A RAREFIED FERMI GAS AND THE TWO-BODY SCATTERING PROBLEM

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The possibility of investigating the properties of a rarefied Fermi-gas is restricted by the lack of a solution of the three-body problem. Using the example of a gas of hard spheres it is shown that the expansion of the energy of the ground state in terms of the small parameter $\gamma = p_F a$ (p_F is the Fermi momentum, a is the sphere diameter) can be carried out only up to terms of order γ^4. The evaluation of the latter terms is equivalent to the solution of the three-body scattering problem at zero energy. The term in the expansion preceding γ^4 and proportional to $\gamma^4 \ln \gamma$ is found and evaluated. The preceding terms in the expansion have been evaluated previously[1,2]. The results are generalized to a gas with an arbitrary repulsive interaction.

THE properties of a Fermi-gas of low density ρ (for example, a hard-sphere gas) are evidently determined by two-body collisions. The characteristics of the two-body scattering problem determine the first three corrections to the energy of the ground state of an ideal Fermi-gas[1,2] (the expansion is made in powers of the Fermi momentum $p_F \sim \rho^{1/3}$). As the density increases three-body collisions become significant. In the expansion of the energy of the ground state there appear terms containing characteristics of the three-body scattering problem. In this paper the term of lowest order in p_F is found and the preceding terms of the expansion (other than those evaluated previously[1,2]), which depend only on the parameters of the two-body problem, are calculated.

The system of units is used in which $\hbar = 2m = 1$ (m is the fermion mass).

1. THE ENERGY OF THE GROUND STATE

The energy of the ground state is expressed in terms of the two-body scattering amplitude t_2 in the medium[2]:

$$\frac{E}{\rho\Omega} = \frac{3}{5}p_F^2 + \frac{1}{2\rho\Omega}\sum_{pp'}\langle pp'|t_2|\nu^2 pp' - \nu p'p\rangle \equiv \frac{3}{5}p_F^2 + \delta\varepsilon. \tag{1}$$

Here Ω is the normalization volume; $\langle \mathbf{r}|\mathrm{p}\rangle = \Omega^{-1/2}e^{i\mathbf{p}\cdot\mathbf{r}}$; the symbol $|\mathrm{p}\rangle$ is used to denote a state with momentum smaller than p_F, in the opposite case $|\mathrm{m}\rangle$ is used; ν is the number of the spin and isospin degrees of freedom; $t_2 = V(1 + M_2)$, where M_2 is the pair correlator; $(1 + M_2)|pp'\rangle$ is the pair wavefunction. The equation for $M_2|pp'\rangle$ differs from the equation describing two-body scattering in vacuo by the many-body corrections $I|pp'\rangle$; moreover, the Pauli exclusion principle is taken into account.

At low densities small relative orbital angular momenta are significant. We retain in (1) the contribution of only the s-wave. For the sake of simplicity we shall consider a specific model—a gas of hard spheres of diameter a. We shall then extend the results to the general case. For the hard spheres

$$\delta\varepsilon = \frac{3(\nu-1)}{\sqrt{4\pi}(2\pi)^3 p_F^3}\int d\mathbf{p}\, d\mathbf{p}'\, c_{00}{}^{\mathbf{gu}} j_0(ua), \tag{2}$$

where $j_0(ua)$ is a spherical Bessel function; **g** and **u** are respectively the total and the relative momenta of the pair pp′; the expression for $c_{00}^{\mathbf{gu}}$ was obtained in reference[2].

Terms which lead to integral powers of $p_F a \equiv \gamma$ in $\delta\epsilon$ are of no interest to us. Up to γ^3 they have been calculated before; for the calculation of the contribution proportional to γ^4 one must have, as is shown below, the solution of the three-body scattering problem. Omitting unessential terms we have

$$c_{00}{}^{\mathbf{gu}} = -\sqrt{4\pi}a\bar{I}^{\mathbf{gu}}(a), \tag{3a}$$

$$\bar{I}^{\mathbf{gu}}(a) = \frac{1}{(2\pi)^6}\int\frac{d\mathbf{m}\,d\mathbf{m}'\, j_0(|\mathbf{m}-\mathbf{m}'|a/2)}{m^2+m'^2-p^2-p'^2}\Omega^2\langle mm'|I|pp'\rangle. \tag{3b}$$

