PARAMETRIC RESONANCE IN A PLASMA

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The parametric resonance conditions are derived for a plasma in an external periodic electric field. The largest oscillation growth rate, which is of the order of the cube root of the electron-ion mass ratio multiplied by the electron Langmuir frequency, obtains at wavelengths comparable with the oscillation amplitude of the electrons in the external field if the external frequency (or its harmonics) are near the electron Langmuir frequency. At the same wavelengths, but far from harmonic resonances (harmonics of the external field below the electron Langmuir frequency) the growth rate is of the order of the ion Larmor frequency. If the amplitude of the electron excursion is small compared with the plasma oscillation wavelength the reduction factor for the growth rates is given by the excursion/wavelength ratio to the two-thirds power.

INTRODUCTION

In experiments on radiation acceleration of plasma [1] it frequently happens that the velocities associated with electron oscillations in the external electric field are appreciably greater than the thermal velocity. Under these conditions the oscillation amplitudes can be appreciably greater than the Debye radius and it is of interest to investigate the stability and characteristics of these oscillations of a plasma in a strong high-frequency electric field. The plasma can be described by a two-fluid hydrodynamic model without including the effect of thermal motion. It has been shown by Aliev and the author [2] that when the frequency of the external field is appreciably greater than the electron Langmuir frequency the plasma can exhibit some new effects which are quite sensitive to the external field. In the present work we investigate one aspect of this problem which is of interest from an experimental point of view—this is the case in which the frequency of the external electric field is comparable with the electron Langmuir frequency.

It will be shown below that oscillations are indeed excited in a plasma located in a uniform high-frequency electric field. The maximum growth rate of these oscillations is of the order of the electron Langmuir frequency multiplied by the cube root of the electron-ion mass ratio; this maximum growth rate obtains near resonance at harmonics of the external frequency (harmonic resonance). Far from this resonance the maximum growth rate is of the order of the ion Langmuir frequency. The maximum growth rates are found at oscillation wavelengths approximately equal to the amplitude of the electron oscillations. In weak electric fields, where the wavelengths of the plasma oscillation are appreciably greater than the excursion of the electron in the external field, the growth rates are found to be much lower than the maximum value cited above. The reduction is given by the excursion/wavelength ratio to the two-thirds power.

Finally, when the fields are very strong and the plasma oscillation wavelengths are very small (large excursion/wavelength ratio) the coupling between the plasma components is reduced and independent oscillations of the electrons and ions are possible. In this case the ions oscillate at the ion Langmuir frequency. This result agrees with that obtained in [2] in which the case of a high-frequency external field was treated. We may note that the low-frequency branch of the oscillations considered in the present work, which corresponds to parametric excitation, goes over to the low-frequency spectrum found earlier [2] in the limit of high-frequency external fields.

1. INITIAL EQUATIONS

Assuming that the plasma is cold so that the thermal motion of the particles can be neglected we base our analysis on the hydrodynamic equations:

$$\frac{\partial n_\alpha}{\partial t} + \operatorname{div} n_\alpha \mathbf{v}_\alpha = 0, \qquad \frac{\partial \mathbf{v}_\alpha}{\partial t} + \left(\mathbf{v}_\alpha \frac{\partial}{\partial \mathbf{r}}\right)\mathbf{v}_\alpha = \frac{e_\alpha}{m_\alpha}\mathbf{E}. \quad (1.1)$$

No magnetic field appears in these equations because we are not interested in a magnetized plasma, (no strong constant field); moreover, the subsequent analysis is concerned only with irrotational oscillations ($\delta\mathbf{E} = -\nabla\delta\varphi$). The equilibrium state of the plasma is uniform in space: $n_\alpha = n_\alpha^{(0)}$. In this case the particle velocities in the equilibrium state are given by

$$\mathbf{u}_\alpha = -\frac{e_\alpha}{m_\alpha\omega_0}\mathbf{E}_0\cos\omega_0 t, \tag{1.2}$$

where $\mathbf{E}_0$ is the amplitude of the external ac field.[1)]

For small oscillations about the equilibrium state we may assume that the displacements from equilibrium are proportional to $e^{i\mathbf{k}\cdot\mathbf{r}}$; then, eliminating the nonequilibrium electric field potential by the use of Poisson's equation we obtain from (1.1) the following equation, which describes the time variation of the particle number density δn_α:

$$\left[\frac{\partial}{\partial t} + i\mathbf{k}\mathbf{u}_\alpha\right]^2\delta n_\alpha = -\frac{4\pi e_\alpha}{m_\alpha}n_\alpha^{(0)}\sum_\beta e_\beta\delta n_\beta. \tag{1.3}$$

For a plasma consisting of electrons and one ion species the functions

$$\nu_\alpha = e_\alpha\delta n_\alpha\exp\left\{-i\frac{e_\alpha\mathbf{E}_0\mathbf{k}}{m_\alpha\omega_0^2}\sin\omega_0 t\right\} \tag{1.4}$$

are governed by the following two equations:[2]

$$\nu_e'' + \omega_{Le}^2[\nu_e + \nu_i\exp(-ia\sin\omega_0 t)] = 0,$$
$$\nu_i'' + \omega_{Li}^2[\nu_i + \nu_e\exp(ia\sin\omega_0 t)] = 0. \tag{1.5}$$

Here, $\omega_{L\alpha} = (4\pi e_\alpha^2 n_\alpha^{(0)}/m_\alpha)^{1/2}$ represents the Langmuir frequency of the electrons and ions respectively and

$$a = \left(\frac{e}{m_e} - \frac{e_i}{m_i}\right)\frac{1}{\omega_0^2}\mathbf{k}\mathbf{E}_0 \approx \frac{e\mathbf{k}\mathbf{E}_0}{m_e\omega_0^2}. \tag{1.6}$$

We shall investigate the solutions of Eq. (1.5) for the case in which the frequency of the external field ω_0 is appreciably greater than the ion-Langmuir frequency but comparable with the electron-Langmuir frequency. The small parameter in (1.5) is the ratio of the ion-Langmuir frequency to the electron-Langmuir frequency or, what is the same thing, the mass ratio. This small parameter is used in investigating the solutions of (1.5).

