QUASILINEAR THEORY OF A WEAKLY TURBULENT PLASMA WITH ACCOUNT OF CORRELATION OF ELECTRIC FIELDS

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Equations of quasilinear theory are derived by taking into account the finite time of electric and magnetic microfield correlation in the plasma. Cases of electrostatic oscillations and an electromagnetic wave moving along an external magnetic field are considered. Heating of the plasma by an external random field is investigated with the aid of the obtained equations.

1. To construct a consistent theory of turbulent plasma, it is necessary to solve a system of nonlinear equations for the particles and for the field This is a difficult problem. It is therefore sensible to attempt to solve the problem semi-phenomenologically, assuming that the correlation function for the Fourier components of the electric field is specified or experimentally determined. This can be justified by the fact that in solving a number of problems in nonlinear plasma theory, only the integral characteristics of the correlation function are important, such as the correlation time, and knowledge of the exact form of the correlation function is not essential. In the existing quasilinear theory [1] the correlation time is assumed infinite.

The purpose of the present paper is to take account of the influence of the finite correlation time of the electric microfields within the framework of the quasi-linear approximation. This approach to the investigation of a turbulent plasma is applicable for the case of a plasma situated in an external electric field with randomly varying parameters (amplitude, phase). It can be used to considerable degree also for self-consistent fields. It is essential here, of course, that the model representations from which the correlation function is determined, be in correspondence with the degree of employed nonlinear approximation. Thus, for example, in the present paper the nonlinearity of the oscillations is taken into account only by introducing the correlator of the amplitudes of the electric-field Fourier components, and no other effects connected with the nonlinearity of the oscillations are taken into account. By way of one of the models of a turbulent plasma one can use the one proposed by Stix,[2] in which the plasma consists of alternating neighboring regions, where the phase has a specified value but varies randomly on going from one region into another.

The arbitrary phase jumps can occur also when the particles go out of resonance, in the case when the oscillators move in an inhomogeneous magnetic field. An interesting possibility of occurrence of stochastic behavior when electric fields act on a nonlinear oscillator was investigated by Chirikov and Zaslavskiĭ.[3] In many cases the stochastic nature of the variation of the phase is connected with a strong nonlinearity, and an account of the considered effects signifies essentially departure outside the limits of the approximation of weakly-turbulent plasma. The results obtained can therefore be only qualitative in character.

2. Let us consider first the case of a circularly polarized electromagnetic wave, propagating along an external magnetic field. The initial system of equations consists of a kinetic equation without a collision integral and Maxwell's equations. As in the quasilinear theory, the distribution functions is conveniently represented in the form

$$f(t,\mathbf{r},\mathbf{v}) = f_0(t,\mathbf{v}) + f_1(t,\mathbf{r},\mathbf{v}), \tag{1}$$

where $f_0(t, \mathbf{v})$ is a function that varies slowly in time and determines the homogeneous background, against which the oscillations develop; $f_1(t, \mathbf{r}, \mathbf{v})$ describes the vibrational process.

The equation for f_0 has the form which is usual for quasilinear theory (see, for example [1]):

$$\frac{\partial f_0}{\partial t} - \frac{e}{m}\left\langle \mathbf{E}\frac{\partial f_1}{\partial \mathbf{v}}\right\rangle - \frac{e}{mc}\left\langle [\mathbf{vH}]\frac{\partial f_1}{\partial \mathbf{v}}\right\rangle - \frac{e}{mc}[\mathbf{vH}_0]\frac{\partial f_0}{\partial \mathbf{v}} = 0. \tag{2*}$$

In this equation $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field of the oscillations, the angle brackets denote averaging in space and in time, as well as over the ensemble of random values of the Fourier components of the electromagnetic field.

*$[\mathbf{vH}] = \mathbf{v}\times\mathbf{H}$.

We expand f_1, **E**, and **H** in a Fourier series:

$$f_1 = \frac{1}{2}\sum_k f_k(t, \mathbf{v})\, e^{i(kz-\omega_k t)} + \text{c. c.}$$
$$\mathbf{E} = \frac{1}{2}\sum_k \mathbf{E}_k(t)\, e^{i(kz-\omega_k t)} + \text{c. c.}$$
$$\mathbf{H} = \frac{1}{2}\sum_k \mathbf{H}_k(t)\, e^{i(kz-\omega_k t)} + \text{c. c.} \qquad (3)$$

$\mathbf{H}_k$ is connected with $\mathbf{E}_k$ by the relation $\mathbf{H}_k \approx c\omega_k^{-1}(\mathbf{k}\times\mathbf{E}_k)$; $E_k(t)$ are the random amplitudes of the electric field, the correlator of which is of the form

$$\overline{E_k(t)E_k^*(t')} = |E_k(t)|^2 U(t-t'), \qquad U(-t) = U(t). \quad (4)$$

The superior bar denotes averaging over the ensemble, and the process is assumed to be quasi-stationary, that is, $U(t)$ varies much faster than $|E_k(t)|^2$.

