NONLINEAR DYNAMICS OF ANTIFERROMAGNETS WITH ANISOTROPY OF THE "EASY PLANE" TYPE

V. I. OZHOGIN

Submitted to JETP editor October 5, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) **48**, 1307-1318 (May, 1965)

A phenomenological quantum-mechanical calculation of various types of nonlinear dynamic phenomena is carried out for antiferromagnets with anisotropy of the "easy plane" type; these phenomena are interesting because of their susceptibility to detailed investigation by modern experimental methods. It is shown, in particular, that the threshold field for occurrence of parallel-pumping instability is inversely proportional to the Dzyaloshinskiĭ field responsible for weak ferromagnetism. Also calculated are effects of a combination type, connected with interaction of two branches of the spin-wave spectrum (low- and high-frequency).

1. INTRODUCTION

THERE is at present a growing interest in the nonlinear dynamics of antiferromagnetic structures, which open up unique possibilities for detailed study of the processes of interaction of antiferromagnetic spin waves both with each other and with other elementary excitations of the crystal. Nonlinear dynamical phenomena are observed, as a rule, at appreciable amplitudes of the alternating magnetic field used to excite oscillations of the system. The frequencies of characteristic oscillations of the magnetic sublattices of typical antiferromagnets lie in the long-wave infrared (IR) region of the spectrum ($\lambda \sim 100$ to $2000\,\mu$), in which there so far exist no sufficiently powerful and convenient sources of radiation. Therefore from the point of view of nonlinear-process investigation, the most interesting antiferromagnets at present are those whose characteristic frequencies of oscillation, for one reason or another, are low enough to permit study of them by the well-developed methods of microwave spectroscopy, including methods that use high power levels.

The simplest of the nonlinear effects—"premature" saturation of antiferromagnetic resonance (AFMR), produced by instability of spin waves degenerate with the uniform precession—was discovered and was calculated quasiclassically by Heeger[1] in $KMnF_3$, in which the AFMR frequency is low because of the small magnitude of the anisotropy field. A similar effect was observed by Borovik-Romanov and Prozorova[2] in a single crystal of $MnCO_3$. The latter substance possesses two resonance frequencies,

$$\omega_{10} = \mu[H_0(H_0 + H_D)]^{1/2}/\hbar,$$
$$\omega_{20} = \mu[H^2_{AE} + H_D\,(H_0 + H_D)]^{1/2}/\hbar$$

(all notation here and below agrees with that used in [3]); the first is essentially determined by the external field H_0 and therefore, at the usual field strengths, lies in the microwave range, whereas the second is close to the submillimeter range ($\mu H_{AE}/2\pi\hbar \sim 4.13$ cm^{-1}[4]). This structure of the AFMR spectrum of $MnCO_3$ and of antiferromagnets similar to it is due to the special form of the anisotropy of these substances; in the absence of a magnetic field, for sublattices aligned antiparallel, there is a whole plane of "easy" directions (it is supposed that the anisotropy within this plane itself is negligibly small).

Besides the availability of the experimental methods of microwave spectroscopy and the relative simplicity of the sublattice motions at the frequency ω_{10}, antiferromagnets with anisotropy of the "easy plane" type are of interest, from the point of view of nonlinear-phenomena study, also because of the presence in them of the high-frequency branch of the oscillations. The fact is that the two branches—the low-frequency and the high-frequency—are coupled with each other by nonlinear terms in the equations of motion of the sublattices. This coupling, naturally, manifests itself only at appreciable amplitudes of oscillation in one of the branches (or in both at once). But the lower branch (ω_{10}) can be sufficiently strongly excited by present methods; therefore microwave-submillimeter experiments of "combination" type can give a significant volume of information about the magnitude of the coupling between oscillations in the different branches.

In the present work, a quantum-mechanical calculation is made of the various nonlinear effects that are possible in antiferromagnets with anisotropy of the "easy plane" type.

2. THE HAMILTONIAN

The Hamiltonian of the antiferromagnets under consideration can be written in the form

$$\mathcal{H} = \int dV \Big\{ \gamma \mathbf{M}_1\mathbf{M}_2 - \mathbf{H}(\mathbf{M}_1 + \mathbf{M}_2) + b_1(M_{1z}^2 + M_{2z}^2) + 2b_2 M_{1z}M_{2z} + 2\beta(M_{1x}M_{2y} - M_{1y}M_{2x}) + \frac{\alpha}{2}\Big(\frac{\partial M_{1i}}{\partial x_k}\Big)^2 + \frac{\alpha}{2}\Big(\frac{\partial M_{2i}}{\partial x_k}\Big)^2 + \alpha_{12}\frac{\partial M_{1i}}{\partial x_k}\frac{\partial M_{2i}}{\partial x_k}\Big\}, \quad (1)$$

where the magnetic field $\mathbf{H}$ is composed of the constant external field $\mathbf{H}_0$ and the self-field $\mathbf{H}_S$ of the spin waves; the latter is described by the equations of magnetostatics,

$$\operatorname{rot} \mathbf{H}_s = 0, \quad \operatorname{div} \mathbf{H}_s = -4\pi \operatorname{div}(\mathbf{M}_1 + \mathbf{M}_2). \quad (1a)^*$$

Anisotropy in the basal plane, as before, is not taken into account.

For the most interesting case ($\mathbf{H}_0$ in the basal plane xy, for example along the x axis), we get the ground state ($T \ll T_N$)

$$M_{1x}^{(0)} = M_{2x}^{(0)} = M_0 \sin\psi, \quad M_{1y}^{(0)} = -M_{2y}^{(0)} = M_0\cos\psi, \quad (2)$$

$$H_E \sin 2\psi - H_D \cos 2\psi - H_0\cos\psi = 0. \quad (2a)$$

If, as is usual, the exchange interaction is large, then

$$\sin\psi \approx \psi = (H_0 + H_D)/2H_E. \quad (3)$$

On carrying out the usual procedure, with use of the auxiliary system of coordinates introduced in [3], we transform the Hamiltonian (1) to the form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}^{(3)} + \mathcal{H}_{int}^{(4)}, \quad (4)$$

where the last two terms describe the interaction of the spin waves, and where the first term is

$$\mathcal{H}_0 = \sum_{\mathbf{k}} \Big\{ \frac{1}{2}A_{1\mathbf{k}}\, a_{1\mathbf{k}}^+ a_{1\mathbf{k}} + \frac{1}{2}A_{2\mathbf{k}}\, a_{2\mathbf{k}}^+ a_{2\mathbf{k}} + B_{\mathbf{k}}\, a_{1\mathbf{k}}^+ a_{2-\mathbf{k}}^+ + C_{\mathbf{k}}\, a_{1\mathbf{k}}^+ a_{2\mathbf{k}} + \frac{1}{2}D_{1\mathbf{k}}\, a_{1\mathbf{k}}^+ a_{1-\mathbf{k}}^+ + \frac{1}{2}D_{2\mathbf{k}}\, a_{2\mathbf{k}}^+ a_{2-\mathbf{k}}^+ + \text{herm. conjugate}\Big\};$$