2. NONRESONANCE CASE AND SUB-HARMONICS (HIGH-FREQUENCY OSCILLATIONS)

In a number of cases the system in (1.5) can be investigated directly using methods developed by Bogolyubov and Mitropolskiĭ.[3] In particular this holds for the nonresonance case in which the equalities

$$\omega_0 \approx (p/q)\omega_{Le} \tag{2.1}$$

do not hold (p and q are simple integers). The solution of (1.5) can then be written in the form

$$\nu_e(t) = u_-(t)e^{-i\omega_{Le}t} + u_+(t)e^{i\omega_{Le}t} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \ldots$$
$$\nu_i(t) = \varepsilon w^{(1)}(t) + \varepsilon^2 w^{(2)} + \ldots, \tag{2.2}$$

where $\epsilon = (\omega_{Li}/\omega_{Le})^2$.

The characteristic time for changes in the functions u_- and u_+ is appreciably greater than $2\pi/\omega_{Le}$. It is then evident that

$$w^{(1)} = u_- e^{-i\omega_{Le}t}\sum_{l=-\infty}^{+\infty}J_l(a)\frac{\omega_{Le}^2}{(l\omega_0 - \omega_{Le})^2}e^{il\omega_0 t} + u_+ e^{i\omega_{Le}t}\sum_{l=-\infty}^{+\infty}J_l(a)\frac{\omega_{Le}^2}{(l\omega_0 + \omega_{Le})^2}e^{il\omega_0 t}, \tag{2.3}$$

$$u^{(1)} = \frac{\omega_{Le}^4}{\omega_0}\sum_{n\neq 0}\frac{e^{-in\omega_0 t}}{n(n\omega_0 + 2\omega_{Le})}\sum_{l=-\infty}^{+\infty}\frac{J_l(a)J_{n+l}(a)}{(l\omega_0 - \omega_{Le})^2}u_- e^{-i\omega_{Le}t} + \frac{\omega_{Le}^4}{\omega_0}\sum_{n\neq 0}\frac{e^{-in\omega_0 t}}{n(n\omega_0 - 2\omega_{Le})}\sum_{l=-\infty}^{+\infty}\frac{J_l(a)J_{n+l}(a)}{(l\omega_0 + \omega_{Le})^2}u_+ e^{i\omega_{Le}t} \tag{2.4}$$

(J_l is the Bessel function). Furthermore, in the first approximation in ϵ we have

$$\frac{d}{dt}u_\pm = \pm iu_\pm\frac{1}{2}\omega_{Le}\omega_{Li}^2\sum_{l=-\infty}^{+\infty}\frac{J_l^2(a)}{(l\omega_0 - \omega_{Le})^2}. \tag{2.5}$$

The last equation allows us to write the following approximate expression for the characteristic frequency of the plasma oscillations:

$$\omega = \omega_{Le}\left\{1 + \frac{1}{2}\sum_{l=-\infty}^{+\infty}J_l^2(a)\frac{\omega_{Li}^2}{(\omega_{Le} - l\omega_0)^2}\right\}. \tag{2.6}$$

In particular, when $\omega_0 \gg \omega_{Le}$ we obtain the oscillations spectrum given in [2].

The relations in (2.3)—(2.5) indicate that nonresonance approximation can become poor near certain critical points

$$\omega_0 = \pm\omega_{Le}/n \quad (n = 1, 2, \ldots), \tag{2.7}$$
$$\omega_0 = \pm\omega_{Le}/(n + 1/2) \quad (n = 0, 1, 2, \ldots). \tag{2.8}$$

It is evident that taking account of the higher approximations in powers of ϵ will require a careful investigation of the higher subharmonics if the thermal motion is taken into account:

$$\omega_0 = \pm\omega_{Le}/(n + 1/l) \quad (n = 0, 1, 2, \ldots;\ l = 3, 4, \ldots). \tag{2.9}$$

The investigation of resonance cases is of special interest for the theory of stability of plasma with respect to perturbations induced by an external high-frequency electric field. It will be shown below that the highest growth rates arise in the

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*The generalization to the case $\mathbf{E}(t) = \Sigma_i \mathbf{E}_i \sin(\omega_0 t + \delta_i)$ is trivial.[2]

vicinity of harmonic resonances of the external frequency (2.7).

It should be emphasized that the method of obtaining the solution in the form (2.2) used here is completely unsuitable for solutions which do not contain high-frequency harmonics. However, before considering this case, which is qualitatively different, we shall consider the possibility of subharmonic resonances. Specifically, in accordance with (2.8) it is necessary to study the possibility of resonances when

$$(n+{}^1\!/_2)\,\omega_0=\omega_{Le}(1+\varepsilon\Delta), \qquad (2.10)$$

where Δ is small compared with ϵ^{-1}. As in the nonresonance case, the solution of (1.5) can be written in the form [3]

$$v_e(t)=u_-e^{-i(n+{}^1\!/_2)\omega_0 t}+u_+e^{i(n+{}^1\!/_2)\omega_0 t}+\varepsilon u^{(1)}+\ldots,$$

$$v_i(t)=\varepsilon w^{(1)}+\ldots, \qquad (2.11)$$

where u_- and u_+ are slowly varying functions of time.

