VORTEX RING FORMATION IN A SUPERFLUID

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The production of vortex rings in a superfluid with relative motion between the normal and the superfluid components is considered. The corresponding Fokker-Planck equation is derived which describes the appearance of vortex rings due to thermal fluctuations and a formula is obtained for the rate of formation of vortex rings.

1. In the original theory of superfluidity due to Landau (cf., for example, [1]) it was assumed that the superfluid component can flow with velocities lower than the critical velocity without experiencing friction either against the walls of the capillary or against the normal component. However, as experiment has shown, the critical velocities are very small and depend in an essential manner on the diameter of the capillary falling off rapidly as the diameter is increased. The explanation of this fact is related to the possibility of formation of localized excitations—vortex rings in the superfluid component—and the appearance of corresponding friction between the normal and the superfluid components (cf., the review in [2]).

In this paper we do not consider real experimental conditions under which, apparently, an important role is played by the phenomena at the boundaries of the fluid, but we shall study an idealized homogeneous problem when the fluid occupies the whole volume and initially $\mathbf{v}_s \neq \mathbf{v}_n$. When the Landau criterion is satisfied

$$|\mathbf{v}_s - \mathbf{v}_n| < \min\left[\varepsilon(p)/p\right] = v_{\text{cr}},$$

where $\epsilon(p)$ and p represent the energy and the momentum of the elementary excitations, such a state will be metastable, since a state without relative motion of the components, but with the same total momentum, is thermodynamically more advantageous.

The mechanism of approach to equilibrium consists of the formation of vortex filaments in the super-fluid component and the subsequent reduction in the relative motion between the normal and the super-fluid components. Since the vortex filaments are macroscopic structures and not elementary excitations, the solution of the problem of the approach to equilibrium by the method of Green's functions is in fact impossible since it requires the introduction of macroscopic Green's functions. However, the macroscopic nature of the excitations makes possible the solution of the problem by means of a hydrodynamic description.

Owing to thermal fluctuations vortex filaments can be formed in the superfluid component. It can be easily seen that the formation of closed filaments, i e., vortex rings, is most probable since they can be formed reversibly requiring the least work A for a given length of the vortex filament. Such rings play in this case the role of Gibbs' nuclei. Their critical size R_1, below which the ring tends to diminish and above which it tends to grow without limit, is determined by the equilibrium between the force of friction and the hydrodynamic forces acting on the vortex filament.

Since the diameter of a vortex filament in a superfluid [2] is of the order of atomic dimensions, which is much smaller than the critical radius of the ring for $|\mathbf{v}_s - \mathbf{v}_n| \ll v_{cr}$ one can easily evaluate the magnitude of $A(R)$. The minimum work is obtained in the case of a reversible process; in this case the force of friction of the vortex filament against the normal component must be absent, and for this (cf. [2,3]) the velocity of the vortex filament must be equal to the velocity of the normal component $\mathbf{v}_n$ which we shall take to be equal to zero. In this case the only force acting on the ring will be the hydrodynamic force acting per unit length of the filament in an ideal fluid. The value of A will be simply the energy of a vortex ring in an ideal fluid having a velocity at infinity. Utilizing the Galilean transformation we obtain

$$A = \varepsilon(R) + \mathbf{p}\mathbf{v}_s, \tag{1.1}$$

where $\epsilon(R)$ and $\mathbf{p}(R)$ are the energy and the momentum of a vortex ring in a fluid at rest [4]:

$$\varepsilon(R) = \rho_s \frac{\varkappa^2}{2} R\left(\ln\frac{8R}{a} - \frac{7}{4}\right), \quad p(R) = |\varkappa|\pi R^2\rho_s, \tag{1.2}$$

where $\kappa = h/m$ is the circulation, and a is a quantity of the order of atomic dimensions. The critical radius is obtained from the condition

$$F = \frac{\partial A}{\partial R} = \rho_s \frac{\varkappa^2}{2}\left(\ln\frac{8R}{a} - \frac{3}{4}\right) - 2\pi\varkappa R \rho_s v_s = 0 \quad (1.3)$$

or

$$R_1 = \frac{\varkappa}{4\pi v_s}\left(\ln\frac{8R_1}{a} - \frac{3}{4}\right), \quad (1.4)$$

from which it can be seen that we must consider vortex rings with $\kappa > 0$.

