac{4\pi e_j}{(\mathbf{k}_1+\ldots+\mathbf{k}_n)^2}\int\mu^{j(n)}_{\mathbf{k}_1\omega_1,\ldots,\mathbf{k}_n\omega_n}(\mathbf{v})\,d\mathbf{v}. \qquad (4)$$

To find the distribution-function expansion coefficients $\mu^{(n)}$ we use the iteration formula

$$\mu^{j(n)}_{\mathbf{k}_1\omega_1,\ldots,\mathbf{k}_n\omega_n}\varphi_{\mathbf{k}_1\omega_1}\cdots\varphi_{\mathbf{k}_n\omega_n}$$

$$=\frac{e_j}{m}\int_{-\infty}^{t}dt'\,\nabla\varphi_{\mathbf{k}_1\omega_1}\frac{\partial}{\partial \mathbf{v}}\mu^{j(n-1)}_{\mathbf{k}_2\omega_2,\ldots,\mathbf{k}\ \omega}\varphi_{\mathbf{k}_2\omega_2}\cdots\varphi_{\mathbf{k}\ \omega}, \qquad (5)$$

which follows from Boltzmann's equation integrated over the particle trajectories. We are interested in a dense plasma:

$$\Omega_p \gg \Omega_H; \qquad \Omega_p = \sqrt{4\pi e^2 n/M}, \quad \Omega_H = eH_0/Mc, \qquad (6)$$

when the development of a stronger instability is possible[5]. Therefore, just as in[5], we neglect the influence of the magnetic field on the ion motion and confine ourselves to a description of the electrons in the drift approximation, assuming that the electrons are cold. This is valid when

$$\omega_H \gg \omega \gg \Omega_H, \qquad kR_H \gg 1 \gg k\rho_H, \qquad (7)$$

where $R_H = v_{Ti}/\Omega_H$ is the Larmor radius of ions with thermal velocities v_{Ti}; the corresponding quantities for the electrons are denoted by lowercase letters: ω_p, ω_H, and ρ_H.

3. Under the assumptions (6) and (7), the dispersion equation (3) for the oscillation frequency ω is written in the linear approximation in the form

$$\varepsilon_{\mathbf{k}}^{(1)}(\omega) \equiv 1 + \frac{\omega_p{}^2}{\omega_H{}^2} - \frac{\omega_p{}^2}{\omega^2}\frac{k_z{}^2}{k^2}$$

$$+\frac{\Omega_p{}^2}{k^2 v_{T_i}{}^2}\Big[\psi(0) + F\Big(\frac{\omega}{kv_{T_i}}\Big)\Big] = 0; \qquad (8)$$

$$\psi(x) = v_{Ti}{}^2\int_{-\infty}^{+\infty} f_{0j}(v_\perp{}^2, v_z{}^2)\,dv_z, \quad F(y) = 2\int_0^\infty dx\,\frac{d\psi/dx}{(1-x/y^2)^{1/2}} \qquad (8')$$

Here $\psi(x)$ is the ion distribution with respect to the dimensionless velocities $x = v_\perp^2/v_{Ti}^2$, and satisfies the normalization condition

$$\int_0^\infty dx\,\psi(x) = 1.$$

In the integrand of F(y), the required branch of the root on the real axis $y = y_r + i\epsilon$ is chosen in the following manner (see[5]):

$$(1-x/y^2)^{-1/2} = |y_r|\,(y_r{}^2 - x)^{-1/2}, \quad y_r{}^2 > x;$$
$$(1-x/y^2)^{-1/2} = -iy_r(x - y_r{}^2)^{-1/2}, \quad y_r{}^2 < x. \qquad (9)$$