$A_{1\mathbf{k}} \equiv A + Z^2 + (X-Y)^2$, $A_{2\mathbf{k}} \equiv A + Z^2 + (X+Y)^2$,
$B_{\mathbf{k}} \equiv B + X^2 + (Z+iY)^2$,
$C_{\mathbf{k}} \equiv C + Y^2 + (Z-iX)^2$, $D_{1\mathbf{k}} \equiv D + (Z - iX + iY)^2$,
$D_{2\mathbf{k}} \equiv D + (Z + iX + iY)^2$,

$$A = \mu M_0(\gamma + 2\beta \operatorname{tg}\psi + b_1 + \alpha k^2),$$
$$B = \mu M_0(\gamma\cos^2\psi + \beta\sin 2\psi + b_2 + \alpha_{12}k^2),$$
$$C = \mu M_0(\gamma \sin^2\psi - \beta\sin 2\psi + b_2), \quad D = \mu M_0 b_1,$$
$$Z = (2\pi\mu M_0)^{1/2}\frac{k_z}{k}, \quad X = (2\pi\mu M_0)^{1/2}\frac{k_x\cos\psi}{k},$$
$$Y = (2\pi\mu M_0)^{1/2}\frac{k_y \sin\psi}{k}. \quad (5)^*$$

For the interaction Hamiltonians, which describe processes participated in by three and four spin waves (magnons) respectively, we get

$$\mathcal{H}_{int}^{(3)} = \Big(\frac{\mu^3}{2M_0V}\Big)^{1/2}\sum_{123}\{[u(\mathbf{k}_1) + i\xi(\mathbf{k}_1) + i\eta(\mathbf{k}_1)]\, a_{2\mathbf{k}_1}^+ a_{1\mathbf{k}_2}^+ a_{1\mathbf{k}_3} + [v(\mathbf{k}_1) - i\xi(\mathbf{k}_1) + i\eta(\mathbf{k}_1)]\, a_{1\mathbf{k}_1}^+ a_{2\mathbf{k}_2}^+ a_{2\mathbf{k}_3} + [u(\mathbf{k}_1) - i\zeta(\mathbf{k}_1) - i\eta(\mathbf{k}_1)\cos 2\psi]\, a_{1\mathbf{k}_1}^+ a_{1\mathbf{k}_2}^+ a_{1\mathbf{k}_3} + [v(\mathbf{k}_1) + i\zeta(\mathbf{k}_1) - i\eta(\mathbf{k}_1)\cos 2\psi]\, a_{2\mathbf{k}_1}^+ a_{2\mathbf{k}_2}^+ a_{2\mathbf{k}_3}\}\,\Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) + \text{h.c.}, \quad (6)$$

where

$$u(\mathbf{k}) \equiv 4\pi M_0 k_z(k_x\sin\psi + k_y\cos\psi)/k^2,$$
$$v(\mathbf{k}) \equiv 4\pi M_0 k_z(k_x \sin\psi - k_y\cos\psi)/k^2,$$
$$\xi(\mathbf{k}) \equiv H_0\cos\psi + M_0\alpha_{12}k^2\sin 2\psi + 2\pi M_0 \sin 2\psi\cdot(k_x^2 + k_y^2)/k^2,$$
$$\eta(\mathbf{k}) \equiv 4\pi M_0 k_x k_y/k^2, \quad \zeta(\mathbf{k}) \equiv 2\pi M_0\sin 2\psi\cdot(k_x^2 - k_y^2)/k^2;$$

$$\mathcal{H}_{int}^{(4)} = -\frac{\mu}{4M_0V}\sum_{1234}\{[B_{\mathbf{k}_1}(a_{1\mathbf{k}_1}^+ a_{2\mathbf{k}_2}^+ a_{2\mathbf{k}_3}^+ a_{2\mathbf{k}_4} + a_{2\mathbf{k}_1}^+ a_{1\mathbf{k}_2}^+ a_{1\mathbf{k}_3}^+ a_{1\mathbf{k}_4}) + D_{1\mathbf{k}_1} a_{1\mathbf{k}_1}^+ a_{1\mathbf{k}_2}^+ a_{1\mathbf{k}_3}^+ a_{1\mathbf{k}_4} + D_{2\mathbf{k}_1} a_{2\mathbf{k}_1}^+ a_{2\mathbf{k}_2}^+ a_{2\mathbf{k}_3}^+ a_{2\mathbf{k}_4}] \times \Delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4)$$
$$+ [C_{\mathbf{k}_1}(a_{1\mathbf{k}_1}^+ a_{2\mathbf{k}_2}^+ a_{2\mathbf{k}_3} a_{2\mathbf{k}_4} + a_{2\mathbf{k}_1} a_{1\mathbf{k}_2} a_{1\mathbf{k}_3}^+ a_{1\mathbf{k}_4}^+) + (A_{1\mathbf{k}_1} - \mu H_E - \mu H_D \operatorname{tg}\psi)\, a_{1\mathbf{k}_1}^+ a_{1\mathbf{k}_2}^+ a_{1\mathbf{k}_3} a_{1\mathbf{k}_4} + (A_{2\mathbf{k}_1} - \mu H_E - \mu H_D\operatorname{tg}\psi)\, a_{2\mathbf{k}_1}^+ a_{2\mathbf{k}_2}^+ a_{2\mathbf{k}_3} a_{2\mathbf{k}_4}$$
$$+ 2(\mu H_E\cos 2\psi + \mu H_D \sin 2\psi + r(\mathbf{k}_1 - \mathbf{k}_3))\, a_{1\mathbf{k}_1}^+ a_{2\mathbf{k}_2}^+ a_{1\mathbf{k}_3} a_{2\mathbf{k}_4} + p(\mathbf{k}_1 - \mathbf{k}_3)\, a_{1\mathbf{k}_1}^+ a_{1\mathbf{k}_2}^+ a_{1\mathbf{k}_3} a_{1\mathbf{k}_4} + q(\mathbf{k}_1 - \mathbf{k}_3) \times a_{2\mathbf{k}_1}^+ a_{2\mathbf{k}_2}^+ a_{2\mathbf{k}_3} a_{2\mathbf{k}_4}]\,\Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\} + \text{h.c.} \quad (7)$$

where

$$r(\mathbf{k}) \equiv \mu M_0\alpha_{12}k^2\cos 2\psi - 4\pi\mu M_0(k_x^2\sin^2\psi - k_y^2\cos^2\psi)/k^2,$$
$$p(\mathbf{k}) \equiv -\mu M_0\alpha k^2 - 4\pi\mu M_0(k_x\sin\psi + k_y\cos\psi)^2/k^2,$$
$$q(\mathbf{k}) \equiv -\mu M_0 \alpha k^2 - 4\pi\mu M_0(k_x \sin\psi - k_y\cos\psi)^2/k^2.$$

*rot ≡ curl.