From (1.1) it can be seen that the energy A(R) is of the form of a potential barrier of height

$$A_1 \sim \frac{\rho_s \varkappa^2 R_1}{4} \ln\frac{R_1}{a}. \quad (1.5)$$

Thus, if due to fluctuations a vortex ring attains the size R_1 then its further growth will be energetically favorable and, in fact, takes place with probability unity.

The object of our further calculations is to determine the number of vortex rings reaching critical dimensions per unit time per unit volume.

3. We are interested in the initial phase of formation of vortex rings in a superfluid when there are still only very few of them and their interaction can be neglected. This problem is analogous to a problem considered earlier by Zel'dovich[5] and Kagan [6] dealing with the boiling of a pure liquid (on the basis of the general method due to Kramers [7]).

Vortex rings can attain critical size both by means of random repeated expansion as a result of fluctuations related to the presence of excitations in the fluid, and also by means of a single random increase directly to critical size. However, the latter is not very probable, since it requires the organization of motion directly of critical dimensions. We shall, therefore, consider only the slow diffusion process. In this case the process of growth of a ring can be represented as the motion of a vortex ring in the presence of an external medium (excitation gas) randomly colliding with the vortex rings, i.e., as a special case of Brownian movement.

We introduce a distribution function for the vortex rings $f(\{n\})$ which depends on the set of quantum numbers $\{n\}$ characterizing the vortex ring which can oscillate and can be oriented in a random manner. Let the probability of transition from the state $\{n'\}$ into the state $\{n\}$ during a time τ be $w_\tau(\{n'\}, \{n' - n\})$. Then the number of vortex rings in the state $\{n\}$ at the time $t + \tau$ will be given by

$$f(\{n\}, t+\tau) = \sum_{n'} w_\tau(\{n'\}, \{n' - n\}) f(\{n'\}, t), \quad (2.1)$$

where the summation extends over all $\{n'\}$, including $\{n'\} = \{n\}$. Equation (2.1) must have the following property: the Gibbs distribution $f(\{n\}) = \exp(-E(\{n\})/kT)$ must be its stationary solution.

For a vortex ring we pick out the quantum numbers n_χ corresponding to its oscillations about its equilibrium circular shape and the quantum numbers N_i corresponding to the motion of the ring as a whole. The energy of oscillations of a ring of radius R will be given by $(1/2 + n_\chi)\hbar\omega(\chi, R)$ where $\omega(\chi, R)$ are obtained from the solution of the problem of small oscillations, χ is the propagation vector for the oscillations or the ordinal number of the corresponding harmonic, n_χ are the corresponding occupation numbers. The total energy of the ring can be written in the form

$$E(N_i, n_\chi) = \sum_\chi \hbar\omega(\chi, R) n_\chi + E(N_i), \quad (2.2)$$

where $E(N_i)$ is the energy of a circular ring including the energy of the zero point oscillations.

We assume that there exist vortex rings of a definite radius R. The equilibrium with respect to the numbers n_χ is established fairly rapidly because energy fluctuations of the order $\hbar\omega$ occur sufficiently frequently. Therefore, the distribution function will have the form

$$f(N_i, n_\chi, t) = f_1(N_i, t)\exp\left\{-\sum_\chi \frac{\hbar\omega_\chi n_\chi}{kT}\right\}. \quad (2.3)$$

The quantum numbers N_i change during a sufficiently small time τ on the average by relatively small amounts, and equation (2.1) taking (2.3) into account can be transformed into a Fokker-Planck equation for the function f_1.