It follows from this definition that Eq. (8) has solutions with $\operatorname{Im}\omega > 0$, corresponding to growing disturbances, only if

$$\int d\psi\, x^{-1/2} > 0.$$

Rosenbluth and Post[5] assumed that the particles have time to escape from the "loss cone" through the magnetic mirrors and consequently there are no particles with $v_\perp = 0$ in the trap. Then $\psi(0) = 0$. Because of this, there have always existed long-wave solutions $k\lambda_D \lesssim |F_i|$ ($\lambda_D = v_{Ti}/\Omega_p$ is the Debye radius and $F_i = \max \operatorname{Im} F(y)$), which grow with a large increment $\gamma \sim \omega$. In real traps of finite length L, the time of ion escape through the magnetic mirrors is finite and is of the order of $T \approx L/v_z$, where v_z is the ion velocity along the magnetic-field force line. It may turn out now that when the ''loss cone'' is not full turbulent diffusion causes it to be filled within times τ shorter than the time of escape of the particles through the mirror, $\tau < T$. Consequently, the assumption that $\psi(0) = 0$ is no longer valid, and we must consider the more general case $\psi(0) \neq 0$ [2)]. When the distribution function $\psi(x)$ is raised at zero, the positive maximum of the function $-\operatorname{Im} y_r F(y)$ decreases quite rapidly in magnitude and shifts towards small y. Finally, if

$$\int\limits_0 dx \frac{d\psi}{dx} x^{-1/2} \leqslant 0$$

even disturbances with very small phase velocity $\omega / k v_{Ti} \equiv y_k \to 0$ cannot build up. We can therefore assume that in the filled-cone mode this integral differs little from zero, and therefore $\psi(0) > F$ for positive values of the function $-\operatorname{Im} yF(y)$.

From (8) we find for this case that the ratio of the increment γ to the frequency ω

$$\gamma / \omega = -F_i / [k^2\lambda^2_D(1 + \omega_p / \omega_H^2) + F_r + \psi(0)] \quad (10)$$

attains a maximum at wavelengths

$$k\lambda_D < (F_r + \psi)^{1/2}(1 + \omega_p^2 / \omega_H^2)^{-1/2}$$

and remains smaller than unity ($\gamma < \omega$). In the short-wave region of the spectrum

$$k\lambda_D > \psi^{1/2}(1 + \omega_p^2 / \omega_H^2)^{-1/2}$$

the ratio γ/ω decreases in inverse proportion to $\sim k^2$. Thus, the filling of the ''loss cone'' by diffusion due to the development of the instability suppresses the instability itself.

This attenuation of the instability can be readily understood from an examination of the ion distribution relative to one of the velocity components, say v_x, transverse to the magnetic field H_0. It can be seen from Fig. 1 that when $\psi(0) = 0$ the distribution has the form of two ion ''beams'' displaced relative to each other; when the cone becomes filled ($\psi(0)$ increases) the difference in the velocities of these two groups of particles becomes smeared out and the instability weakens. The weakening of the instability, and consequently of the turbulent diffusion, continues until the ion flux into the ''loss cone,'' due to the turbulent diffusion, becomes equal to the flux of ions leaving the trap through the magnetic mirrors. For purposes of controlled thermonuclear fusion, practical interest is attached only to plasma confinement times longer than the time of departure of the electrons from the trap as a result of Coulomb collisions. We can therefore assume that the electrons leave

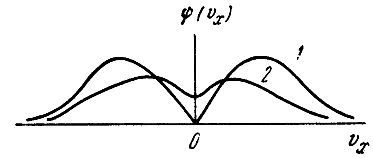

FIG. 1. Distribution of the ions with respect to the velocities v_x in the case of empty (1) and full (2) "loss cone."

the trap more rapidly than the ions, and the plasma becomes positively charged. This in turn leads to repulsion of the very slow ions, along the force lines, towards the ends of the system. We can therefore assume that the time of departure of the plasma from the ''loss cone'' to the ends is determined by the time of flight T of the thermal ions with velocities $v_z \sim v_{Ti}$ between the magnetic mirrors. This time of flight is of the order of

$$T \approx L / v_{Ti}. \quad (11)$$

Thus, in such systems the plasma distribution relaxes rapidly under the influence of the instability to a more stable state. The confinement time is then determined by the ion time of flight between the magnetic mirrors, and we need not consider the detailed spectrum of the resultant turbulence. Some of the parameters of the newly established ion distribution could be determined from the condition that the time of turbulent diffusion $\tau(F_i, \psi(0))$, expressed in terms of the distribution characteristics F_i and $\psi(0)$, be equal to the time of flight T. The calculation of the times τ for the case of a suppressed instability is also useful because in the limiting case of an empty ''loss cone,'' $\psi(0) \leq F_i$, it gives an order-of-magnitude estimate of the time of turbulent diffusion τ_{min} for the case $T < \tau_{min}$ and the value of the critical length $L_* = v_{Ti}\tau_{min}$.