Since (2.3) does not exhibit any singularities in the region of the point (2.8) it is obvious that $w^{(1)}$ will be given by

$$w^{(1)}=\left(n+\frac{1}{2}\right)^2\sum_{l=-\infty}^{+\infty}J_l(a)\,e^{il\omega_0 t}\left\{\frac{u_-}{(l-n-{}^1\!/_2)^2}e^{-i(n+1/2)\omega_0 t}+\frac{u_+}{(l+n+{}^1\!/_2)^2}e^{i(n+1/2)\omega_0 t}\right\}. \qquad (2.12)$$

Using the principle of harmonic balance it is easy to show that

$$u^{(1)}=\left(n+\frac{1}{2}\right)^4\sum_{s\neq 0,\,2n+1}\sum_{l=-\infty}^{+\infty}\frac{J_l(a)}{s(s-2n-1)}\times\left\{\frac{u_-J_{l-s}(a)}{(l-n-{}^1\!/_2)^2}e^{i(s-n-1/2)\omega_0 t}+\frac{u_+J_{l+s}(a)}{(l+n+{}^1\!/_2)^2}e^{-i(s-n-1/2)\omega_0 t}\right\}, \qquad (2.13)$$

and

$$\frac{du_-}{dt}=i\varepsilon\Delta\omega_0\left(n+\frac{1}{2}\right)u_--\frac{i\varepsilon}{2}\omega_0\left(n+\frac{1}{2}\right)^3\times\sum_{l=-\infty}^{+\infty}\frac{1}{(l+n+{}^1\!/_2)^2}\{J_l^2(a)u_-+J_lJ_{l+2n+1}u_+\}, \qquad (2.14)$$

$$\frac{du_+}{dt}=-i\varepsilon\Delta\omega_0\left(n+\frac{1}{2}\right)u_++\frac{i\varepsilon}{2}\omega_0\left(n+\frac{1}{2}\right)^3\times\sum_{l=-\infty}^{+\infty}\frac{1}{(l+n+{}^1\!/_2)^2}\{J_l^2u_+-J_lJ_{l+2n+1}u_-\}. \qquad (2.15)$$

The system (2.14) and (2.15) has a solution $\sim e^{-i\delta\omega t}$. Hence

$$\omega=(n+{}^1\!/_2)\,\omega_0+\delta\omega=\omega_{Le}\left\{1+\varepsilon\left(\Delta\mp\left\{\left[\Delta-\frac{1}{2}\left(n+\frac{1}{2}\right)^2\sum_{l=-\infty}^{+\infty}\frac{J_l^2(a)}{(l+n+{}^1\!/_2)^2}\right]^2+\left[\frac{1}{2}\left(n+\frac{1}{2}\right)^2\sum_{l=-\infty}^{+\infty}\frac{J_lJ_{l+2n+1}}{(l+n+{}^1\!/_2)^2}\right]^2\right\}^{1/2}\right)\right\}. \qquad (2.16)$$

We then see that oscillations are not excited in this approximation. In the limit $\Delta \gg 1$ (2.16) with the minus sign becomes (2.6). With the plus sign the right side of (2.16) becomes $(2n+1)\omega_0 - \omega$ where ω is determined by (2.6).

In the vicinity of the higher subharmonics (2.9) it is possible that frequency corrections of order $\omega_{Le}\epsilon^{(l-1)}$ will arise. In particular, when $(n+{}^1\!/_3)\omega_0 \approx \omega_{Le}$ the order of this correction is smaller than $\omega_{Le}\epsilon^2 = \omega_{Li}{}^4/\omega_{Le}{}^3$. In other words, the time in which the subharmonic resonances can develop is six orders of magnitude greater than the period of the electron Langmuir oscillations.

We wish to emphasize again that the analysis in this section refers, as is obvious from (2.2)—(2.4) and (2.11)—(2.13), to the case in which the solution of (1.5) does not contain the zeroth harmonic of the external field. Or, to put the matter more precisely, the harmonic expansion of the solutions does not contain a term which varies slightly in a period of the external field. In the following section we consider the opposite limit, in which it is necessary to take account of the zeroth harmonic of the external field in forming the solutions of (1.5).

3. RESONANCE AT HARMONICS OF THE EXTERNAL FREQUENCY; LOW-FREQUENCY OSCILLATIONS AND NONRESONANCE EXCITATION OF OSCILLATIONS

We first consider the harmonic resonance case $\omega_{Le}{}^2 \approx (n\omega_0)^2$ (resonance at a harmonic of the external frequency). The solution of (1.5) is written in the form

$$v_e=e^{\nu t}\left\{u_{+n}\,e^{-in\omega_0 t}+u_{-n}\,e^{in\omega_0 t}+\sum_{l\neq\pm n}u_le^{-il\omega_0 t}\right\}, \qquad (3.1)$$

$$v_i=e^{\nu t}\left\{w_0+\sum_{l\neq 0}w_le^{il\omega_0 t}\right\}. \qquad (3.2)$$

The necessity for taking account of the fundamental is obvious in this case because the coefficients in (1.5) contain all harmonics of the external field. In this connection substitution of the solution of the form (3.2) even without the fundamental will give rise to a fundamental term. Substituting these expansions in (1.5) and using the principle of harmonic balance, we find

$$w_m\left\{1+\left(\frac{im\omega_0+\gamma}{\omega_{Li}}\right)^2\right\}+u_nJ_{m+n}+u_{-n}J_{m-n}+\sum_{l\neq\pm n}u_lJ_{m+l}=0\quad(m\neq 0),\tag{3.3}$$

$$u_m\left\{1+\left(\frac{-im\omega_0+\gamma}{\omega_{Le}}\right)^2\right\}+w_0J_m+\sum_{l\neq 0}w_lJ_{l+m}=0\quad(m\neq\pm n),\tag{3.4}$$

$$u_{+n}\left\{1+\left(\frac{-in\omega_0+\gamma}{\omega_{Le}}\right)^2\right\}+w_0J_n+\sum_{l\neq 0}w_lJ_{l+n}=0,\tag{3.5}$$

$$u_{-n}\left\{1+\left(\frac{in\omega_0+\gamma}{\omega_{Le}}\right)^2\right\}+w_0J_{-n}+\sum_{l\neq 0}w_lJ_{l-n}=0,\tag{3.6}$$

$$w_0\left\{1+\left(\frac{\gamma}{\omega_{Li}}\right)^2\right\}+u_nJ_n+u_{-n}J_{-n}+\sum_{l\neq\pm n}w_lJ_l=0.\tag{3.7}$$