Since transitions with a change in the numbers n_χ do not alter the form of the distribution function, then in the sum over n' we can leave only the sum over N_i; carrying out the summation over n_χ we obtain from this

$$\sum_{n_\chi} f(N_i, n_\chi, t+\tau) = \sum_{n_\chi, N_i'} w_\tau(N_i', n_\chi, \{N_i' - N_i\}) f(N_i', n_\chi, t). \quad (2.4)$$

Expanding in terms of $N_i' - N_i$ in the first argument of w_τ and in f, and taking τ to be small and the functions w_τ and f to be sufficiently smooth, we obtain a Fokker-Planck equation in the space of the quantum numbers N_i which can be written in the form

$$\sum_{n_\chi} \frac{\partial f(N_i, n_\chi)}{\partial t} = \sum_{n_\chi, i} \frac{\partial}{\partial N_i}\left(-\frac{dN_i}{dt} f + \sum_j D_{ij}\frac{\partial f}{\partial N_j}\right), \quad (2.5)$$

where the quantity dN_i/dt is the average rate of change of the quantum number N_i, while the second term in brackets describes the diffusion process associated with fluctuations. The average rate of change of N_i can be written in the form

$$\frac{dN_i}{dt} = -\sum_j \eta_{ij}\frac{\partial E}{\partial N_j}, \tag{2.6}$$

where η_{ij} are certain kinetic coefficients.

Since the Gibbs distribution must be a solution of (2.5) the following relations hold

$$D_{ij} = kT\eta_{ij}, \tag{2.7}$$

and (2.5) assumes the form

$$\sum_{n_\chi}\frac{\partial f}{\partial t} = \sum_{n_\chi, i, j}\frac{\partial}{\partial N_i}\left[\eta_{ij}\left(\frac{\partial E}{\partial N_j}f + kT\frac{\partial f}{\partial N_j}\right)\right]. \tag{2.8}$$

The energy of a circular ring depends on its radius R and on its orientation defined by the angles ϑ, φ. According to (1.1) and (2.2) for an arbitrary ring the energy can be written in the form

$$E(N_i, n_\chi) = E(R) + E'(n_\chi, \vartheta, R), \tag{2.9}$$

where

$$E(R) = \varepsilon(R) - p(R)v_s,$$

$$E'(n_\chi, \vartheta, R) = \sum_\chi \hbar\omega(\chi, R)\, n_\chi + p(R)v_s(1-\cos\vartheta), \tag{2.10}$$

ϑ is the angle between $-\mathbf{v}_s$ and the momentum of the ring $p(R)$.

We go over from the variables N_i to new independent variables N_i', including R, ϑ, φ. It can be easily shown that (2.8) in the new variables assumes the form

$$\sum_{n_\chi}\frac{\partial(fJ)}{\partial t} = \sum_{n_\chi}\frac{\partial}{\partial N_l'}\left\{\eta_{ij}\frac{\partial N_m'}{\partial N_j}\frac{\partial N_l'}{\partial N_i}\left[\left(\frac{\partial E}{\partial N_m'} - kT\frac{\partial \ln J}{\partial N_m'}\right)fJ + kT\frac{\partial(fJ)}{\partial N_m'}\right]\right\}, \tag{2.11}$$

where

$$J = \partial(N_1 \ldots N_n)/\partial(N_1' \ldots N_n') \tag{2.12}$$

is the Jacobian of the transformation.

Equilibrium with respect to orientations is established rapidly since for small R the energy practically does not depend on ϑ, while for large R all the rings are concentrated near $\vartheta = 0$, and, therefore, we can take

$$Jf = Jf'(R)\exp(-E'/kT). \tag{2.13}$$

We are interested only in the formation of vortex rings of a definite radius, and it is immaterial in which particular quantum state these rings are found. Therefore, we can integrate (2.11) over the angles ϑ and φ, and also over the remaining N_i' with the exception of $R = N_{i_1}'$. Substituting (2.13) into (2.11) and carrying out the summations and the integrations we obtain

$$\frac{\partial\sigma}{\partial t} = \frac{\partial}{\partial R}\left\{\eta_{RR}\left[\left(\frac{\partial E(R)}{\partial R} - kT\frac{\partial\ln\Gamma}{\partial R}\right)\sigma + kT\frac{\partial\sigma}{\partial R}\right]\right\}, \tag{2.14}$$

where the function $\sigma(R, t)$ is the distribution function for the vortex rings over the radii, Γ is the statistical weight of the state of a vortex ring of radius R:

$$\ln\Gamma = \ln\sum_{n_\chi}\int Je^{-E'/kT}\,d\vartheta\, d\varphi\, dN'. \tag{2.15}$$