*tg ≡ tan.

To diagonalize the Hamiltonian $\mathcal{H}_0$, we transform from the operators a^+_{sk}, a_{sk} to the operators c^+_{sk}, c_{sk} (s = 1, 2):

$$a_{s\mathbf{k}} = \sum_{r=1}^{2} (U_{sr} c_{r\mathbf{k}} e^{-i\varepsilon_r t/\hbar} + V^+_{sr} c^+_{r-\mathbf{k}} e^{i\varepsilon_r t/\hbar}) \tag{8}$$

Then, by using in the usual way the equations of motion of the operators a^+_{sk}, a_{sk} [5], we get the two branches of the oscillations:

$$\varepsilon_{1,2}^2 = A^2 + C^2 - B^2 - D^2 + 2(\alpha_0 Z^2 + \beta_0 X^2 + \gamma_0 Y^2)$$
$$\pm 2\{[(AC - BD) + (\alpha_0 Z^2 - \beta_0 X^2 + \gamma_0 Y^2)]^2$$
$$+ 4\alpha_0\beta_0 X^2 Z^2 + 4\gamma_0\beta_0 X^2 Y^2\}^{1/2}, \tag{9}$$

where for conciseness of notation we have introduced the abbreviations

$$\alpha_0 = A + C - B - D,\ \beta_0 = A - C - B + D,$$
$$\gamma_0 = A + B + C + D.$$

For $2\pi/\gamma \ll 1$, it is possible to obtain in a more convenient form the spectrum of the spin waves of an antiferromagnet with anisotropy of the "easy plane" type, with the field of the spin waves taken into account:

$$\varepsilon_{1\mathbf{k}}^2 = [\mu^2 H_0 (H_0 + H_D) + \Theta_c^2 (ak)^2] \Big[1 + \frac{4\pi}{\gamma} \frac{k_z^2}{k^2}$$
$$+ \frac{4\pi}{\gamma} \frac{\mu^2 (H_0 + H_D)^2}{\mu^2 H_0 (H_0 + H_D) + \Theta_c^2 (ak)^2} \frac{k_y^2}{k^2} \Big], \tag{9a}$$

$$\varepsilon_{2\mathbf{k}}^2 = [\mu^2 H_{AE}^2 + \mu^2 H_D (H_0 + H_D) + \Theta_c^2 (ak)^2]$$
$$\times \Big(1 + \frac{4\pi}{\gamma} \frac{k_x^2}{k^2}\Big). \tag{9b}$$

For the transformation coefficients U_{sr}, V_{sr} with the field of the spin waves taken into account, cumbersome expressions are obtained. These differ from those calculated earlier, [3] however, only by factors of the order of $[1 + 2\pi\Omega(\mathbf{k})/\gamma]$, where $|\Omega(\mathbf{k})| \sim 1$. Therefore below, as a rule, the values of U_{sr} and V_{sr} used will be those calculated without taking account of the field of the spin waves:

$$U_{11} = U_{21} = [(A + C + \varepsilon_{1\mathbf{k}})/4\varepsilon_{1\mathbf{k}}]^{1/2} \equiv U_{1\mathbf{k}},$$
$$U_{12} = -U_{22} = [(A - C + \varepsilon_{2\mathbf{k}})/4\varepsilon_{2\mathbf{k}}]^{1/2} \equiv U_{2\mathbf{k}},$$
$$V_{11} = V_{21} = -[(A + C - \varepsilon_{1\mathbf{k}})/4\varepsilon_{1\mathbf{k}}]^{1/2} \equiv V_{1\mathbf{k}},$$
$$V_{12} = -V_{22} = [(A - C - \varepsilon_{2\mathbf{k}})/4\varepsilon_{2\mathbf{k}}]^{1/2} \equiv V_{2\mathbf{k}}. \tag{10}$$

3. THRESHOLD SATURATION OF LOW- AND HIGH-FREQUENCY AFMR

Threshold saturation of low-frequency AFMR, in the antiferromagnets considered, occurs because of parametric excitation of short spin waves of the lower branch, which interact with a spin wave of the same branch but with $\mathbf{k} = 0$ (that is, with the low-frequency uniform precession of the magnetic moments of the sublattices). A quantum-mechanical calculation of the amplitude of the alternating field at which saturation begins consists, according to White and Sparks [6], in a determination of the number of quanta of the uniform precession (for instability in the case of "parallel pumping," the number of photons) at which only an infinite number of parametrically excited spin waves can maintain a stationary distribution of populations of the spin levels of the system.

First of all it must be mentioned that in an antiferromagnet with anisotropy of the "easy plane" type, just as in $KMnF_3$ [1], three-magnon processes have no great significance in the consideration of instabilities of the lower branch[1]: because of the small influence of the field of the spin waves on the spectrum of the system [cf. (9a), (9b)], the requirement of the law of conservation of energy in collisions of magnons gives threshold fields dependent on processes of odd order within a single branch that are extremely large and, as a rule, practically unattainable.

The instability under investigation is caused by a four-magnon process, in which two quanta of the uniform precession are annihilated and a magnon pair (two spin waves with oppositely directed nonzero momenta) is created. To calculate the corresponding threshold field, we express $\mathcal{H}^{(4)}_{\text{int}}$ in terms of c^+_{sk} and c_{sk} and, keeping only the terms that describe this process, get

$$\mathcal{H}^{(4)} = -\sum \Phi_{00\mathbf{kk}}\, c_{10} c_{10} c_{1\mathbf{k}}^+ c_{1-\mathbf{k}}^+$$
$$\times \exp\Big[-\frac{i}{\hbar}(\varepsilon_{10} + \varepsilon_{10} - \varepsilon_{1\mathbf{k}} - \varepsilon_{1-\mathbf{k}})t\Big] + \text{h.c.}, \tag{11}$$

$$\Phi_{00\mathbf{kk}} \approx \frac{\mu^2\gamma}{16V}[2U_{10}^2 U_{1\mathbf{k}}^2 + 2V_{10}^2 V_{1\mathbf{k}}^2 + 8U_{10} V_{10} U_{1\mathbf{k}} V_{1\mathbf{k}}$$
$$+ 3(U_{10}^2 + V_{10}^2) U_{1\mathbf{k}} V_{1\mathbf{k}} + 3U_{10} V_{10} (U_{1\mathbf{k}}^2 + V_{1\mathbf{k}}^2)]$$
$$\approx \frac{\mu^2\gamma}{16V}\Big(2 - \frac{\varepsilon_{1\mathbf{k}}^2 + \varepsilon_{10}^2}{2\varepsilon_{1\mathbf{k}}\varepsilon_{10}}\Big).$$

The expression for Φ_{00kk} has been obtained to within terms proportional either to k^2 or to $\sin^2 \psi$; the first may be disregarded, since in this process only magnons with k = 0 participate, and the second are small when $\psi \ll 1$.

