SOVIET PHYSICS JETP VOLUME 21, NUMBER 2 AUGUST, 1965

SPATIAL DISPERSION OF INHOMOGENEOUS MEDIA

Yu. A. RYZHOV, V. V. TAMOĬKIN and V. I. TATARSKIĬ

Radio Physics Institute, Gorkiĭ State University; Institute of Atmospheric Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor August 11, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 656-665 (February, 1965)

A spatial-dispersion mechanism connected with random inhomogeneities of a medium is investigated. An expression is obtained for the effective dielectric permittivity tensor of an inhomogeneous medium for strong fluctuations of the dielectric permittivity. It is shown that the presence of spatial dispersion brought about by the inhomogeneity of the medium does not lead to the appearance of longitudinal waves of the mean field. An inhomogeneous plasma is considered as a specific example. The limiting cases of strong and weak fluctuations of the dielectric permittivity are investigated (at frequencies far from and close to the resonance frequency of the plasma).

INTRODUCTION

THE thermal motion of the electrons of a plasma is not the only reason for the appearance of the phenomenon of spatial dispersion. The process of scattering of electromagnetic waves by inhomogeneities of the medium produces a spatially nonlocal coupling between the mean values of the electric field and the electrical induction, which leads to a dependence of the effective dielectric permittivity on the wave vector. Therefore, even in a statistically homogeneous and isotropic medium, the effective dielectric permittivity is a tensor. An expression is obtained in the present work for the effective dielectric permittivity tensor $\epsilon_{ij}^{\text{eff}}(\omega, \mathbf{k})$ of a randomly inhomogeneous medium.

In the case of weak fluctuations of the dielectric permittivity, this expression generalizes the well known results of Kaner,[1] obtained without account of spatial dispersion. Calculation of the tensor $\epsilon_{ij}^{\text{eff}}(\omega, \mathbf{k})$ for the case of strong fluctuations of the dielectric permittivity is based on the results of the research of Finkel'berg[2] and Gertsenshteĭn and one of the authors.[3,4]

The problem considered in[3,4] is that of propagation of waves in a medium with strong fluctuations. Some results of the research of one of the authors[4] need corrections, which will be made in the appropriate place.

Investigation shows that the spatial dispersion of the tensor $\epsilon_{ij}^{\text{eff}}(\omega, \mathbf{k})$ does not lead to the appearance of longitudinal waves of the mean field. A more complete consideration is given for the case of an inhomogeneous plasma. An expression is obtained for $\epsilon_{ij}^{\text{eff}}(\omega, \mathbf{k})$ at frequencies close to the resonance frequency of the plasma, when the plasma is a medium with strong fluctuations of the dielectric permittivity.

1. DERIVATION OF THE EQUATION FOR THE EFFECTIVE DIELECTRIC PERMITTIVITY TENSOR

Let us write down the equation for the field of a point monochromatic dipole in a medium with fluctuating dielectric permittivity $\epsilon(\mathbf{r})$ in the form

$$\Delta\mathbf{E} - \operatorname{grad}\operatorname{div}\mathbf{E} + \omega^2\varepsilon(\mathbf{r})\mathbf{E}/c^2 = \mathbf{n}\delta(\mathbf{r}-\mathbf{r}_0). \tag{1}$$

We shall make use of a technique employed in[2], according to which it is convenient to make use in the case of a singular Green's function of functions $\mathbf{F}(\mathbf{r})$ and $\xi(\mathbf{r})$ defined by

$$\xi = 3\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}, \qquad \mathbf{F} = \frac{\varepsilon+2\varepsilon_0}{3\varepsilon_0}\mathbf{E}, \tag{2}$$

where ϵ_0 is some additional dielectric permittivity defined below.[1)] It follows from (1) and (2) that the vector function $\mathbf{F}(\mathbf{r})$ satisfies the following integral equation:

$$F_i(\mathbf{r}) = G_{ij}^0(\mathbf{r}, \mathbf{r}_0)n_j - k_0^2\varepsilon_0\int G_{ij}'(\mathbf{r}, \mathbf{r}')\xi(\mathbf{r}')F_j(\mathbf{r}')d\mathbf{r}'; \tag{3}$$

$$G_{ij}^0(\mathbf{r}) = G_{ij}'(\mathbf{r}) + \frac{1}{3k_0^2\varepsilon_0}\delta_{ij}\delta(\mathbf{r}),$$

$$G_{ij}'(\mathbf{r}) = -\mathrm{P}\left[\delta_{ij} + \frac{1}{k_0^2\varepsilon_0}\frac{\partial^2}{\partial x_i\partial x_j}\right]\frac{\exp(ik_0\sqrt{\varepsilon_0}r)}{4\pi r}.$$

1)It is easy to see that the auxiliary dielectric permittivity ϵ_0 is a scalar quantity only in the case of homogeneous and isotropic fluctuations $\Delta\epsilon$.

The symbol P means that the principal value is taken.