4. We proceed to consider the turbulence spectrum and the turbulent ion diffusion in velocity space. To this end we turn again to Eq. (2). Under the assumptions (7), the contribution of the ions to the coefficients $\epsilon^{(n)}$ with $n \geq 2$ turns out to be smaller than the contribution of the electrons by a factor $(\Omega_p/\Omega_H)^{n-1}$, and can always be neglected. The contribution of the electrons to $\epsilon^{(2)}$ and $\epsilon^{(3)}$, under conditions (7), was calculated in the review[7] by integrating over the particle trajectories*

$$\varepsilon^{(2)}_{k', k''}(\omega', \omega'') = i \frac{\omega_p^2 e [k'k'']_z}{(k' + k'')^2 m \omega_H} \times \int\limits_{-\infty}^{+\infty} \frac{dv_z \, df_{0e}/dv_z}{\omega' + \omega'' - (k_z' + k_z'')v_z + i0}$$

2)The author is grateful to R. Z. Sagdeev for this remark.

*$[k'k''] = k' \times k''$.

$$\times\left(\frac{k_z''}{\omega''-k_z''v_z}-\frac{k_z'}{\omega'-k_z'v_z}\right),$$

$$\varepsilon^{(3)}_{-\mathbf{k}',\mathbf{k},\mathbf{k}'}(-\omega',\omega,\omega')=-\frac{\omega_p^2e^2[\mathbf{kk}']^2}{k^2m^2\omega_H^2}$$

$$\times\int_{-\infty}^{+\infty}dv_z\frac{df_{0e}/dv_z}{\omega'+\omega''-(k_z'+k_z'')v_z+i0}$$

$$\times\left(\frac{k_z}{\omega-k_zv_z}-\frac{k_z'}{\omega'-k_z'v_z}\right)\frac{1}{\omega-k_zv_z}.$$

Finally, to rewrite Eq. (2) more compactly, we introduce the "number" of oscillations n_k of frequency ω:

$$n_\mathbf{k}(t)=\left|\frac{\partial\varepsilon_\mathbf{k}^{(1)}(\omega)}{\partial\omega}\right|\frac{k^2|\varphi_\mathbf{k}|^2}{8\pi}.$$

Then Eq. (2) takes the form

$$\frac{\partial n_\mathbf{k}}{\partial t}=2\gamma_\mathbf{k}n_\mathbf{k}+\sum_{\mathbf{k}'}R(\mathbf{k},\mathbf{k}')n_\mathbf{k}n_{\mathbf{k}'}$$

$$+2\pi\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}|V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2\delta(\omega-\omega'-\omega'')$$

$$\times(n_{\mathbf{k}'}n_{\mathbf{k}''}-n_{\mathbf{k}'}n_\mathbf{k}\theta_{\mathbf{k}''}\theta_\mathbf{k}-n_{\mathbf{k}''}n_\mathbf{k}\theta_{\mathbf{k}'}\theta_\mathbf{k}),\qquad(12)$$

where the instability increment is

$$\gamma_\mathbf{k}=-\left\{\frac{\Omega_p^2}{k^2v_{Ti}^2}\operatorname{Im}F\left(\frac{\omega}{kv_{Ti}}\right)\right.$$

$$\left.-\frac{\omega_p^2}{k^2}\int_{-\infty}k_z\frac{df_{0e}}{dv_z}\delta(\omega-k_zv_z)dv_z\right\}\Big/\frac{\partial\varepsilon_\mathbf{k}(\omega)}{\partial\omega};$$

θ_k = sign $\omega\,\partial\epsilon_k(\omega)/\partial\omega$—sign of the oscillation energy;

$$V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}=\left|i\frac{\omega_p^2e[\mathbf{k}'\mathbf{k}'']}{8\pi m\omega_H\omega}\left(\frac{k_z'}{\omega'}-\frac{k_z''}{\omega''}\right)\left(\frac{k_z}{\omega}+\frac{k_z'}{\omega'}+\frac{k_z''}{\omega''}\right)\right|$$

$$\times\left[\frac{kk'k''}{(8\pi)^{3/2}}\left|\frac{\partial\varepsilon_\mathbf{k}(\omega)}{\partial\omega}\frac{\partial\varepsilon_{\mathbf{k}'}(\omega')}{\partial\omega'}\frac{\partial\varepsilon_{\mathbf{k}''}(\omega'')}{\partial\omega''}\right|^{1/2}\right]^{-1};$$