We see from (2.14) that in accordance with the thermodynamic fluctuation theory in the equilibrium state the number of vortex rings of radius R is proportional to $\exp(-A_{min}/kT)$, where $A_{min} = \delta F = \delta E - kT\delta S$ (δE and δS are respectively the changes in the energy and the entropy associated with the formation of a vortex ring).

3. Our further problem is the determination of Γ and η_{RR} for a vortex ring.

The kinetic coefficient η_{RR} can be obtained by considering the motion of a vortex ring, since the quantity $-\eta_{RR}\partial E/\partial R$ gives the average rate of change of the vortex ring taking dissipative processes into account. The normal fluid exerts a friction force per unit length of an element of a vortex filament given by [3]

$$\mathbf{F}_1 = -b\frac{\rho_n\rho_s}{\rho\varkappa}[\varkappa[\varkappa\mathbf{v}_L]] + b'\frac{\rho_n\rho_s}{\rho}[\varkappa\mathbf{v}_L], \tag{3.1}$$*

where $\mathbf{v}_L$ is the velocity of the vortex filament; the dimensionless coefficients b and b′ differ somewhat from the coefficients of Hall and Vinen [3] B and B′. In addition to the friction force an element of a vortex filament is also acted upon by the superfluid with a Magnus force [4]

$$\mathbf{F}_2 = \rho_s[\varkappa, \mathbf{v}_s + \mathbf{U}(R) - \mathbf{v}_L], \tag{3.2}$$

where $\mathbf{U}(R)$ is the velocity of motion of the vortex ring in a stationary fluid. Since a vortex filament has no mass, the total force acting on an element of the filament is equal to zero, and we obtain for the radial velocity the expression

$$\frac{dR}{dt} = -\frac{\rho\rho_n b}{\rho_n^2 b^2 + \rho^2(1-b'\rho_n/\rho)^2}\frac{\partial E}{\partial R}\frac{\partial R}{\partial p}, \tag{3.3}$$

from which we can obtain the expression for the coefficient η_{RR}:

$$\eta_{RR} = \frac{\rho\rho_n b}{\rho_n^2 b^2 + \rho^2(1-b'\rho_n/\rho)^2}\frac{\partial R}{\partial p}. \tag{3.4}$$

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*$[\varkappa\mathbf{v}_L] = \varkappa \times \mathbf{v}_L$.

In order to determine the statistical weight Γ it is necessary to know the quantum numbers determining the circular vortex ring. If we regard a vortex ring as an excitation of energy ϵ and momentum p, then the latter must be quantized in such a manner that

$$\mathbf{p} = \left(\frac{2\pi\nu_1\hbar}{L_1}, \frac{2\pi\nu_2\hbar}{L_2}, \frac{2\pi\nu_3\hbar}{L_3} \right),$$

where ν_1, ν_2, ν_3 are integers and L_1, L_2, L_3 are the dimensions of the box. This enables us to evaluate the Jacobian of the transformation (2.12):

$$J = \frac{V}{(2\pi\hbar)^3} p^2 \frac{dp}{dR} \sin\vartheta. \qquad (3.5)$$

Utilizing formula (2.15) we obtain

$$\ln\Gamma = S_L + S_1, \qquad (3.6)$$

where

$$S_1 = \ln \frac{kTV2\pi}{v_s(2\pi\hbar)^3} p \frac{dp}{dR} (1 - e^{-2pv_s/kT}), \qquad (3.7)$$

while the entropy of the filament after summation over n_χ has the form

$$S_L = -\sum_\chi \ln(1 - e^{-\hbar\omega_\chi/kT}). \qquad (3.8)$$

For vortex rings of sufficiently large radius the vextors χ are distributed practically continuously, and for ω one can utilize the hydrodynamic formula (cf., for example, [2])

$$\omega = \frac{\hbar^2}{2m}\chi^2 \ln\frac{1{,}046}{\chi a}. \qquad (3.9)$$

From this we can obtain an approximate expression for

$$S_L = 2R\frac{\sqrt{2mkT}}{\hbar}\left(\ln\frac{1.046\hbar}{a\sqrt{2mkT}}\right)^{-1/2}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right), \qquad (3.10)$$

which is valid for sufficiently low temperatures; ζ is the Riemann zeta function.