The relations in (3.3) and (3.4) yield the following approximate expressions for u_m (for $m\neq\pm n$) and w_m (for $m\neq 0$):

$$w_m=-\frac{\varepsilon\omega_{Le}^2}{(im\omega_0+\gamma)^2+\omega_{Li}^2}\left\{u_nJ_{m+n}(a)+u_{-n}J_{m-n}(a)-w_0\sum_{l\neq\pm n}\frac{\omega_{Le}^2J_lJ_{l+m}}{\omega_{Le}^2+(-il\omega_0+\gamma)^2}\right\}+O(\varepsilon^2),\tag{3.8}$$

$$u_m=-\frac{\omega_{Le}^2}{(-im\omega_0+\gamma)^2+\omega_{Le}^2}\left\{w_0J_m-\sum_{l\neq 0}J_{l+m}\frac{\varepsilon\omega_{Le}^2}{(il\omega_0+\gamma)^2+\omega_{Li}^2}\times\left[u_nJ_{l+n}+u_{-n}J_{l-n}-w_0\sum_{r\neq\pm n}\frac{\omega_{Le}^2J_rJ_{l+r}}{\omega_{Le}^2+(-ir\omega_0+\gamma)^2}\right]\right\}+O(\varepsilon^2).\tag{3.9}$$

Substituting the last two expressions in (3.5)—(3.7) we have

$$u_n\left\{1+\left(\frac{-in\omega_0+\gamma}{\omega_{Le}}\right)^2-\varepsilon A_n\right\}-u_{-n}\varepsilon B_n+w_0\{J_n+\varepsilon C_n\}=0,\tag{3.10}$$

$$u_{-n}\left\{1+\left(\frac{in\omega_0+\gamma}{\omega_{Le}}\right)^2-\varepsilon A_{-n}\right\}-u_n\varepsilon B_n+w_0\{J_{-n}+\varepsilon C_{-n}\}=0,\tag{3.11}$$

$$w_0\left\{1+\left(\frac{\gamma}{\omega_{Li}}\right)^2-D_n+\varepsilon E_n\right\}+u_n\{J_n+\varepsilon C_n\}+u_{-n}\{J_{-n}+\varepsilon C_{-n}\}=0,\tag{3.12}$$

where

$$A_n=\sum_{l\neq 0}J_{l+n}^2(a)\frac{\omega_{Le}^2}{(il\omega_0+\gamma)^2+\omega_{Li}^2}\tag{3.13}$$

$$B_n=\sum_{l\neq 0}J_{l+n}(a)J_{l-n}(a)\frac{\omega_{Le}^2}{(il\omega_0+\gamma)^2+\omega_{Li}^2}\tag{3.14}$$

$$C_n=\sum_{l\neq 0}\frac{\omega_{Le}^2}{(il\omega_0+\gamma)^2+\omega_{Li}^2}\times\sum_{r\neq\pm n}\frac{\omega_{Le}^2}{\omega_{Le}^2+(-ir\omega_0+\gamma)^2}J_{n+l}J_rJ_{r+l},\tag{3.15}$$

$$D_n=\sum_{l\neq\pm n}J_l^2\frac{\omega_{Le}^2}{(il\omega_0+\gamma)^2+\omega_{Le}^2},\tag{3.16}$$

$$E_n=-\sum_{l\neq\pm n}\sum_{r\neq\pm n}\sum_{s\neq 0}\omega_{Le}^6\times\frac{J_l(a)J_r(a)J_{l+s}(a)J_{r+s}(a)}{[(il\omega_0+\gamma)^2+\omega_{Le}^2][(ir\omega_0+\gamma)^2+\omega_{Le}^2][(is\omega_0+\gamma)^2+\omega_{Li}^2]}.\tag{3.17}$$

The solutions of the system in (3.10)—(3.12) will be nontrivial when

$$\left\{\left[\left(\frac{-in\omega_0+\gamma}{\omega_{Le}}\right)^2+1-\varepsilon A_n\right]\left[\left(\frac{in\omega_0+\gamma}{\omega_{Le}}\right)^2+1-\varepsilon A_{-n}\right]-\varepsilon^2B_n^2\right\}\left\{1+\frac{\gamma^2}{\varepsilon\omega_{Le}^2}-D_n+\varepsilon E_n\right\}-[J_n+\varepsilon C_n]^2\times\left[\left(\frac{in\omega_0+\gamma}{\omega_{Le}}\right)^2+1-\varepsilon A_{-n}\right]-[J_{-n}+\varepsilon C_{-n}]^2\times\left[\left(\frac{-in\omega_0+\gamma}{\omega_{Le}}\right)^2+1-\varepsilon A_n\right]-2\varepsilon B_n[J_n+\varepsilon C_n][J_{-n}+\varepsilon C_{-n}]=0.\tag{3.18}$$

Assuming that $|\gamma|\ll\omega_0$, ω_{Le} and $|\omega_{Le}^2-n^2\omega_0^2|\ll\omega_{Le}^2$, we obtain the following expression from (3.18):

$$\frac{\gamma^4}{\omega_{Le}^4}+\frac{\gamma^2}{\omega_{Le}^2}\left\{\frac{1}{4}\left(1-\frac{n^2\omega_0^2}{\omega_{Le}^2}\right)^2+\varepsilon\left[1-D_n^0-\frac{1}{2}J_n^2-\frac{1}{2}A_n^0\left(1-\frac{n^2\omega_0^2}{\omega_{Le}^2}\right)\right]\right\}+\frac{\varepsilon}{4}\left\{\left(1-\frac{n^2\omega_0^2}{\omega_{Le}^2}\right)^2\times(1-D_n^0)-2J_n^2\left(1-\frac{n^2\omega_0^2}{\omega_{Le}^2}\right)+\varepsilon\left[2J_n^2(A_n^0-(-1)^nB_n^0)+\left(1-\frac{n^2\omega_0^2}{\omega_{Le}^2}\right)\times(-4J_nC_n^0+2A_n^0(D_n^0-1))\right]\right\}=0.\tag{3.19}$$