$$R(\mathbf{k},\mathbf{k}')=-\frac{16\pi\Omega_p^2e^2[\mathbf{kk}']^2}{M^2v_{Ti}^2\Omega_H^2\omega^2k^2k'^2|\partial\varepsilon_\mathbf{k}(\omega)/\partial\omega||\partial\varepsilon_{\mathbf{k}'}(\omega')/\partial\omega'|}$$

$$\times\operatorname{Im}F\left(\frac{\omega+\omega'}{(\mathbf{k}+\mathbf{k}')v_{Ti}}\right).$$

The physical meaning of the terms in the right side of (12) is obvious. Thus, the first term describes the resonant buildup of oscillations by ions with increment γ_k, the second the resonant absorption of the energy of the forced oscillations of frequency $\omega+\omega'$ and wave vector $\mathbf{k}+\mathbf{k}'$ by the ions, and the third the damping of the oscillations[3)]

[3)]The scattering of the waves by electrons ($\omega-\omega'\lesssim|k_z-k_z'|v_{Te}$), which is also possible, may turn out to be smaller in order of magnitude for cold electrons because of the mutual cancellation of the second and third terms describing this process, in (2).

The change in the total energy of the oscillations is due only to the first two terms, whereas the damping processes only redistribute the energy among the modes or transfer it to the stable region of the phase space (ω, $\mathbf{k}$), where it is absorbed by the ions.

By comparing in the kinetic equation the terms corresponding to the energy acquired by the oscillations as a result of the instability with the terms corresponding to the energy lost to linear Landau damping and to outflow into the damping regions of the spectrum, we obtain an expression for the spectral energy density of the oscillations in the steady-turbulence mode:

$$\frac{1}{2\pi}k_\perp^2\sum_{k_z}\frac{e^2|\varphi_\mathbf{k}|^2}{M^2v_{Ti}^4}\approx\frac{0.1F_iy_k^2}{k_\perp^2R_H^2\Phi_k},\qquad(13)$$

where

$$\Phi_k=[\psi(0)+F_r+k_\perp^2\lambda_D^2(1+\omega_p^2/\omega_H^2)+\langle\theta^2F\rangle],$$

$$\langle\theta^2F\rangle=\frac{1}{4\pi^3}\int_0^{2\pi}d\theta\int_0^{\varkappa_1}\varkappa d\varkappa\int_x^\infty dx\frac{d\psi}{dx}\frac{\sqrt{x_*}\sin^2\theta}{(x-x_*)^{1/2}},$$

$$\sqrt{x_*}=(y_k+y_{\varkappa k})/(1-2\varkappa\cos\theta+\varkappa^2)^{1/2},\quad\varkappa_1\sim1.\qquad(13')$$

We have changed over here to the continuous variables θ and k_1, in accordance with the rule

$$\sum_{\mathbf{k}_\perp}\approx\frac{1}{4\pi^2}\int_0^{2\pi}d\theta\int_0^\infty k_\perp dk_\perp.\qquad(14)$$

The term $\langle\theta^2F\rangle$ takes into account the simultaneous absorption of two oscillations with wave vectors $\mathbf{k}$ and $\mathbf{k}'$ by the ions (nonlinear Landau damping). The most intense absorption is observed when

$$\omega+\omega'\sim|\mathbf{k}+\mathbf{k}'|v_{Ti}.$$

This is possible if $|\mathbf{k}+\mathbf{k}'|\lesssim\omega/v_{Ti}$ and $\omega\omega'>0$.

Let us consider further that oscillations build up with dimensionless phase velocity $\omega/kv_{Ti}=y_m<1$ near the maximum of the function $-\operatorname{Im}F(y)$, i.e.,

$$F_i=\operatorname{Im}F(y_m).\qquad(15)$$

Then the maximum of $\langle\theta^2F\rangle$ can be approximately estimated at

$$\langle\theta^2F\rangle\sim y_m^4F_r,$$

where F_r is the maximum absolute value of $-\operatorname{Im}F(y)$ in the region where it is negative. We thus arrive at the conclusion that in most real cases, when $y_m\ll1$, nonlinear damping of the oscillations can be neglected compared with decay processes.