We furthermore construct, with the aid of (11), an expression for the probabilities of transitions

[1]If we disregard very small external fields, of order $2\pi H_D/\gamma$; these, however, are without experimental interest.

that increase and decrease the number n_{10} of quanta of the uniform precession by two; thus we can find an equation for the time rates of change of the occupation numbers n_{10}, n_{1k}, and n_{1-k} (cf. [6]):

$$\dot{n}_{10} = \frac{2\pi}{\hbar}\Delta n_{10}|\Phi_{00kk}|^2\frac{1}{\pi\hbar 2\eta_{1k}}[4(n_{10}+1)(n_{10}+2)n_{1k}n_{1-k} - 4n_{10}(n_{10}-1)(n_{1k}+1)(n_{1-k}+1)],$$

$$\dot{n}_{1k} = -\frac{1}{2}\dot{n}_{10} - 2\eta_{1k}(n_{1k}-\bar{n}_{1k}),$$

$$\dot{n}_{1-k} = -\frac{1}{2}\dot{n}_{10} - 2\eta_{1k}(n_{1-k}-\bar{n}_{1-k}). \qquad (12)$$

Here $\Delta n_{10} = 2$, since in every collision the number of quanta of the uniform precession changes by two; $(\pi\hbar 2\eta_{1k})^{-1}$ is the density of the final state expressed in terms of the relaxation frequency η_{ik} of a magnon $(\epsilon_{ik}, \mathbf{k})$ (as in [6], it is assumed that $\eta_{1k} = \eta_{1-k}$); $\bar{n}_{1k}$ and $\bar{n}_{1-k}$ are the equilibrium values of the occupation numbers of the corresponding magnons $(\bar{n}_{1k} = \bar{n}_{1-k})$; furthermore, because of the requirement of the law of conservation of energy $(\epsilon_{1k} = \epsilon_{10})$, $\Phi_{00kk} \approx \mu^2\gamma/16V$.

From the condition of stationarity $(\dot{n}_{1k} = \dot{n}_{1-k} = 0)$, and by taking into account that until the actual moment of onset of instability $1 \ll n_{1k}, n_{1-k} \ll n_{10}$, we get

$$n_{1k} = \bar{n}_{1k}n_{1c}^2/(n_{1c}^2 - n_{10}^2), \qquad (13)$$

where $n_{1c} \equiv \hbar\eta_{1k}/2|\Phi_{00kk}|$. Thus instability occurs when $n_{10} = n_{1c}$.

To find the relation between the number of quanta of the uniform precession and the amplitude of the magnetic field, we notice that from the classical equations of motion for the uniform precession without damping[7] it follows that $(\mathbf{m} \equiv \mathbf{M}_1 + \mathbf{M}_2)$

$$G(t) \equiv (H_0 m_y^2 + (H_0 + H_D)m_z^2)V^2 = \text{const.} \qquad (14)$$

The value of the constant is determined by the "initial" values of the components of $\mathbf{m}$; for these, it is convenient to take the stationary m_y and m_z, calculated with allowance for damping and for the alternating magnetic field that sustains the oscillations. If the alternating field is $\mathbf{h} = (0, 0, h_z e^{-i\omega_{10}t})$, then the corresponding constant is

$$G^{(z)}(0) = H_0(\gamma_m h_{z0} m_0/2\eta_{10})^2 V^2, \qquad (14a)$$

where γ_m is the magnetomechanical ratio, $m_0 \equiv M_{1x}^{(0)} + M_{2x}^{(0)}$, and η_{10} is the relaxation frequency of the uniform precession (connected by the relation $\eta_{10} = \gamma_m\Delta H_{10}/2$ with the AFMR line width ΔH_{10}).

On the other hand, by regarding the combination (14) as an operator and by using the formulas for transformation from the operators $\hat{\mathbf{M}}_S$ to the operators a_S^+, a_S (cf. [3]) together with the equalities (8), we get

$$\hat{G} = 4\mu M_0 V[(H_0 + H_D)(U_{10} + V_{10})^2 + H_0 \sin^2\psi(U_{10} - V_{10})^2](c_{10}^+ c_{10} + 1/2).$$

From this and from (10) it follows that the characteristic values of the operator $\hat{G}$ are

$$G = 2M_0 V\varepsilon_{10}(n_{10} + 1/2)(H_0 + H_D)/H_E. \qquad (14b)$$

Comparison of (14a) with (14b) gives the relation between h_z and n_{10}; this, after substitution into (13), leads to the desired expression for the critical amplitude of the alternating magnetic field:

$$h_{zc} = 2\Delta H_{10}(2\gamma_m\Delta H_{1k}/\omega_{10})^{1/2}, \qquad (15)$$

where

$$\eta_{1k} \equiv \gamma_m\Delta H_{1k}/2, \qquad \omega_{1k} \equiv \varepsilon_{1k}/\hbar.$$

This formula agrees with that derived by Heeger[1] for $KMnF_3$ (in comparing them, it is necessary to take account of the facts that in [1] ΔH denotes not the complete line but its half-width, and that the calculation was made for a circularly polarized alternating field).

If the low-frequency AFMR is excited by a field with y-polarization, then

$$G^{(y)}(0) = (H_0 + H_D)(\gamma_m m_0 h_{y0}/2\eta_{10})^2 V^2$$

and therefore

$$h_{yc} = h_{zc}[H_0/(H_0 + H_D)]^{1/2}. \qquad (16)$$

At a field amplitude above the threshold value, there is established some new stationary state of populations of the spin levels of the system. A calculation of it, however, would be very complicated and must take account of the converse influence of the excited spin waves on the uniform precession[8].