Substituting in (2) the expansion (3) and going over to polar coordinates in **v**-space with an axis along $\mathbf{H}_0$, we can write the equation for f_0 in the form

$$\frac{\partial f_0}{\partial t} = \frac{e}{4m}\sum_k e^{\pm i\theta}\left(1-\frac{kv_z}{\omega_k}\right)\overline{E_k^{\pm *}\left(\frac{\partial f_k^\pm}{\partial v_\perp} \pm \frac{i}{v_\perp}\frac{\partial f_k^\pm}{\partial \theta}\right)}$$
$$+ e^{\pm i\theta}\frac{kv_\perp}{\omega_k}\overline{E_k^{\pm *}\frac{\partial f_k^\pm}{\partial v_z}} + \text{c. c.} \qquad (5)$$

Here $E_k^\pm = E_x^\pm iE_y$ are the amplitudes of a circularly polarized wave, θ is the angle between $\mathbf{v}_\perp$ and the x axis; it is assumed that f_0 does not depend on θ [see (8)].

$f_k(t, \mathbf{v})$ is determined from the following equation:

$$\frac{\partial f_k^\pm}{\partial t} + i(kv_z-\omega_k)f_k^\pm + \omega_H\frac{\partial f_k^\pm}{\partial\theta} - \frac{e}{2m}E_k^\pm e^{\mp i\theta}$$
$$\times\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial f_0}{\partial v_z}\right] = 0. \qquad (6)$$

The solution of this equation is of the form

$$f_k^\pm(t, \mathbf{v}) = \sum_n f_n(0)\exp\left[i(kv_z-\omega_k+n\omega_H)t + in\theta\right]$$
$$+ \frac{e}{2m}e^{\mp i\theta}\int_0^t E_k^\pm(t')\exp\left[i(kv_z-\omega_k\mp\omega_H)(t'-t)\right]$$
$$\times\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial f_0}{\partial v_z}\right]dt',$$
$$f_k^\pm(t=0) = \sum_n f_n(0)\, e^{in\theta}. \qquad (7)$$

Substituting f_k from (7) in (5) and averaging, we obtain for f_0 the following equation:

$$\frac{\partial f_0}{\partial t} = \frac{\pi e^2}{4m^2}\left\{\left(\frac{\partial}{\partial v_\perp}+\frac{1}{v_\perp}\right)\sum_k |E_k^\pm|^2 V(kv_z-\omega_k\mp\omega_H)\right.$$
$$\times\left(1-\frac{kv_z}{\omega_k}\right)\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial f_0}{\partial v_z}\right]$$
$$+\frac{\partial}{\partial v_z}\sum_k |E_k^\pm|^2 V(kv_z-\omega_k\mp\omega_H)\frac{kv_\perp}{\omega_k}$$
$$\left.\times\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial f_0}{\partial v_z}\right]\right\} \qquad (8)$$

where $V(\omega)$ is the Fourier transform of the function $U(t)$:

$$V(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(t)\, e^{-i\omega t}\, dt. \qquad (8')$$

It was assumed in the derivation of the equation that $f_0(t, \mathbf{v})$ and $|E_k^\pm(t)|^2$ are time-dependent functions which vary little over time intervals of the order of $|k\bar{v} - \omega_k \mp \omega_h|^{-1}$ ($\bar{v}$ is the average velocity in the distribution f_0). This condition is customary in the quasilinear theory.

The equation for $|E_k|^2$ can also be obtained by a method analogous to that used in quasilinear theory. Using Maxwell's equations, it is easy to obtain the following relation for the variation of the energy of the k-th harmonic of the electromagnetic field:

$$\frac{\partial |E_k^\pm|^2}{\partial t}\left(1+\frac{k^2c^2}{\omega_k^2}\right)$$
$$= 2\pi e\sum_\alpha \int d\mathbf{v}\, v_\perp e^{\pm i\theta}\, \overline{f_{k\alpha}^\pm E_k^{\pm *}} + \text{c. c.} \qquad (9)$$

In this formula the index α corresponds to different plasma components (beam, thermal particles, etc.). Substituting $f_{k\alpha}^\pm$ from (7) and averaging, we obtain for $|E_k|^2$ the following equation:

$$\frac{\partial |E_k^\pm|^2}{\partial t} = \frac{2\pi^2 e^2}{m}$$
$$\times\frac{\omega_k^2}{k^2c^2+\omega_k^2}\sum_\alpha\int d\mathbf{v}\, v_\perp\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial f_0^\alpha}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial f_0^\alpha}{\partial v_z}\right]$$
$$\times V(kv_z-\omega_k\mp\omega_H)\,|E_k^\pm|^2. \qquad (10)$$