For $\delta\epsilon$ we obtain from (2) and (3)

$$\delta\varepsilon = -\frac{3(\nu-1)\gamma}{(2\pi)^9 p_F^4}\int\frac{d\mathbf{p}\,d\mathbf{p}'\,d\mathbf{m}\,d\mathbf{m}'}{m^2+m'^2-p^2-p'^2} \times j_0\left(|\mathbf{m}-\mathbf{m}'|\frac{a}{2}\right)\Omega^2\langle mm'|I|pp'\rangle. \tag{4}$$

I can be represented in the form of a sequence each term of which contains a certain number of two-body amplitudes of the "ladder" type [1)]:

$$I = I^{(2)} + I^{(3)} + \dots \tag{5}$$

(the superscript denotes the number of amplitudes). The amplitude for $\gamma \to 0$ and momenta of the order of p_F is equal to $8\pi a$. If we substitute into (4) the value of $I^{(2)}$ then we obtain the contribution $\delta\epsilon'$ the expansion of which begins with terms of order γ^3 [2]. We denote the next term in the expansion of this part of $\delta\epsilon$ by $\delta\epsilon_2$. The leading term in the expansion obtained from (4) by substituting I with a larger number of amplitudes will be of order higher than γ^3. We denote this term by $\delta\epsilon_1$. In order to obtain the term in the expansion for $\delta\epsilon$ which comes after γ^3 it is necessary to estimate the quantities $\delta\epsilon_1$ and $\delta\epsilon_2$.

2. THREE-BODY SCATTERING

We estimate the quantity $\delta\epsilon_1$. In going over to each successive term in the sequence (5) we add an amplitude, the law of conservation of momentum, an energy denominator (Green's function) and two integrations. If the latter are m-integrations and if we take the amplitudes to be equal to $8\pi a$ for all momenta, then the integral (4) with $I = I^{(3)}$ will have at large momenta the structure

$$\delta\varepsilon \sim p_F^2 \gamma^4 \int (d\mathbf{m})^2 / m^6. \tag{6}$$

The integral diverges logarithmically. Taking into account the dependence of the amplitudes on the momenta should lead to the cutting off of the integral at the only suitable value 1/a. Therefore,

$$\delta\varepsilon \sim p_F^2 \gamma^4 \ln \gamma. \tag{7}$$

Two new m-integrations are contained in that part of $I^{(3)}$ which describes the effect of triple correlations on pair correlations [2]. The remaining part of $I^{(3)}$ yields $\delta\epsilon \sim \gamma^4$. The most dangerous contribution to $\delta\epsilon$ with $I = I^{(4)}$ diverges linearly:

$$\delta\varepsilon \sim p_F^2 \gamma^4 a \int \frac{(d\mathbf{m})^3}{m^8} \sim p_F^2 \gamma^4. \tag{8}$$

In general, the new amplitude adds the factor a and one degree of divergence (i.e., 1/a) which compensates for this factor. The most dangerous contribution to $\delta\epsilon$ with an arbitrary number of amplitudes in I higher than three is of order γ^4.

Thus, for the evaluation of $\delta\epsilon \sim \gamma^4$ it is necessary to carry out a summation of an infinite number of terms. They all contain three p-integrations. The summation of the selected sequence is equivalent at $\gamma \to 0$ to a solution of the problem of three-body scattering in vacuo at zero energy [3].

We now evaluate the coefficient of the logarithmic term in (7). The part of $I^{(3)}$ describing the effect of triple correlations on pair correlations has been obtained previously [2] (cf., expansion (36) in reference [2]):

$$\langle mm' | I^{(3)} | pp' \rangle = \sum_{p_1} \langle mm'p_1 | J^{(3)} | \nu pp'p_1 - pp_1p' - p_1p'p \rangle, \tag{9a}$$

$$J^{(3)} = \sum_{\alpha\beta\gamma}{}' \bar{t}^{\alpha} \bar{M}_2^{\beta} M_2^{\gamma} \approx \sum_{\alpha\beta\gamma}{}' \bar{t}^{\alpha} G_3 \bar{t}^{\beta} G_2^{\gamma} t^{\gamma}. \tag{9b}$$