Expression (13) with $\psi+F_r>\langle\theta^2F\rangle$ has been obtained from a crude comparison, in the kinetic equation (1), of the linear terms $2\gamma_k n_k$ with the nonlinear terms describing the oscillation damping. If the opposite condition $\langle \theta^2 F \rangle > \psi + F_r$ holds, then (13) is an expansion of the exact solution in powers of $y_m \ll 1$.

5. Knowing the turbulence spectrum, we can describe with the aid of the "quasilinear" equation the turbulent diffusion of the ions in velocity space under the influence of the oscillations that build up in the unstable plasma. Averaging Boltzmann's equation over the time of the rapid oscillations, we obtain a "quasilinear" equation for the averaged ion distribution function[8,9]:

$$\frac{\partial f_{0i}}{\partial t} = \frac{e^2}{M^2} \sum_{\mathbf{k}} \frac{\partial}{\partial \mathbf{v}} \mathbf{k} |\varphi_{\mathbf{k}}|^2 \frac{\gamma_{\mathbf{k}}}{(\omega_{\mathbf{k}} - \mathbf{k}\mathbf{v})^2 + \gamma_{\mathbf{k}}^2} \mathbf{k} \frac{\partial}{\partial \mathbf{v}} f_{0i}. \quad (16)$$

Simultaneous absorption of two oscillations by the ions are accounted for here by summing in (16) not only over the natural oscillations $\varphi_k(\mathbf{r}, t)$, but also over the forced oscillations, with amplitude

$$\varphi^{(2)}_{\mathbf{k}+\mathbf{k}'} = -\frac{\varepsilon^{(2)}_{\mathbf{k},\mathbf{k}'}(\omega, \omega')\varphi_{\mathbf{k}}\varphi_{\mathbf{k}'}}{\varepsilon^{(1)}_{\mathbf{k}+\mathbf{k}'}(\omega + \omega')} \approx \frac{ie[\mathbf{k}\mathbf{k}']_z}{m\omega_H\omega_{\mathbf{k}}} \varphi_{\mathbf{k}}\varphi_{\mathbf{k}'}. \quad (17)$$

The resonant interaction of the ions with the oscillations in (16) can be described more illustratively by integrating (16) with respect to the angles of revolution of the ions along the orbit and with respect to the longitudinal velocities v_z. As a result we obtain the diffusion equation

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{\partial}{\partial x} \Big\{ \sum \frac{|\omega| e^2 |\varphi_{\mathbf{k}}|^2}{M^2 v_{Ti}^4 (x/y_k^2 - 1)^{1/2}} + \sum \frac{|\omega + \omega'| e^4 [\mathbf{k}\mathbf{k}']^2 |\varphi_{\mathbf{k}}\varphi_{\mathbf{k}'}|^2}{M^4 \Omega_H^2 \omega_{\mathbf{k}}^2 v_{Ti}^4} \times \Big[\frac{(k + k')^2 v_{Ti}^2 x}{(\omega + \omega')^2} - 1 \Big]^{-1/2} \Big\} \frac{\partial \psi}{\partial x}, \quad (18)$$

the first sum in (18) being taken in the region $x > y_k^2$ and the second in the region $x > (\omega + \omega')^2/(k + k')^2 v_{Ti}^2$. We see therefore that the slow ions give up their transverse energy (8) to the oscillations and are dumped in the "loss cone." The oscillations in turn give up their energy to the "thermal" ions. Consequently all that takes place is a redistribution of the energy among the ions with different velocities, no resultant force acts on the ions, and their total energy is conserved during the process of turbulent diffusion.

From the expression for the turbulence spectrum and from (18) we obtain the order of magnitude of the coefficient of turbulent diffusion:

$$D \sim 0.1 \int_{k_{min}}^{\infty} dk_\perp \frac{\Omega_H y_k^3 |F_i|}{k^2 R_H \Phi_k (x/y_k^2 - 1)^{1/2}}. \quad (19)$$

It follows therefore that the strongest turbulent diffusion is produced by the long-wave oscillations. For traps not much longer than the critical length L_{c1} indicated in[5], however, oscillations with excessively long wavelengths cannot develop in the plasma. The longest wavelength of the growing perturbations k_{min}^{-1} can be obtained, following[5], from the inequality

$$\mathrm{Im}\, k_z L \gtrsim 10.$$

It expresses the requirement that the perturbations increase to nonlinear effects before they reach the ends, where they are damped as a result of the increased Landau damping by the electrons.