Equations (8) and (10), form, when V is known, a closed system for the determination of f_0^α and $|E_k|^2$. From these equations we easily obtain the energy conservation law:

$$\sum_\alpha\int d\mathbf{v}\frac{mv^2}{2}f_0^\alpha + \sum_k\frac{|E_k|^2+|H_k|^2}{8\pi} = \text{const.} \qquad (11)$$

As can be seen from the system (8) and (10), the quasilinear approximation corresponds to $U(t) = 1$ $[V(\omega) = \delta(\omega)]$. If we denote by τ the characteristic scale of the time correlation, that is, the interval over which U(t) differs from zero (in the quasilinear approximation $\tau = \infty$), then the function $V(\omega)$ has a finite width of the order of τ^{-1}. Thus, an account of the finite nature of τ leads to a broadening of the region of velocities in which the particles exchange energy effectively with the waves.

It follows from this, in particular, that energy can be transferred from the random field to the nonresonant plasma particles for which $kv_z \ll \omega_k^{\pm}\omega_H$[1]. In this case, retaining in the equation for f_0 only the highest-order terms in the parameters

$$kv_z / (\omega_k \pm \omega_H) \ll 1, \quad kv_\perp / \omega_k \ll 1,$$

we obtain

$$\frac{\partial f_0}{\partial t} = \frac{\pi e^2}{4m^2}\left[\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp \sum_k |E_k^{\pm}|^2 V(\omega_k \pm \omega_H)\frac{\partial f_0}{\partial v_\perp}\right)\right]. \quad (12)$$

The solution of this equation subject to the initial condition

$$f_0(t=0, \mathbf{v}) = N_0\left(\frac{m}{2\pi T_0}\right)^{3/2}\exp\left(-\frac{mv^2}{2T_0}\right) \quad (13)$$

is of the form

$$f_0(t, \mathbf{v}) = N_0\left(\frac{m}{2\pi}\right)^{3/2}\frac{1}{T_\perp T_0^{1/2}}\exp\left(-\frac{mv_\perp^2}{2T_\perp} - \frac{mv_z^2}{2T_0}\right), \quad (14)$$

where the transverse temperature $T_\perp(t)$ is defined by the relation

$$T_\perp = T_0 + \frac{1}{2}\frac{\pi e^2}{m}\sum_k V(\omega_k \pm \omega_H)\int_0^t |E_k^{\pm}(t')|^2 dt'. \quad (15)$$

If the correlation function is chosen in the form $U(t) = e^{-t/\tau}$, then

$$V(\omega) = \frac{1}{\pi}\frac{\tau}{1 + \omega^2\tau^2},$$

and we obtain for $T_\perp$ in this case the following formula

$$T_\perp = T_0 + \frac{e^2}{2m}\sum_k \frac{\tau}{1 + (\omega_k \pm \omega_H)^2\tau^2}\int_0 |E_k^{\pm}(t')|^2 dt'. \quad (16)$$

It follows from (15) and (16) that the plasma electrons can be heated by the transverse electromagnetic field, if the correlation time is finite. The energy of the electromagnetic field goes over essentially into the transverse thermal energy of the plasma, and the increase in the longitudinal thermal energy occurs in the next higher order in the small parameter. We note that formula (16) has the same form as when the plasma is heated by an electromagnetic field with fixed phase in the presence of collisions (then τ is the time between the collisions). The physical nature of this analogy is connected with the fact that in the case which we are considering the phase relations between the field and the particle are violated as a result of random jumps in the field phase, whereas in the presence of collisions the phase relations are violated when the electrons collide with any scattering centers.

The heating of the plasma is accompanied by damping of the electromagnetic field. For $|E_k|^2$ we obtain from (10), under the condition $kv_z/(\omega_k \pm \omega_H) \ll 1$, the following equation:

$$\frac{\partial |E_k^{\pm}|^2}{\partial t} + \omega_0^2\frac{\omega_k^2}{k^2c^2 + \omega_k^2}\frac{\tau}{1 + (\omega_k \pm \omega_H)^2\tau^2}|E_k^{\pm}|^2 = 0. \quad (17)$$

Integrating (17), we get

$$|E_k^{\pm}(t)|^2 = |E_k^0|^2 e^{-2\nu_k t}, \quad (18)$$

$$\nu_k = \frac{1}{2}\omega_0^2\frac{\omega_k^2}{k^2c^2 + \omega_k^2}\frac{\tau}{1 + (\omega_k \pm \omega_H)^2\tau^2}.$$

In deriving the initial system (8) and (10), we have assumed that within a time on the order of the correlation time τ the value of $|E_k|^2$ does not change appreciably. This calls for satisfaction of the condition

$$\nu\tau = \frac{\omega_0^2}{(\omega_k \pm \omega_H)^2}\frac{\omega_k^2}{k^2c^2 + \omega_k^2} \ll 1. \quad (19)$$

This condition is satisfied over a rather wide range of plasma and field parameters.