Threshold saturation of high-frequency AFMR, in contrast to low-frequency, can arise also from a process in which only three magnons participate: the law of conservation of energy does not prohibit a quantum of the uniform precession, of frequency ω_{20}, from splitting into two spin waves of the lower branch, with frequencies $\omega_{1\kappa} = \omega_{1-\kappa} = \omega_{20}/2$ and wave vectors κ and $-\kappa$, where

$$|\varkappa| = (\varepsilon_{20}^2 - 4\varepsilon_{10}^2)^{1/2}/2\Theta_c a.$$

Processes of this nature, as has already been remarked[3,9], are a peculiar characteristic of antiferromagnets with anisotropy of the "easy plane" type.

To calculate the threshold field due to this process, we write that part of the Hamiltonian (6), transformed by means of (8), which corresponds to annihilation or creation of a quantum of the high-frequency uniform precession:

$$\mathcal{H}_{int}^{(3)} = \sum \Phi_{\mathbf{kk}0} c_{1-\mathbf{k}}^{+} c_{1\mathbf{k}}^{+} c_{20} \exp\left[\frac{i}{\hbar}(\varepsilon_{1\mathbf{k}} + \varepsilon_{1-\mathbf{k}} - \varepsilon_{20})t\right] + \text{h.c.}; \quad (17)$$

here $\Phi_{\mathbf{kk}0}$ denotes the following expression:

$$\Phi_{\mathbf{kk}0} = (2\mu^3/M_0V)^{1/2}\{4\pi M_0 \cos\psi(U_{1\mathbf{k}} + V_{1\mathbf{k}}) \times (U_{1\mathbf{k}} U_{20} + V_{1\mathbf{k}} V_{20}) \times 2k_z k_y/k^2 + i[H_0(U_{1\mathbf{k}}^2 U_{20} - V_{1\mathbf{k}}^2 V_{20}) + (M_0\alpha_{12}k^2 + 4\pi M_0 k_y^2/k^2)\sin 2\psi (U_{1\mathbf{k}} - V_{1\mathbf{k}})(U_{1\mathbf{k}} U_{20} + V_{1\mathbf{k}} V_{20}) + 4\pi M_0 \sin 2\psi\, U_{1\mathbf{k}} V_{1\mathbf{k}} (U_{20} - V_{20}) \lim_{\mathbf{k}_3\to 0} (k_{3x}^2/k_3^2)]\}. \quad (17a)$$

When account is taken of the requirement of the law of conservation of energy ($\epsilon_{20} = 2\epsilon_{1K}$), expressed by the presence in (17) of the time exponent, the expressions for the occupation numbers take the form

$$\dot{n}_{20} = (4|\Phi_{\varkappa\varkappa 0}|^2/\hbar\eta_{1\varkappa})[(n_{20}+1)n_{1\varkappa}n_{1-\varkappa} - n_{20}(n_{1\varkappa}+1) \times (n_{1-\varkappa}+1)],$$

$$\dot{n}_{1\varkappa} = -\dot{n}_{20} - 2\eta_{1\varkappa}(n_{1\varkappa} - \bar{n}_{1\varkappa}),$$

$$\dot{n}_{1-\varkappa} = -\dot{n}_{20} - 2\eta_{1\varkappa}(n_{1-\varkappa} - \bar{n}_{1-\varkappa}). \quad (18)$$

Consequently, instability occurs when

$$n_{20} = n_{2c} \equiv \hbar^2\eta_{1\varkappa}^2 / 4|\Phi_{\varkappa\varkappa 0}|^2. \quad (19)$$

To find an explicit form for Φ_{KK0}, we use formula (10), since introduction of the values of the coefficients U_{sk} and V_{sk} calculated with allowance for the field of the spin waves gives a correction of only small relative size (of order $2\pi\mu M_0/\epsilon_{20}$); furthermore, we are interested in the field range $2\pi H_D/\gamma \ll H_0$, therefore we neglect the terms with $\sin 2\psi$ in (17a); we get

$$\Phi_{\varkappa\varkappa 0} \approx \mu^2(\gamma/2\varepsilon_{20}V)^{1/2}(iH_0 + 3\pi M_0\varepsilon_{20}\varkappa_z\varkappa_y/\mu H_E\varkappa^2). \quad (17b)$$

If we furthermore use the equality

$$2H_E(m_x - m_0)^2 + [H_A + H_D\ (H_0 + H_D)/2H_E]l_z^2 \approx \text{const}, \quad (20)$$

which follows from the classical equations of motion for the uniform precession ($\mathbf{l} \equiv \mathbf{M}_1 - \mathbf{M}_2$), we find

$$h_{x0}^2 = 8\hbar^2\eta_{20}H_E n_{20}/\varepsilon_{20}VM_0, \quad (21)$$

and then also the expression for the threshold amplitude (due to the three-magnon process),

$$h_{xc}(\varepsilon_{20} \to \varepsilon_{1\varkappa} + \varepsilon_{1-\varkappa}) = \frac{1}{2}\Delta H_{20} \times \min\left\{\Delta H_{1\varkappa}\left[H_0^2 + 18\pi^2 M_0^2 \frac{H_A}{H_E}\left(\frac{\varkappa_z\varkappa_y}{\varkappa^2}\right)^2\right]^{-1/2}\right\}, \quad (22)$$

which, naturally, is applicable only for those H_0's for which the requirement $\epsilon_{20} = 2\epsilon_{1k}$ can be satisfied.

Four-magnon instability of the high-frequency AFMR can arise from processes to which correspond terms in $\mathcal{H}_{int}^{(4)}$ that contain

$$c_{20}c_{20}c_{2\mathbf{k}}^{+}c_{2-\mathbf{k}}^{+}, \quad c_{20}c_{20}c_{1\mathbf{q}}^{+}c_{1-\mathbf{q}}^{+}, \quad c_{20}c_{1\mathbf{k}_1}^{+}c_{1\mathbf{k}_2}^{+}c_{1\mathbf{k}_3}^{+}$$

(and their conjugates). The coefficient of the third combination of operators vanishes by Eq. (10); to the first combination, as is easily shown, corresponds a threshold field

$$h_{xc}(\varepsilon_{20} + \varepsilon_{20} \to \varepsilon_{2\mathbf{k}} + \varepsilon_{2-\mathbf{k}}) = 2\Delta H_{20}(2\gamma_m\Delta H_{2\mathbf{k}}/\omega_{20})^{1/2}, \quad (23a)$$

and to the second a field

$$h_{xc}(\varepsilon_{20} + \varepsilon_{20} \to \varepsilon_{1\mathbf{q}} + \varepsilon_{1-\mathbf{q}}) = \Delta H_{20}(2\gamma_m\Delta H_{1\mathbf{q}}/\omega_{20})^{1/2}, \quad (23b)$$

where, of course,

$$|\mathbf{q}| = \mu(H_{AE}^2 + H_D^2 - H_0^2)^{1/2}/\Theta_c a.$$

Comparison of the amplitudes (22) and (23a, b) shows that for typical values of ΔH_{20} and ΔH_{sk} and in the fields usually applied, the former is much smaller than the latter.