Here

$$A_n^0=n^2\sum_{l\neq 0}l^{-2}J_{l+n}^2,\qquad B_n^0=-n^2\sum_{l\neq 0}l^{-2}J_{l+n}J_{l-n},$$

$$C_n{}^0 = -n^2 \sum_{l\neq 0}\sum_{r\neq\pm n} \frac{J_{l+n}J_{l+r}J_r}{(n^2-r^2)\,l^2},$$

$$D_n{}^0 = n^2 \sum_{l\neq\pm n} (n^2-l^2)^{-1} J_l^2.$$

The roots of the biquadratic equation (3.19) represent the answer to the problem. We shall consider a number of simple limiting cases, which are actually sufficient for understanding the behavior of the plasma at resonances of harmonics of the external frequency. In the strong field limit, where the electron excursion in the external field is much greater than the wavelength of the plasma oscillations, the solutions of (3.19) can be written

$$\gamma / \omega_{Le} = +{}^1\!/_2 i(1 - n^2\omega_0{}^2 / \omega_{Le}{}^2), \qquad (3.20)$$

$$\gamma / \omega_{Le} = \pm i\varepsilon^{1/2} = \pm i\omega_{Li} / \omega_{Le}. \qquad (3.21)$$

It is evident that oscillations are not excited in this limit. These formulas correspond to independent oscillations of the electrons and ions.

In the opposite limit, the weak field case, the solution can be written by employing an expansion in powers of α, the ratio of the electron excursion to the wavelength of the plasma oscillations. In this case, when $n = 1$

$$\frac{\gamma^2}{\omega_{Le}{}^2} = \frac{1}{8}\Big\{-\Big(1-\frac{\omega_0{}^2}{\omega_{Le}{}^2}+\varepsilon\Big)^2 - \frac{3}{2}\varepsilon a^2 \pm \Big[\Big[\Big(1-\frac{\omega_0{}^2}{\omega_{Le}{}^2}+\varepsilon\Big)^2 - \frac{3}{2}\varepsilon a^2\Big]^2 + 8\varepsilon a\Big(1-\frac{\omega_0{}^2}{\omega_{Le}{}^2}+\varepsilon\Big)\Big]^{1/2}\Big\}. \qquad (3.22)$$

If the following inequality is satisfied

$$(\varepsilon a^2)^{1/3} \gg |1 - \omega_0{}^2 / \omega_{Le}{}^2 + \varepsilon| \gg \varepsilon a^2, \qquad (3.23)$$

(3.22) is simplified, and assumes the form

$$\frac{\gamma^2}{\omega_{Le}{}^2} = \pm\Big(\frac{\varepsilon a^2}{8}\Big[1-\frac{\omega_0{}^2}{\omega_{Le}{}^2}+\varepsilon\Big]\Big)^{1/2} = \pm\frac{\omega_{Li}a}{2\sqrt{2}\,\omega_{Le}{}^2}(\omega_{Le}{}^2 + \omega_{Li}{}^2 - \omega_0{}^2)^{1/2}. \qquad (3.24)$$

It then follows that oscillations can be excited, 1) when $\omega_{Le}{}^2 + \omega_{Li}{}^2 < \omega_0^2$ and 2) when the plus sign is used in $\omega_{Le}{}^2 + \omega_{Li}{}^2 > \omega_0^2$.

If (3.23) is violated, then (3.22) indicates the possibility of two roots, one of which is appreciably greater than the other. The larger root is of the form

$$\frac{\gamma^2}{\omega_{Le}{}^2} = -\frac{1}{4}\Big(1-\frac{\omega_0{}^2}{\omega_{Le}{}^2}+\varepsilon\Big)^2 - \frac{3}{8}\varepsilon a^2. \qquad (3.25)$$

It is evident that this root does not correspond to a growing wave. The small roots are given by the following:

$$\frac{\gamma^2}{\omega_{Le}{}^2} = \varepsilon a^2 \Big/ 2\Big[1+\varepsilon-\frac{\omega_0{}^2}{\omega_{Le}{}^2}\Big] \quad \text{for} \quad \Big|1-\frac{\omega_0{}^2}{\omega_{Le}{}^2}+\varepsilon\Big| \gg (\varepsilon a^2)^{1/3}, \qquad (3.26)$$

$$\frac{\gamma^2}{\omega_{Le}{}^2} = \frac{2}{3}\Big[1+\varepsilon-\frac{\omega_0{}^2}{\omega_{Le}{}^2}\Big] \quad \text{for} \quad \Big|1-\frac{\omega_0{}^2}{\omega_{Le}{}^2}+\varepsilon\Big| \ll \varepsilon a^2. \qquad (3.27)$$

According to the last two expressions, excitation is possible when $\omega_{Le}{}^2 + \omega_{Li}{}^2 > \omega_0^2$. It is evident that for a given $a^2 \ll 1$ the growth rate is a maximum when

$$1 - \omega_0{}^2 / \omega_{Le}{}^2 + \varepsilon = (\varepsilon a^2)^{1/3}\eta, \qquad (3.28)$$

where η is of order unity. In this case

$$\gamma^2 / \omega_{Le}{}^2 = {}^1\!/_8 (\varepsilon a^2)^{2/3}\{-\eta^2 \pm (\eta^4 + 8\eta)^{1/2}\}. \qquad (3.29)$$

In particular, for the plus sign the maximum growth rate obtains when $\eta = +1$. On the other hand, for the minus sign excitation is possible if $-2 < \eta < (\epsilon a)^{2/3}$ and the maximum growth rate obtains when $\eta = 2^{-1/3}$.