Using the dispersion equation (8), we rewrite this inequality in the form

$$k\lambda_D \gtrsim 10 \frac{\lambda_D}{y_m |F_i| L} \sqrt{\frac{M}{m}} \Big[\Big(1 + \frac{\omega_p^2}{\omega_H^2} \Big) k^2 \lambda_D^2 + F_r + \psi(0) \Big]^{1/2}. \quad (20)$$

It is assumed here, of course, that the trap diameter R is not very small, so that $\mathrm{Im}\, k_\perp R \gg 10$. This takes place when $R/L \gg \sqrt{\psi m/M}$. In the opposite case the wavelength will be limited for the condition $\mathrm{Im}\, k_\perp R \gtrsim 10$.

It follows therefore that when

$$L < L_{c1} = 10 \frac{\lambda_D}{y_m |F_i|} \Big[\frac{M}{m} \Big(1 + \frac{\omega_p^2}{\omega_H^2} \Big) \Big]^{1/2} \quad (21)$$

no oscillations can build up at all. When the critical length $L \approx L_{c1}$ is reached, the oscillations that increase most rapidly are those with wavelength

$$\sqrt{2} k\lambda_D = (\psi + F_r)^{1/2} (1 + \omega_p^2/\omega_H^2)^{-1/2}.$$

The time required for the ions to fill the empty "loss cone" as a result of turbulent diffusion is

$$\Omega_H \tau_{c1} \approx \frac{10 \Omega_p \Phi \sqrt{F_r}}{(\Omega_H^2 + m\Omega_p^2/M)^{1/2} y_m^4 |F_i|} \Big|_{\psi(0)=0}$$

and turns out to be much longer than the time of flight of the ions $T = L_{c1}/v_{Ti}$ at densities smaller than

$$\frac{\Omega_p}{(\Omega_H^2 + m\Omega_p^2/M)^{1/2}} < y_m^{3/2} \Big(\frac{M}{m F_r \Phi^2} \Big)^{1/4} \Big|_{\psi(0)=0}. \quad (22)$$

Under these conditions, when $L > L_{c1}$ the development of instability always causes the ions to leave the trap within a time on the order of the time of flight of the ions between the magnetic mirrors, $T \approx L/v_{Ti}$.

If condition (22) is not satisfied, there exists a length interval $L_* > L > L_{c1}$ in which the time of diffusion of the ions into the "loss cone" τ exceeds the time of flight T and determines the time of confinement of the ions in the trap. Here, as follows from (20), the wavelengths of oscillations that develop increase with increasing trap length L, and maximum diffusion occurs. From (19) and (20) we have for this case

$$\Omega_H\tau \approx 10^2 R_H \sqrt{\frac{M}{m}}\,\frac{\Phi\sqrt{F_r}}{y_m^{5/2}F_i^2 L}\bigg|_{\psi(0)=0},$$

$$L_{c1} < L < L_* = 10\frac{R_H}{y_m|F_i|}\left(\frac{M\Phi^2 F_r}{m}\right)^{1/4}. \qquad (23)$$

With further increase in length, $L > L_*$, we again go over to the filled "loss cone" mode, and the time is $\tau \approx L/v_{Ti}$. (A qualitative plot of $\tau(L)$ is shown in Fig. 2.) We note that when L increases from L_{c1} to L_* the wavelength of the growing oscillations remains much shorter than the Larmor radius ($k(L_*)R_H \approx (MF_r/m\Phi^2)^{1/4}y_m^{3/2} > 1$), the increment is

$$\gamma_* = \Omega_H y_m^{5/2} F_i (M/mF_r^5)^{1/4} \gtrsim \Omega_H,$$

and the frequency $\omega \gg \Omega_H$. We therefore remain at all times within the limits of applicability of the theory.