4. MAGNON-PHOTON INSTABILITY

Magnon-photon instability is a threshold excitation of a magnon pair with destruction of one or more photons. According to White and Sparks [6], it is an adequate description of parametric excitation of spin waves by an alternating magnetic field without aid of a uniform precession (for example, in ferrites, "parallel pumping").

To calculate similar effects in antiferromagnets with anisotropy of the "easy plane" type, we write that part of the addition to the Hamiltonian (1), due to the external alternating field $\mathbf{h}(h_x, h_y, h_z)$, which is quadratic in the operators a_{sk}^{+} and a_{sk}:

$$\delta\mathcal{H} = \mu H_D \delta\psi \sum_{\mathbf{k}} (a_{1\mathbf{k}}^{+}a_{1\mathbf{k}} + a_{2\mathbf{k}}^{+}a_{2\mathbf{k}}) + \mu h_y \sum_{\mathbf{k}} (a_{1\mathbf{k}}^{+}a_{1\mathbf{k}} - a_{2\mathbf{k}}^{+}a_{2\mathbf{k}}); \quad (24)$$

here, by virtue of (3), $\delta\psi = h_x/2H_E$. Hence it is clear that from the standpoint of the phenomena under study, two polarizations of the alternating magnetic field are of interest: parallel to the equilibrium ferromagnetic vector $\mathbf{m}^{(0)}$, and parallel to the equilibrium antiferromagnetic vector $\mathbf{l}^{(0)}$.

We consider the case $\mathbf{h} = (h_0 e^{i\omega_\nu t}, 0, 0)$. Following (6), we express this field in terms of creation and annihilation operators of photons (A_ν is a normalization factor):

$$h_x = iA_\nu(2\pi\hbar\omega_\nu)^{1/2}(c_\nu e^{-i\omega_\nu t} - c_\nu^{+} e^{i\omega_\nu t}). \quad (25)$$

Then, using (8) and (10), we transform the first term in (24) to the form

$$\delta\mathcal{H}^{(x)} = A_\nu(2\pi\hbar\omega_\nu)^{1/2}\sum_{\mathbf{k}}\Big\{P_{11}\Big(ic_\nu c_{1\mathbf{k}}^+ c_{1-\mathbf{k}}^+ \times \exp\Big[\frac{i}{\hbar}(\varepsilon_{1\mathbf{k}}+\varepsilon_{1-\mathbf{k}}-\hbar\omega_\nu)t\Big]+\text{h.c.}\Big)+P_{22}\Big(ic_\nu c_{2\mathbf{k}}^+ c_{2-\mathbf{k}}^+ \times\exp\Big[\frac{i}{\hbar}(\varepsilon_{2\mathbf{k}}+\varepsilon_{2-\mathbf{k}}-\hbar\omega_\nu)t\Big]+\text{h.c.}\Big)\Big\};$$

$$P_{11} = \mu H_D(U_{11}^*V_{11}^* + U_{21}^*V_{21}^*)/2H_E \approx \mu H_D/4\varepsilon_{1\mathbf{k}},$$
$$P_{22} = \mu H_D(U_{12}^*V_{12}^* + U_{22}^*V_{22}^*)/2H_E \approx \mu H_D/4\varepsilon_{2\mathbf{k}}, \quad (26)$$

here terms containing the operators $c_\nu^+ c_{s\mathbf{k}}^+ c_{s\mathbf{k}}$, $c_\nu c_{s\mathbf{k}} s_{s-\mathbf{k}}$ and their conjugates have been omitted as unimportant.

The expression obtained, which agrees with (17) as regards its structure, enables us to draw the conclusion that the energy of an alternating magnetic field parallel to the constant field can be intensively absorbed by the specimen if its frequency is equal to $2\omega_{1k}$ or $2\omega_{2k}$, and if its amplitude exceeds a certain threshold value, which is smaller, the larger the Dzyaloshinskiĭ interaction that determines the weak ferromagnetic moment of an antiferromagnet.

A procedure similar to that used in the determination of the three-magnon instability threshold leads to the following values of the threshold amplitudes (the relation between the amplitude h_0 and the number of photons n_ν is given by the formula $h_0^2 = 8\pi\hbar\omega_\nu A_\nu^2 n_\nu$):

$$h_{0s}^{\text{cr}} = \min\{\Delta H_{s\mathbf{k}}\cdot 2\varepsilon_{s\mathbf{k}}/\mu H_D\}, \quad (27)$$

where s is equal to 1 or to 2, according as "parallel pumping" is carried out at a frequency ω_ν equal to $2\omega_{1k}$ or to $2\omega_{2k}$. A calculation on the basis of the classical equations of motion gives the same result.

The presence of the Dzyaloshinskiĭ field in the denominator of (27) leads us to expect a very small value of the threshold field in antiferromagnets in which H_D is large (for example, in $CoCO_3$, with $H_D \sim 25$ kOe [10,11]). Furthermore, the effect differs basically from threshold absorption of a parallel field in a ferromagnet [12]: the occurrence of magnon-photon instability with excitation of uniform precession, in an antiferromagnet with anisotropy of the "easy plane" type, is due to internal (anisotropy) interactions, and not to a special shape of the specimen.

5. COMBINATION EFFECTS

Combination effects include phenomena caused by an alternating magnetic field of frequency $\omega_{2k} \pm \omega_{1k}$. Analysis of expression (24), transformed with the aid of (8) and (10), shows that the coefficients of the corresponding combinations of operators in the perturbation Hamiltonian differ from zero only in the presence of a y component of the alternating field. Considering, therefore, the case $h = (0, h_0 e^{i\omega_\nu t}, 0)$ by the method just described, we get (unimportant terms have again been omitted)

$$\delta\mathcal{H}^{(y)} = iA_\nu(2\pi\hbar\omega_\nu)^{1/2}\sum_{\mathbf{k}}\Big\{P_{12}c_\nu c_{1\mathbf{k}} c_{2\mathbf{k}}^+ \times\exp\Big[\frac{i}{\hbar}(\varepsilon_{2\mathbf{k}}-\varepsilon_{1\mathbf{k}}-\hbar\omega_\nu)t\Big] + Q_{12}c_\nu c_{1\mathbf{k}}^+ c_{2-\mathbf{k}}^+\exp\Big[\frac{i}{\hbar}(\varepsilon_{1\mathbf{k}}+\varepsilon_{2-\mathbf{k}}-\hbar\omega_\nu)t\Big]\Big\}+\text{h.c.};$$

$$P_{12} = (U_{12}^*U_{11} + V_{12}^*V_{11} - U_{22}^*U_{21} - V_{22}^*V_{21}) \approx \mu(\varepsilon_{1\mathbf{k}}+\varepsilon_{2\mathbf{k}})/2(\varepsilon_{1\mathbf{k}}\varepsilon_{2\mathbf{k}})^{1/2},$$
$$Q_{12} = (U_{11}^*V_{12}^* + U_{12}^*V_{11}^* - U_{21}^*V_{22}^* - U_{22}^*V_{21}^*) \approx \mu(\varepsilon_{1\mathbf{k}}-\varepsilon_{2\mathbf{k}})/2(\varepsilon_{1\mathbf{k}}\varepsilon_{2\mathbf{k}})^{1/2}. \quad (28)$$