Here the Greek indices denote variables acted upon by operators: $\alpha, \beta, \gamma = (12), (23), (31)$. The prime on the summation denotes the condition $\alpha \neq \beta$, $\beta \neq \gamma$. The amplitudes $\bar{t}$ and t are determined by the equations

$$t = V + VG_2 t, \tag{10a}$$

$$\bar{t}^{\alpha} = V^{\alpha} + V^{\alpha} G_3 \bar{t}^{\alpha}. \tag{10b}$$

G_2 and G_3 are Green's functions in which the Pauli principle is taken into account by the operator P representing projection on the Fermi sphere:

$$G_2 = -(1 - P_1)(1 - P_2)(T_1 + T_2 - p_1^2 - p_2^2)^{-1}, \tag{11a}$$

$$G_3 = -(1 - P_1)(1 - P_2)(1 - P_3) \times (T_1 + T_2 + T_3 - p_1^2 - p_2^2 - p_3^2)^{-1}. \tag{11b}$$

If the p-momenta and the dependence of the amplitudes on the momenta are taken into account the integral (6) can be schematically written in the form

$$\int_{\sim p_F} \frac{dm}{m - p} f(m), \tag{12}$$

where f(m) is the cut-off function; f(0) = 1. It can be easily seen that the logarithmic term in the expansion (12) in terms of γ is equal to $-f(0) \ln \gamma = -\ln \gamma$. It does not depend on the form of f(m) nor on the value of the p-momenta. Formally it can be obtained by assuming in (12) that the p-momenta are equal to zero, f(m) = f(0) and by integrating between the limits (p_F, 1/a). Applying this rule to the evaluation of $\delta\epsilon$ in accordance with formulas (4), (9)—(11), we obtain [2)]

$$\delta\varepsilon_1 = \frac{64}{27\pi^2} p_F^2 (\nu - 1)(\nu - 2) \gamma^4 \ln \gamma. \tag{13}$$

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1)The "ladder" amplitude is determined by an equation of the form (10) (cf., below) with the Green's function G_n ($n > 2$).

2)The factor $(\nu-1)(\nu-2)$ is also contained in the contribution of order γ^4 determined by triple collisions).

3. CALCULATION OF $\delta\epsilon_2$

In this section we shall show that the contribution $\delta\epsilon_2$ is of order $\gamma^4 \ln \gamma$ and can be calculated. For large m-momenta the integral determining $\delta\epsilon$ with $I = I^{(2)}$ has the form

$$\delta\varepsilon \sim p_F{}^3\gamma^3 \int \frac{dm}{m^2} f(m). \qquad (14)$$

The function f(m) symbolically represents the dependence of the amplitudes on the momenta. We expand (14) in terms of γ:

$$\delta\varepsilon \sim p_F{}^2\gamma^3 f(0) - p_F{}^2\gamma^3(p_F f'(0)) \ln \gamma + \ldots \qquad (15)$$

It can be easily verified that the coefficient of the logarithmic term does not depend on the p-momenta in the Green's functions and the conservation laws, just as in the preceding section.

All the terms of $I^{(2)}$ are given in reference [2]. We take one of them and carry out the calculation of $\delta\epsilon$ using it as an example. We have

$$\langle mm'|I^{(2)}|pp'\rangle = -\sum_{p_1}\langle mm'p_1|\bar{t}^{23}M_2{}^{12}|pp_1p'\rangle,$$
$$M_2 \approx G_2 t. \qquad (16)$$