We now turn our attention to the intermediate case, in which the electron excursion is comparable with the wavelength of the plasma oscillation. We assume that J_n, A_n^0, D_n^0 and B_n^0 are of order unity. In this case

$$\frac{\gamma^2}{\omega_{Le}{}^2} = -\frac{1}{2}\Big\{\frac{1}{4}\Big(1-\frac{n^2\omega_0{}^2}{\omega_{Le}{}^2}\Big)^2 + \varepsilon\Big(1-D_n{}^0-\frac{1}{2}J_n{}^2\Big) \pm \Big(\Big[\frac{1}{4}\Big(1-\frac{n^2\omega_0{}^2}{\omega_{Le}{}^2}\Big)^2 + \varepsilon\Big(1-D_n{}^0-\frac{1}{2}J_n{}^2\Big)\Big]^2 + 2\varepsilon J_n{}^2\Big[1-\frac{n^2\omega_0{}^2}{\omega_{Le}{}^2} - \varepsilon(A_n-(-1)^n B_n)\Big]\Big)^{1/2}\Big\}. \qquad (3.30)$$

This expression assumes a simple form in several limiting cases:

$$\left.\begin{aligned} \frac{\gamma^2}{\omega_{Le}^2} &= -\frac{1}{4}\Big(1-\frac{n^2\omega_0{}^2}{\omega_{Le}^2}\Big)^2 \qquad (3.31)\\ \frac{\gamma^2}{\omega_{Le}^2} &= 2\varepsilon J_n{}^2 \Big/ \Big(1-\frac{n^2\omega_0{}^2}{\omega_{Le}^2}\Big) \qquad (3.32)\end{aligned}\right\} \text{for } 1 \gg \Big|1-\frac{n^2\omega_0{}^2}{\omega_{Le}^2}\Big|^3 \gg \varepsilon,$$

$$\frac{\gamma^2}{\omega_{Le}{}^2} = \mp\Big[\frac{\varepsilon}{2}J_n{}^2\Big(1-\frac{n^2\omega_0{}^2}{\omega_{Le}{}^2}\Big)\Big]^{1/2} \quad \text{for} \quad \Big|1-\frac{n^2\omega_0{}^2}{\omega_{Le}{}^2}\Big|^3 \ll \varepsilon \ll \Big|1-\frac{n^2\omega_0{}^2}{\omega_{Le}{}^2}\Big|, \qquad (3.33)$$

$$\frac{\gamma^2}{\omega_{Le}{}^2} = \frac{\varepsilon}{2}\Big\{1-D_n{}^0-\frac{J_n{}^2}{2} \pm \Big[\Big(1-D_n{}^0-\frac{J_n{}^2}{2}\Big)^2 - 2J_n{}^2(A_n{}^0-(-1)^n B_n{}^0)\Big]^{1/2}\Big\} \quad \text{for} \quad \Big|1-\frac{n^2\omega_0{}^2}{\omega_{Le}{}^2}\Big| \ll \varepsilon. \qquad (3.34)$$

The highest growth rate is found in the region

$$1 - (n\omega_0 / \omega_{Le})^2 = (\varepsilon J_n{}^2)^{1/3}\zeta.$$

In this case

$$\gamma^2 / \omega_{Le}{}^2 = -{}^1\!/_8 (\varepsilon J_n{}^2)^{2/3}\{\zeta^2 \pm (\zeta^4 + 32\zeta)^{1/2}\}. \qquad (3.35)$$

For the minus sign the maximum growth rate is obtained when $\zeta = 2^{2/3}$. For the plus sign excitation is possible if $-2^{5/3} < \zeta < 0$.

The expressions (3.21), (3.26) and (3.32), which we have obtained for the vicinity of resonances at harmonics of the external frequency, are also suitable (as is evident from the conditions for their application) at remote distances from the resonance point (2.7). This fact indicates the need for analyzing the solutions of (3.18) without assuming that $|1 - (n\omega_0/\omega_{Le})^2|$ is small. Since the case of large γ is actually considered in Sec. 2, the quantity γ can be regarded as small as compared with both the electron-Langmuir frequency and the frequency of the external field. It is evident that in this case (3.18) becomes

$$\left[\frac{1}{4}\frac{\omega_{Le}^2}{n^2\omega_0^2}\left(1-\frac{n^2\omega_0^2}{\omega_{Le}^2}\right)^2+\frac{\gamma^2}{\omega_{Le}^2}\right]\times\left[\frac{\gamma^2}{\omega_{Le}^2}+\varepsilon(1-D_n)\right]-\frac{\varepsilon}{2}\left[\frac{\omega_{Le}^2}{n^2\omega_0^2}-1\right]J_n^2(a)=0,$$

whence

$$\frac{\gamma^2}{\omega_{Le}^2}=-\varepsilon\left\{1-\sum_{l=-\infty}^{+\infty}\frac{\omega_{Le}^2J_l^2(a)}{\omega_{Le}^2-(l\omega_0)^2}\right\}=-\varepsilon\left\{1-\frac{\pi\omega_{Le}/\omega_0}{\sin(\pi\omega_{Le}/\omega_0)}J_{\omega_{Le}/\omega_0}(a)J_{-\omega_{Le}/\omega_0}(a)\right\}. \quad (3.36)$$

In the strong field limit this expression becomes (3.21). When the wavelength of the oscillations is comparable with the electron excursion in the external field (3.36) yields (3.32). Finally, in the weak-field limit (3.26) follows from (3.36).

It is important to note that if the frequency of the external field is appreciably greater than the electron Langmuir frequency (3.36) assumes the form $\gamma^2 = -\omega_{Li}^2[1 - J_0^2(a)]$, corresponding to the spectrum of low-frequency oscillations found earlier for the case $\omega_0 \gg \omega_{Le}$.[2]

The results of this section indicate that far from harmonic resonances the possible growth rate given by (3.36) is of the order of the ion-Langmuir frequency if the wavelength of the plasma oscillations is comparable with the electron excursion in the external field. At long wavelengths, (3.36) yields

$$\gamma^2 = {}^1\!/_2 a^2\omega_{Li}^2\omega_0^2 / (\omega_{Le}^2 - \omega_0^2).$$

In other words, oscillations can be excited if the frequency of the external field is smaller than the electron-Langmuir frequency. In this case the growth rate is of the order of the ion-Langmuir frequency multiplied by the ratio of the electron excursion to the oscillation wavelength.