The term with coefficient Q_{12} obviously describes a parametric sum excitation: a photon of frequency $\omega_\nu = (\epsilon_{1k} + \epsilon_{2-k})/\hbar$ decomposes into a pair, consisting of magnons of the different branches. In constructing an expression for the probability of this process, we take into account that the final state is none other than these two magnons; therefore for the density of the final states we take the expression used by White and Sparks [6] and by us (cf. above),

$$\rho(E) = \eta_f/\pi\hbar(\eta_f^2 + (\omega_f - \omega_i)^2];$$
$$\eta_f = \eta_{1\mathbf{k}} + \eta_{2-\mathbf{k}}, \quad \omega_f = \omega_i = \omega_{1\mathbf{k}} + \omega_{2-\mathbf{k}}.$$

For the occupation numbers we get the following equations:

$$\dot{n}_\nu = \frac{4\pi\omega_\nu Q_{12}^2}{\hbar(\eta_{1\mathbf{k}}+\eta_{2\mathbf{k}})}[(n_\nu+1)n_{1\mathbf{k}}n_{2\mathbf{k}} - n_\nu(n_{1\mathbf{k}}+1)(n_{2\mathbf{k}}+1)],$$
$$\dot{n}_{1\mathbf{k}} = -\dot{n}_\nu - 2\eta_{1\mathbf{k}}(n_{1\mathbf{k}} - \bar{n}_{1\mathbf{k}}),$$
$$\dot{n}_{2\mathbf{k}} = -\dot{n}_\nu - 2\eta_{2\mathbf{k}}(n_{2\mathbf{k}} - \bar{n}_{2\mathbf{k}}), \quad (29)$$

from these, the requirement of stationarity ($\dot{n}_{1k} = \dot{n}_{2k} = 0$) allows us to obtain the threshold amplitude of the field of frequency $\omega_\nu = \omega_{1k} + \omega_{2k}$:

$$h_{\nu 0}^{\text{cr}} = \min\{2\hbar(\eta_{1\mathbf{k}}\eta_{2\mathbf{k}})^{1/2}/|Q_{12}|\} \approx \min\{2(\Delta H_{2\mathbf{k}}\Delta H_{1\mathbf{k}})^{1/2}(\varepsilon_{2\mathbf{k}}\varepsilon_{1\mathbf{k}})^{1/2}/(\varepsilon_{2\mathbf{k}}-\varepsilon_{1\mathbf{k}})\}. \quad (30)$$

Beginning with this value of the amplitude, the amount of energy absorbed by the specimen increases abruptly.

The influence on the antiferromagnet of an alternating magnetic field h_y of the difference frequency is taken into account by the term in (28) with the combination of operators $c_\nu c_{1k} c_{2k}^+$, which describes a process of simultaneous annihilation of a photon and of a low-frequency magnon and creation of a high-frequency magnon. The equations for the rate of change of the occupation numbers, determined by this process and the process inverse to it, have the form [2)]

$$\dot{n}_\nu = \frac{f_{12}^2}{2(\eta_{1k}+\eta_{2k})}\left[n_\nu (n_{2k}-n_{1k}) + n_{2k}(n_{1k}+1)\right],$$

$$\dot{n}_{1k} = \dot{n}_\nu - 2\eta_{1k}(n_{1k}-\bar{n}_{1k}), \quad \dot{n}_{2k} = -\dot{n}_\nu - 2\eta_{2k}(n_{2k}-\bar{n}_{2k}), \tag{31}$$

where

$$f_{12} \equiv 2A_\nu (2\pi\hbar\omega_\nu)^{1/2} P_{12}/\hbar.$$

From the condition of stationarity ($\dot{n}_{1k} = \dot{n}_{2k} = 0$), we find

$$n_{1k} = \bar{n}_{1k} - \frac{f_{12}^2 n_\nu (\bar{n}_{1k}-\bar{n}_{2k})}{(1+\eta_{1k}/\eta_{2k})(4\eta_{1k}\eta_{2k}+f_{12}^2 n_\nu)}$$

$$n_{2k} = \bar{n}_{2k} + \frac{f_{12}^2 n_\nu (\bar{n}_{1k}-\bar{n}_{2k})}{(1+\eta_{2k}/\eta_{1k})(4\eta_{1k}\eta_{2k}+f_{12}^2 n_\nu)}. \tag{32}$$

Hence, in particular, it is clear that the values of n_{1k} and n_{2k} do not become infinite for any positive value of n_ν. This indicates that an alternating magnetic field of the difference frequency cannot excite parametric resonance. The equalities (32), however, show that the presence of difference pumping causes a redistribution of the populations of the spin waves (with those **k**'s for which $\omega_\nu = \omega_{2k} - \omega_{1k}$) and is accompanied by absorption of power:

$$Q = (\varepsilon_{2k}-\varepsilon_{1k})\dot{n}_\nu$$

$$= -\frac{h_0^2 P_{12}^2 (\varepsilon_{2k}-\varepsilon_{1k})(\bar{n}_{1k}-\bar{n}_{2k})}{2\hbar^2[\eta_{1k}+\eta_{2k}+h_0^2 P_{12}^2(\eta_{1k}^{-1}+\eta_{2k}^{-1})/4\hbar^2]} \tag{33}$$

Absorption of power at a difference frequency for an antiferromagnet with anisotropy of the "easy axis" type was treated by Kaganov and Chupis [13]. Comparison of results shows that in antiferromagnets with anisotropy of the "easy plane" type, the amount of nonlinear coupling between branches is much greater, and therefore experimental observation of the effect is more probable. Further it should be mentioned that by $\bar{n}_{sk}$ in formula (33) may be understood not necessarily the temperature-equilibrium value of the occupation numbers, but a stationary distribution established by an arbitrary method without participation of difference pumping. In the antiferromagnets considered, because of the favorable value of ω_{10} (in the usual fields, in the microwave range), it is experimentally simple to create an appreciable above-equilibrium population n_{10} (for example, by a field $h_z \cos\omega_{10}t$), and this gives a possibility of controlling absorption of power at frequency $\omega_{20}-\omega_{10}$ (usually in the submillimeter range).