This term expresses the effect of triple correlations on pair correlations. Substituting (16) into (4) we obtain

$$\delta\varepsilon = \frac{8}{9\pi^3}\left(\frac{f'(0)}{a}\right)p_F{}^2(\nu-1)\gamma^4\ln\gamma, \qquad (17)$$

where in this case

$$f'(0) = \frac{1}{8\pi a}\frac{\partial}{\partial m}\Omega\langle m0|\bar{t}(m)|m0\rangle|_{m=0}, \qquad (18)$$

while $\bar{t}(m)$ is determined by the equation which follows from (10b) and (11b) for $\gamma \to 0$:

$$\bar{t}(m) = V - V(T_1 + T_2 + m^2)^{-1}\bar{t}(m). \qquad (19)$$

Writing (19) in matrix form as a series in V, differentiating with respect to m and again collecting the series in V into the amplitudes $\bar{t}$ one can easily obtain the formula

$$\frac{\partial}{\partial \mathbf{m}}\Omega\langle m0|\bar{t}(m)|m0\rangle|_{m=0} = [\Omega\langle 00|\bar{t}(0)|00\rangle]^2 \lim_{m\to 0}\int\frac{\partial G}{\partial \mathbf{m}}d\mathbf{m}_2, \qquad (20)$$

where G is the Green's function for equation (19):

$$G = -\frac{1}{2(2\pi)^3}\frac{1}{m^2 - \mathbf{m}\mathbf{m}_2 + m_2^2}$$

Thus, the desired derivative is expressed in terms of the vacuum scattering amplitude at zero energy. From (20) and (18) we find that $f'(0) = a\sqrt{3}/2$. From this we have

$$\delta\varepsilon = \frac{4\sqrt{3}}{9\pi^3}p_F{}^2(\nu-1)\gamma^4\ln\gamma. \qquad (21)$$

The contribution of all the terms of $I^{(2)}$ expressing the effect of supercorrelations on pair correlations is obtained from (21) by multiplying it by $-4(\nu - 2)$:

$$\delta\varepsilon_2 = -\frac{16\sqrt{3}}{9\pi^3}p_F{}^2(\nu-1)(\nu-2)\gamma^4\ln\gamma. \qquad (22)$$

In $I^{(2)}$ there are also terms which express the mutual effect of pair correlations and the change in the dispersion law for the particles [2]. In their case we convince ourselves in a similar manner that $f(m) = j_0^2(ma)$, and consequently they introduce nothing into the logarithmic contribution.

Formula (4) gives the contribution of the s-wave to $\delta\epsilon$. From an analogous formula for the p-wave it follows that the p-wave gives no contribution of order $\gamma^4 \ln \gamma$. Thus, the total contribution of order $\gamma^4 \ln \gamma$ to $\delta\epsilon$ is equal to 3)

$$\delta\varepsilon = \delta\varepsilon_1 + \delta\varepsilon_2 = \frac{16}{27\pi^3}(4\pi - 3\sqrt{3})p_F{}^2(\nu-1)(\nu-2)\gamma^4\ln\gamma. \qquad (23)$$

The results of the preceding paper [2] and of the present paper can be easily generalized to a Fermi gas with an arbitrary repulsive interaction of finite range. Without reproducing the intermediate calculations we give the results of the evaluation of $\delta\epsilon$.

1. The contribution to the energy, which in the case of hard spheres is proportional to γ^3, for an arbitrary potential is equal to

$$\delta\varepsilon = p_F{}^2\left\{(\nu-1)\left[\frac{1}{10\pi}(p_F a_0)^2(p_F r_0) + (p_F a_0)^3(0.064 + 0.012 + (\nu-3)0.059)\right] + (\nu+1)\frac{1}{5\pi}(p_F a_1)^3\right\}. \qquad (24)$$

Here a_0, a_1 are the s- and p-scattering lengths for a particle of reduced mass $1/4$ scattered by a potential V, r_0 is the effective range. 4)

2. The logarithmic term is equal to

$$\delta\varepsilon = \frac{16}{27\pi^3}(4\pi - 3\sqrt{3})p_F{}^2(\nu-1)(\nu-2)(p_F a_0)^4 \ln p_F R, \qquad (25)$$

where R is any quantity of the dimensions of length characterizing the potential.

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3) In a letter by the present author[4] perturbation theory graphs are given which correspond to the logarithmic contribution (23). The letter contains a misprint involving the sign of $\delta\epsilon_2$.

4) The parameters a_0, r_0, and a_1 are determined by the relations

$$k\cot\delta_0 = -\frac{1}{a_0} + \frac{1}{2}r_0k^2, \quad k^3\cot\delta_1 = -\frac{3}{a_1{}^3} \quad \text{for} \quad k \to 0.$$

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