Finally, in accordance with considerations given above, it is found that a marked increase in the growth rates occurs at resonances on harmonics of the external frequency. In the long-wave (or weak-field) case, (3.9) indicates that the maximum growth rate is of the order of the electron-Langmuir frequency multiplied by the cube root of the electron-ion mass ratio and the ratio of the electron excursion to the oscillation wavelength to the two-thirds power. This same estimate holds for the case in which the electron excursion in the external field is comparable with the oscillation wavelength.

4. KINETIC THEORY OF PARAMETRIC RESONANCE

In describing the parametric resonance effect above we have used a simple plasma model, neglecting the thermal motion of the particles. Below we present the basis for a corresponding kinetic theory; this analysis can be used to introduce the thermal motion of the plasma particles and to delineate the range of applicability of the hydrodynamic analysis.

We use as a basis for the theory of irrotational oscillations of a plasma the kinetic equation with a self-consistent field and consider small deviations from a spatially uniform particle distribution $f_{\alpha 0}$; for a plasma in a uniform high-frequency electric field $\mathbf{E} = \mathbf{E}_0 \sin\omega_0 t$ it is then convenient to write the nonequilibrium correction to the particle distribution function in the form (cf. [2])

$$\delta f_\alpha(\mathbf{p},\mathbf{r},t)=\exp\left\{i\mathbf{kr}+\frac{e_\alpha\mathbf{E}_0\mathbf{k}}{m_\alpha\omega_0^2}\sin\omega_0 t\right\}\times e^{\gamma t}\sum_{l=-\infty}^{+\infty}e^{-il\omega_0 t}\psi_{\alpha,l}\left(\mathbf{p}-\frac{e_\alpha}{\omega_0}\mathbf{E}_0\cos\omega_0 t\right). \quad (4.1)$$

The function $\psi_{\alpha,l}$ is described by the following equations:

$$\{\gamma-i(s\omega_0-\mathbf{kv})\}\psi_{e,s}(\mathbf{p})-i\mathbf{k}\frac{\partial f_{e0}}{\partial\mathbf{p}}\frac{4\pi e^2}{k^2}\int d\mathbf{p}'\left\{\psi_{e,s}(\mathbf{p}')+\frac{e_i}{e}\sum_{r=-\infty}^{+\infty}J_r(a)\psi_{i,s-r}(\mathbf{p}')\right\}=0,$$

$$\{\gamma-i(s\omega_0-\mathbf{kv})\}\psi_{i,s}(\mathbf{p})-i\mathbf{k}\frac{\partial f_{i0}}{\partial\mathbf{p}}\frac{4\pi e_i^2}{k^2}\int d\mathbf{p}'\left\{\psi_{i,s}(\mathbf{p}')+\frac{e}{e_i}\sum_{r=-\infty}^{+\infty}J_r(a)\psi_{e,s+r}(\mathbf{p}')\right\}=0. \quad (4.2)$$

It is evident that the infinite system of integral equations (4.2) allows us to obtain for the functions

$$u_n=e\int d\mathbf{p}\,\psi_{e,n}(\mathbf{p}),\qquad w_n=e_i\int d\mathbf{p}\,\psi_{i,-n}(\mathbf{p}) \quad (4.3)$$

the following system of linear algebraic equations:

$$u_s\{1+\delta\varepsilon_e(s\omega_0+i\gamma,\mathbf{k})\}$$
$$+\sum_{l=-\infty}^{+\infty}J_{l+s}(a)\,w_l\delta\varepsilon_e(s\omega_0+i\gamma,\mathbf{k})=0,$$
$$w_s\{1+\delta\varepsilon_i(-s\omega_0+i\gamma,\mathbf{k})\}$$
$$+\sum_{l=-\infty}^{+\infty}J_{l+s}(a)\,u_l\delta\varepsilon_i(-s\omega_0+i\gamma,\mathbf{k})=0. \quad (4.4)$$

Here,

$$\delta\varepsilon_\alpha(\omega+i\gamma,\mathbf{k})=\frac{4\pi e_\alpha^2}{k^2}\int\frac{d\mathbf{p}}{\omega+i\gamma-\mathbf{kv}}\,\mathbf{k}\frac{\partial f_{\alpha 0}}{\partial\mathbf{p}}, \quad (4.5)$$

while $f_{\alpha 0}(\mathbf{p})$ can be a Maxwellian distribution.

We shall be interested in obtaining results which apply in the vicinity of resonances of harmonics of the external frequency (2.7); we are also interested in a small parameter m_e/m_i so that all the functions u_s and w_s can be expressed approximately in terms of the three functions w_0 and $u_{\pm n}$. The kinetic analogs for the hydrodynamic expressions (3.8) and (3.9) are of the form

$$w_m=-\left[1-\frac{1}{1+\delta\varepsilon_i(-m\omega_0+i\gamma,\mathbf{k})}\right]\Big\{u_nJ_{m+n}(a)$$
$$+u_{-n}J_{m-n}(a)-w_0\sum_{l\neq\pm n}J_{l+m}(a)J_l(a)$$
$$\times\left[1-\frac{1}{1+\delta\varepsilon_e(l\omega_0+i\gamma,\mathbf{k})}\right]\Big\},\quad m\neq 0, \quad (4.6)$$
$$u_m=-\left[1-\frac{1}{1+\delta\varepsilon_e(m\omega_0+i\gamma,\mathbf{k})}\right]\Big\{w_0J_m-\sum_{l\neq 0}J_{l+m}$$
$$\times\left[1-\frac{1}{1+\delta\varepsilon_i(-l\omega_0+i\gamma,\mathbf{k})}\right]\Big[u_nJ_{l+n}+u_{-n}J_{l-n}$$
$$-w_0\sum_{r\neq\pm n}J_rJ_{r+l}\Big(1-\frac{1}{1+\delta\varepsilon_e(r\omega_0+i\gamma,\mathbf{k})}\Big)\Big]\Big\},$$
$$m\neq\pm n. \quad (4.7)$$

It is evident that (4.6) and (4.7) allow us to obtain by means of (4.4) a system of three equations for three functions. The compatibility condition for this system of equations, which is the dispersion relation we see, becomes (3.18) in the limit of zero temperature for the electrons and ions.