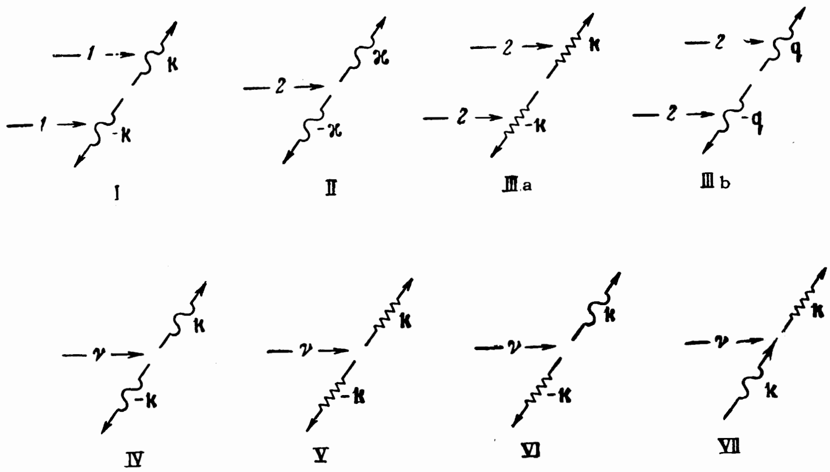
6. CONCLUSION

In conclusion, for best review of the results, we present a table in which are collected all the effects considered above. In the process diagrams only the direct process is represented; the straight arrows with numbers 1 or 2 denote quanta of the appropriate uniform precession, the different wavy arrows denote spin waves of the different branches with

Process diagram	Polarization of field h	Frequency of field h	Threshold amplitude
			Four-magnon instability of low-frequency AFMR
I	z	ω_{10}	$2\Delta H_{10}\left(\frac{2\gamma_m \Delta H_{1k}}{\omega_{10}}\right)^{1/2}$
	y	ω_{10}	$2\Delta H_{10}\left(\frac{2\gamma_m \Delta H_{1k}}{\omega_{10}}\right)^{1/2}\left(\frac{H_0}{H_0+H_D}\right)^{1/2}$
			Three-magnon instability of high-frequency AFMR
II	x	ω_{20}	$\Delta H_{20}\Delta H_{1\varkappa}/2H_0$, $2\pi M_0 (H_A/H_E)^{1/2} \ll H_0 \lesssim H_{AE}/2$
			Four-magnon instability of high-frequency AFMR
III a	x	ω_{20}	$2\Delta H_{20}\left(\frac{2\gamma_m \Delta H_{2k}}{\omega_{20}}\right)^{1/2}$
III b	x	ω_{20}	$\Delta H_{20}\left(\frac{2\gamma_m \Delta H_{1q}}{\omega_{20}}\right)^{1/2}$
			Low-frequency parallel pumping
IV	x	$2\omega_{1k}$	$\min\left\{\frac{2\omega_{1k}\Delta H_{1k}}{\gamma_m H_D}\right\}$
			High-frequency parallel pumping
V	x	$2\omega_{2k}$	$\min\left\{\frac{2\omega_{2k}\Delta H_{2k}}{\gamma_m H_D}\right\}$
			Parametric sum resonance
VI	y	$\omega_{1k}+\omega_{2k}$	$\min\left\{\frac{2(\omega_{1k}\omega_{2k})^{1/2}}{\omega_{2k}-\omega_{1k}}(\Delta H_{1k}\Delta H_{2k})^{1/2}\right\}$
			Parametric difference resonance
VII	y	$\omega_{2k}-\omega_{1k}$	—

[2)]The expression for $\dot{n}_\nu$ was obtained on the assumption that the relaxational "washing out" of the branch is due more to the influence of the field of the spin waves, that is $\hbar\eta_{1k}/\epsilon_{1k} > 2\pi/\gamma$ (in the contrary case, the calculation is carried out by analogy with [13]). The appearance of the sum $\eta_{1k}+\eta_{2k}$ in the denominator of this expression can be established with the aid of the general formula for the probability of a transition with allowance for the "line width" not only of the final, but also of the initial state.

Process Diagrams



$\mathbf{k} \neq 0$, the straight arrow with the sign ν denotes a photon; ΔH_{S0}, ΔH_{Sk} are the complete "line widths". The formulas obtained may prove useful in the use of nonlinear effects to determine relaxation constants of the magnetic materials considered.

I gratefully thank A. S. Borovik-Romanov and V. G. Bar'yakhtar for helpful discussions.

[1] A. J. Heeger, Phys. Rev. **131**, 608 (1963).

[2] A. S. Borovik-Romanov and L. A. Prozorova, JETP **46**, 1151 (1964), Soviet Phys. JETP **19**, 778 (1964).

[3] V. I. Ozhogin, JETP **46**, 531 (1964), Soviet Phys. JETP **19**, 362 (1964).

[4] P. L. Richards, J. Appl. Phys. **35**, 850 (1964).

[5] E. A. Turov, Fizicheskie svoĭstva magnito-uporyadochennykh kristallov (Physical Properties of Magnetically Ordered Crystals), AN SSSR, 1963.

[6] R. M. White and M. Sparks, Phys. Rev. **130**, 632 (1963).

[7] A. S. Borovik-Romanov, JETP **36**, 766 (1959), Soviet Phys. JETP **9**, 539 (1959).

[8] H. Suhl, J. Phys. Chem. Solids **1**, 209 (1957).

[9] V. G. Bar'yakhtar and O. V. Kovalev, Tezisy dokladov IX Vsesoyuznogo soveshchaniya po fizike nizkikh temperatur (Abstracts of the Reports of the Ninth All-Union Conference on the Physics of Low Temperatures), Leningrad, 1962.

[10] A. S. Borovik-Romanov and V. I. Ozhogin, JETP **39**, 27 (1960), Soviet Phys. JETP **12**, 18 (1961).

[11] E. G. Rudashevskiĭ, JETP **46**, 134 (1964), Soviet Phys. JETP **19**, 96 (1964).

[12] Schlömann, Green, and Milano, J. Appl. Phys. **31**, 386S (1960).

[13] M. I. Kaganov and I. E. Chupis, JETP **44**, 1695 (1963), Soviet Phys. JETP **17**, 1141 (1963).

Translated by W. F. Brown, Jr.
195