Thus, we have determined all the quantities appearing in (2.14).

Generally speaking, (2.14) must be solved subject to certain boundary conditions when there are no vortex rings of radii exceeding a certain R_0. However, the process of growth of the vortex rings is a very slow one due to the large value of the potential barrier, and in fact it can be treated as a stationary one (cf.[7,5]). At the same time we have the boundary conditions: for small $R \sim R_0$ equilibrium exists and

$$\sigma(R_0) = \Gamma(R_0)\exp(-E(R_0)/kT); \qquad (3.11)$$

for $R \to \infty$ there are no vortex rings and $\sigma = 0$.

Solving the stationary equation (2.14) we obtain

$$j_R = \left\{ kT\sigma(R)\Gamma^{-1}(R)\exp\left[\frac{E(R)}{kT}\right]\Big|_{R_0}^{\infty} \right\} \times \left\{ \int_{R_0} \frac{1}{\eta_{RR}} \exp\left[\frac{E(R)}{kT} - \ln\Gamma(R)\right] dR \right\}^{-1}, \qquad (3.12)$$

where the constant j_R gives us the flux of vortex rings along the R axis or the number of vortex rings of radius above the critical radius formed per unit time. The integrand has a maximum at $R = R_{cr}$ defined by the equation

$$\frac{\partial E(R)}{\partial R} - kT\frac{\partial\ln\Gamma(R)}{\partial R} = 0 \qquad (3.13)$$

or

$$R_{cr} = \frac{\hbar}{2mv_s}\left[\ln\frac{8R_{cr}}{a} - \frac{2}{4}\right] Å - \frac{kT}{\hbar}\sqrt{2mkT}\left(\ln\frac{1.046\hbar}{a\sqrt{2mkT}}\right)^{-1/2}\frac{4.4m}{4\pi^2\rho_s v_s\hbar}. \qquad (3.14)$$

It should be noted that this equation does not always have a solution. Calculations show that for He II at T = 1.8°K, a = 19 Å and $v_s > 70$ cm/sec Eq. (3.14) no longer has a root since in this case there is no barrier and vortex rings can be formed easily (unfortunately the accuracy of formula (3.10) for S_L is not great, and the numbers are not very reliable). The quantity a is also determined quite roughly [2]. In the case when (3.14) has a solution the principal contribution to the integral comes from the neighborhood of the point $R = R_{cr}$. Utilizing the boundary conditions we obtain

$$j_R = \frac{2\pi kT\sqrt{kT}\,V}{v_s(2\pi\hbar)^3\sqrt{\pi}}\frac{\hbar}{m}\rho_s R_{cr}^2 \exp\left[-\frac{E(R_{cr}) - kTS_L(R)_{cr}}{kT}\right] \times \frac{\rho\rho_n b}{\rho_n^2 b^2 + (1 - b'\rho_n/\rho)^2\rho^2}\left[\frac{\partial^2}{\partial R^2}(E - kTS_L)\Big|_{R_{cr}}\right]^{1/2}.$$

This formula enables us on taking into account formulas (2.10), (3.10), and (3.14) to evaluate the number of vortex rings of radius larger than the critical radius formed per unit time in volume V. The reciprocal quantity defines the time of relaxation towards the equilibrium state with $v_s = v_n$. Calculations show that an appreciable rate of formation of vortex rings in He II occurs for $v_s > 60$ cm/sec and falls off sharply towards unobservable values of the order of exp (−1000) at velocities in the neighborhood of 40 cm/sec. The quantity j_R varies very rapidly with temperature, particularly because of the variation in ρ_s for T > 1.4°K.

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