It is evident from (4.4) that the hydrodynamic theory of parametric resonance will not apply if the wavelength of the oscillations is comparable with the Debye radius. The hydrodynamic theory is actually limited by another condition. This condition is easily seen, for example, from an analysis of the spectrum of low-frequency oscillations far from harmonic resonances. For this case we obtain the following dispersion equation

$$1+\frac{1}{\delta\varepsilon_i(i\gamma,\mathbf{k})}=\sum_{l=-\infty}^{+\infty}J_l^2(a)\left[1-\frac{1}{1+\delta\varepsilon_e(l\omega_0+i\gamma,\mathbf{k})}\right]. \quad (4.8)$$

It is obvious that when $|\gamma|\gg kv_{Te}$ (v_{Te} is the electron thermal velocity) (4.8) yields the hydrodynamic result (3.36). Hence, we can write (4.8) when $\gamma\sim kv_{Te}$. Then, under the assumption that the Debye radius is small compared with the wavelength of the plasma oscillations, we find

$$\gamma^2=\omega_{Li}^2\Big\{-1+J_0^2(a)\frac{1-J_+(i\gamma/kv_{Te})}{(kr_{De})^2+1-J_+(i\gamma/kv_{Te})}$$
$$+\sum_{l\neq 0}\frac{\omega_{Le}^2J_l^2(a)}{\omega_{Le}^2-l^2\omega_0^2}\Big\}; \quad (4.9)$$
$$J_+(\beta)=\beta e^{-\beta^2/2}\int_{+i\infty}^{\beta}dt\,e^{t^2/2},\qquad r_{De}^2=\frac{\varkappa T_e}{4\pi e^2n_e^0}. \quad (4.10)$$

It is clear that the transition from (4.9) to (3.36) requires that the Debye radius r_{De} be small compared with the electron excursion in the external field.

It is easy to see that this same requirement leads to (3.36) when $|\gamma|\ll kv_{Te}$. Actually, it is precisely this requirement that allows us to satisfy the inequality $|\nu|\gg kv_{Ti}$, which was found to be sufficient to obtain (3.36) in an isothermal plasma. Inasmuch as the growth rates are larger in the vicinity of a resonance of a harmonic of the external frequency then in the nonresonance region, under actual conditions the applicability of the hydrodynamic analysis is generally less stringent. However, in this case as well, if the hydrodynamic theory of parametric resonance is to apply we require that the Debye radius be small compared with the electron excursion. In the case being considered, in which the frequency of the external field is close to the electron-Langmuir frequency, it is convenient to write this condition in the form

$$E_0^2/4\pi>N\varkappa T(\omega_0/\omega_{Le})^4.$$

One of the mechanisms which limits possible exponential growth of the amplitudes is heating of the plasma, which leads to violation of the inequality that has been written and leads to dissipative effects due to thermal motion. In this case, considering the nonlinear effects of the oscillations on the spatially uniform velocity distribution function it is evident that this function will have a rapidly varying part in addition to the slowly varying part

$$f_a(\mathbf{p},t)=\sum_l e^{-il\omega_0t}f_{a,l}(\mathbf{p},t).$$

For example, for the electron function we find

$$\Big(-is\omega_0+\frac{\partial}{\partial t}\Big)f_{e,s}\cong-\frac{\partial}{\partial p_n}D_{nm}{}^e\frac{\partial f_{e,0}}{\partial p_m}$$

$$D_{nm}{}^e = \frac{(2\pi)^3}{V}\int d\mathbf{k}\left(\frac{4\pi e}{k^2}\right)^2 \sum_l \frac{k_m k_n w_0(\mathbf{k}) w_0(-\mathbf{k})}{[\gamma(-\mathbf{k}) - i(l\omega_0 + \mathbf{kv})]} \frac{J_l(a(-\mathbf{k}))J_{s-l}(a(\mathbf{k}))}{[1+\delta\varepsilon_e(l\omega_0 + i\gamma(-\mathbf{k}), -\mathbf{k})][1+\delta\varepsilon_e((s-l)\omega_0 + i\gamma(\mathbf{k}), \mathbf{k})]}$$

The function w_0 includes the slow dependence on time ($\sim e^{\gamma t}$). It then follows that

$$\left(-is\omega_0 + \frac{\partial}{\partial t}\right)\int d\mathbf{p}\frac{p^2}{2m}f_{e,s} \cong -i\int d\mathbf{k} \sum_{l=-\infty}^{+\infty}\frac{[l\omega_0 + i\gamma(-\mathbf{k})]\delta\varepsilon_e(l\omega_0 + i\gamma(-\mathbf{k}), -\mathbf{k}) W_0(\mathbf{k}) J_l(a(-\mathbf{k}))J_{s-l}(a(\mathbf{k}))}{[1+\delta\varepsilon_e(l\omega_0 + i\gamma(-\mathbf{k}), -\mathbf{k})][1+\delta\varepsilon_e((s-l)\omega_0 + i\gamma(\mathbf{k}), \mathbf{k})]},$$

where

$$W_0(\mathbf{k}) = (4\pi / k^2) w_0(\mathbf{k}) w_0(-\mathbf{k}) (2\pi)^3 / V$$

is the spectral density of the wave energy. In the vicinity of a resonance on a harmonic of the external frequency

$$\int d\mathbf{p}\frac{p^2}{2m}f_{e,s} \approx \int d\mathbf{k}\, W_0(\mathbf{k}) \sim \left(\frac{m\omega_0^2}{eE_0}\right)^3 \frac{\varkappa T}{N} e^{2\gamma t}.$$

It is then evident that for times greater than

$$\frac{1}{2\gamma_{max}} \ln\left[\left(\frac{eE_0}{m\omega_0^2}\right)^3 \frac{E_0^2}{4\pi\varkappa T}\right],$$

the hydrodynamic analysis becomes less applicable, as does the linear approximation. The rapidly varying anisotropic electron velocity distribution that arises at this point can lead to the appearance of new instabilities; these can, in turn, lead to further heating of the plasma.

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