In order to obtain a more realistic idea of the order of magnitude of the critical lengths L_{c1} and L_* and of the confinement time, we use for the ion distribution function the approximation[5]:

$$f_{0i} = (v_\perp^2 - v_z^2)^{1/2}\exp(-Mv_\perp^2/2T),$$
$$v_\perp > |v_z|,$$
$$f_{0i} = 0, \quad v_\perp < |v_z|. \qquad (24)$$

The function F can be expressed in terms of a well known function[10,11]

$$F(y) = -y(1-2y^2)Z(y) + 2y^2,$$

$$Z(y) = \pi^{-1/2}\int_{-\infty}^{+\infty}\frac{e^{-t^2}}{t-y}\,dt.$$

The maximum of Im F(y) < 0 is reached when $y_m = 0.367$, and is equal to $F_i = -0.31$. The real part of $F_r(y_m) \approx 0.44$, and the constant $\Phi \approx F_r$. Using this expression for F(y) we have in the case of hydrogen

$$\Omega_H\tau \approx 3\cdot 10^6 R_H L^{-1},$$

$$L_{c1} \approx 4\cdot 10^3\lambda_D(1+\omega_p^2/\omega_H^2)^{1/2}, \; L_* \approx 1.5\cdot 10^3 R_H. \quad (25)$$

Let us consider, finally, the evolution of the turbulence spectrum as a function of the trap length L. As already noted, in the case of large densities

$$\Omega_p/\Omega_H \gtrsim (M/mF_r\Phi^2)^{1/4}$$

there exists a wavelength interval $L_{c1} < L < L_*$ in

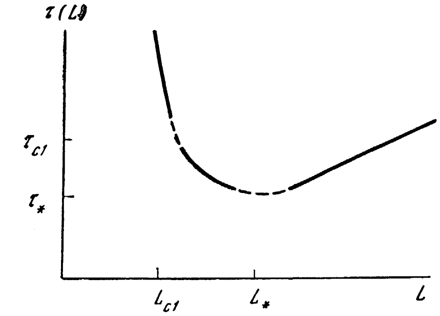

FIG. 2. Dependence of the lifetime τ of the ions in the trap on the trap length L.

which the turbulence spectrum is described approximately by expression (13) under the conditions $\psi(0) = 0$ and $\langle\theta^2 F\rangle \ll F_r$:

$$\sum_k \frac{e^2|\varphi_k|^2}{M^2 v_{Ti}^4} \approx -\frac{0.1F_i y_k^2}{[k^2\lambda_D^2(1+\omega_p^2/\omega_H^2)+F_r]k^2R_H^2},$$

$$kL > \sqrt{MF_r/m}\,y_m^{-1}F_i^{-1}, \qquad (26)$$

where the lower limit of the wave-number spectrum is obtained from condition (21). This estimate remains valid in order of magnitude also for $L > L_*$, up to lengths

$$L_{c2} = R_H\sqrt{M/m\psi},$$

at which oscillations can build up with wavelengths of the order of the Larmor radius $kR_H \sim 1$. In that case, however, it is necessary to take into account the effect of the filling of the cone $\psi(0) \neq 0$.

To consider the spectrum of the long-wave oscillations $kR_H \lesssim 1$ it would be necessary to take into account the effect of the magnetic field on the motion of the ions in such oscillations and to obtain in this manner a new kinetic equation for the waves. However, even an examination of the linear approximation shows that oscillations with very large wavelengths, $kR_H \ll 1$, with frequencies $\omega \approx l\Omega_H$, can not be excited. Indeed, it is now necessary to replace the function F(y) in the dispersion equation (8) by

$$F(\omega, k) = \int_{-\infty}^{+\infty} dv_z \int_0^\infty v_{Ti}^2\,dx\,\frac{\partial f_{0i}(v_z^2, x)}{\partial x}\,\frac{\omega J_l^2(k_\perp R_H\sqrt{x})}{\omega - l\Omega_H - k_z + i0}.$$

The region of small velocities x, where $\partial f_{0i}/\partial x > 0$, enters with a smaller weight when $kR_H \ll 1$. Therefore

$$\int df_{0i} J_l^2 < 0,$$

and the plasma distribution is stable to such long wave oscillations even in the presence of a "loss cone." For this reason, the estimates (26), where we must put $kR_H \gtrsim 1$ at all times, suffice for a description of the turbulence spectrum.

Summarizing our results, we can state that for a sufficiently long magnetic-mirror trap, with $L > L_{c1}$, the plasma leaves the trap practically within the time of flight of the ions between the magnetic mirrors.

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