Substituting (18) in (16), we obtain for the maximum temperature for $t \to \infty$, when $|E_k|^2 \to 0$, the following formula:

$$T_\perp^{max} = T_0 + \frac{1}{8\pi N_0}\sum_k (|E_k^0|^2 + |H_k^0|^2). \quad (20)$$

3. We now proceed to consider longitudinal waves in the absence of a magnetic field. In this case, the spectrum of the oscillations is assumed three dimensional. In analogy with (8) and (10), we can obtain the following equations for f_0 and $|E_k|^2$:

$$\frac{\partial f_0^\alpha}{\partial t} = \pi\frac{e^2}{m^2}\frac{\partial}{\partial v_i}\left[\sum_{\mathbf{k}} |E_{\mathbf{k}}|^2\frac{k_i k_k}{k^2}V(\mathbf{kv} - \omega_{\mathbf{k}})\frac{\partial f_0^\alpha}{\partial v_k}\right], \quad (21)$$

$$\frac{\partial |E_{\mathbf{k}}|^2}{\partial t} = \frac{8\pi^2 e^2}{mk^2}\sum_\alpha \int d\mathbf{v}(\mathbf{kv})\mathbf{k}\frac{\partial f_0^\alpha}{\partial \mathbf{v}}V(\mathbf{kv} - \omega_{\mathbf{k}})|E_{\mathbf{k}}|^2. \quad (22)$$

These equations describe, in particular, the heating of the plasma by longitudinal waves.

[1]The acceleration of the resonant particles ($kv_z = \omega_k \pm \omega_H$) by an electromagnetic field was considered by Tsytovich.[4]

Assuming the spectrum of the longitudinal waves to be isotropic, we obtain in the highest order in $\mathbf{k}\cdot\mathbf{v}/\omega_k$ the following relation for f_0:

$$f_0 = N_0\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right),$$

where the change in the plasma temperature with time is determined by the formula

$$T = T_0 + \frac{2e^2}{3m}\sum_{\mathbf{k}} \frac{\tau}{1+\omega_k^2\tau^2}\int_0^t |E_{\mathbf{k}}(t')|^2\,dt'. \tag{23}$$

In the next higher order in $\mathbf{k}\cdot\mathbf{f}/\omega_k$, a change takes place in the directional velocity of the plasma particles (acceleration):

$$m\frac{du_i}{dt} = \frac{1}{N_0}\int d\mathbf{v}\, mv_i \frac{\partial f_0}{\partial t}$$
$$= 4\frac{e^2}{m}\sum_{\mathbf{k}} k_i |E_{\mathbf{k}}|^2 \frac{\omega_k\tau^3}{[1+\omega_k^2\tau^2]^2}. \tag{24}$$

We note that a similar effect is possible also with transverse waves.

The attenuation of the longitudinal waves, accompanying the plasma heating, is determined by the same formula (18) with

$$\nu = \nu^l = \omega_0^2 \frac{\tau}{1+\omega_k^2\tau^2}.$$

In this case $\nu^l\tau \lesssim 1$ ($\omega_k \approx \omega_0$), in which connection all the results of this section are qualitative in character.

Extension of the velocity region in which the particles exchange energy effectively with the random field can lead to stabilization of the two-stream instability. This can occur in the case when $k\Delta v\tau \le 1$ (Δv is the width of the beam distribution function). Then, the k-harmonic, the phase velocity of which lies in the interval where $\partial f_0/\partial v > 0$, will interact not only with the particles for which $\partial f_0/\partial v > 0$, that is, the particles which transfer energy to the field, but also with those for which $\partial f_0/\partial v < 0$, that is, with particles which absorb field energy, and the role of the latter can turn out to be more important. Indeed, assuming for simplicity that V can be regarded as constant in an interval Δv ($V = V_0$), we obtain from (22) in one-dimensional case

$$\gamma_b = -\frac{8\pi^2 e^2}{m} N_b V_0 < 0,$$

where γ_b is the contribution of the beam to the increment of the plasma wave, and N_b is the density of the beam. We see that the instability is indeed stabilized.

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