If Eq. (3) is solved by successive approximations, then we get a series in powers of $\xi(\mathbf{r})$. Therefore, in order that this series converge in the best manner, it is convenient to impose on the quantity the additional condition $\langle\xi\rangle = 0$, which is an equation determining the auxiliary dielectric permittivity ϵ_0:

$$\langle\xi\rangle = \left\langle \frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right\rangle = 0. \qquad (4)$$

We introduce into consideration the quantity $\hat{\xi}_{ij}^{\text{eff}}$, defined by the equation

$$\langle\xi F_i\rangle = \hat{\xi}_{ij}^{\text{eff}}\langle F_j\rangle, \qquad (5)$$

where the operator $\hat{\xi}_{ij}^{\text{eff}}$ is an integral operator:

$$\langle\xi F_i\rangle = \int \xi_{ij}^{\text{eff}}(\mathbf{r}-\mathbf{r}')\langle F_j(\mathbf{r}')\rangle d\mathbf{r}'. \qquad (6)$$

The following relations, which are useful below, derive from Eq. (2):

$$\langle F_i\rangle = (\hat{\varepsilon}_{ij}^{\text{eff}}/3\varepsilon_0 + {}^2/_3\delta_{ij})\langle F_j\rangle,$$
$$\langle\xi F_i\rangle = (\hat{\varepsilon}_{ij}^{\text{eff}}/\varepsilon_0 - \delta_{ij})\langle E_j\rangle. \qquad (7)$$

We get for fields of the form $e^{-i\mathbf{k}\cdot\mathbf{r}}$

$$f_i(\mathbf{k}) = \left[\frac{1}{3\varepsilon_0}\varepsilon_{ij}^{\text{eff}}(\omega,\mathbf{k}) + \frac{2}{3}\delta_{ij}\right]e_j(\mathbf{k}),$$
$$\xi_{ij}^{\text{eff}}(\omega,\mathbf{k})f_j(\mathbf{k}) = \left[\frac{1}{\varepsilon_0}\varepsilon_{ij}^{\text{eff}}(\omega,\mathbf{k}) - \delta_{ij}\right]e_j(\mathbf{k}), \qquad (8)$$

where e_j, f_j are the Fourier components of the fields E_j, F_j,

$$\xi_{ij}^{\text{eff}}(\omega,\mathbf{k}) = \int \xi_{ij}^{\text{eff}}(\mathbf{r})e^{-i\mathbf{kr}}d\mathbf{r}. \qquad (9)$$

A connection between $\xi_{ij}^{\text{eff}}(\omega, \mathbf{k})$ and $\epsilon_{ij}^{\text{eff}}(\omega, \mathbf{k})$ follows from (8):

$$\frac{1}{3\varepsilon_0}\varepsilon_{jk}^{\text{eff}}\xi_{ij}^{\text{eff}} - \frac{1}{\varepsilon_0}\varepsilon_{ik}^{\text{eff}} = -\delta_{ik} - \frac{2}{3}\xi_{ik}^{\text{eff}}. \qquad (10)$$

In the case of homogeneous and isotropic fluctuations of the dielectric permittivity tensor, $\epsilon_{ij}^{\text{eff}}$ and ξ_{ij}^{eff} have the form

$$\varepsilon_{ij}^{\text{eff}}(\omega,\mathbf{k}) = \left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\varepsilon_{\text{eff}}^{tr}(\omega,k) + \frac{k_ik_j}{k^2}\varepsilon_{\text{eff}}^{l}(\omega,\mathbf{k}),$$
$$\xi_{ij}^{\text{eff}}(\omega,\mathbf{k}) = \left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\xi_{\text{eff}}^{tr}(\omega,k) + \frac{k_ik_j}{k^2}\xi_{\text{eff}}^{l}(\omega,\mathbf{k}). \qquad (11)$$

With account of (11), we have from Eq. (10)

$$\varepsilon_{\text{eff}}^{l}(\omega,k) = \varepsilon_0\frac{1+{}^2/_3\xi_{\text{eff}}^{l}(\omega,k)}{1-{}^1/_3\xi_{\text{eff}}^{l}(\omega,k)}$$

$$\varepsilon_{\text{eff}}^{tr}(\omega,k) = \varepsilon_0\frac{1+{}^2/_3\xi_{\text{eff}}^{tr}(\omega,k)}{1-{}^1/_3\xi_{\text{eff}}^{tr}(\omega,k)}. \qquad (12)$$

It was shown previously[4] by analysis of the perturbation-theory series for Eq. (3) that $\langle F_i\rangle$ satisfies the following integral equation:

$$\langle F_i(\mathbf{r})\rangle = G_{ij}^0(\mathbf{r},\mathbf{r}_0)n_j + \iint G_{il}'(\mathbf{r},\mathbf{r}_1)Q_{ln}(\mathbf{r},\mathbf{r}_2)\langle F_n(\mathbf{r}_2)\rangle d\mathbf{r}_1 d\mathbf{r}_2, \qquad (13)$$

where the function $Q_{ln}(\mathbf{r}_1, \mathbf{r}_2)$ is defined by the series

$$Q_{ij}(\mathbf{r}_1,\mathbf{r}_2) = k_0^4\varepsilon_0^2B_\xi(\mathbf{r}_1,\mathbf{r}_2)G_{ij}'(\mathbf{r}_1,\mathbf{r}_2) + k_0^8\varepsilon_0^4\iint G_{il}'(\mathbf{r}_1,\mathbf{r}_3) \times G_{ln}'(\mathbf{r}_3,\mathbf{r}_4)G_{nj}'(\mathbf{r}_4,\mathbf{r}_2)B_\xi(\mathbf{r}_1,\mathbf{r}_4)B_\xi(\mathbf{r}_3,\mathbf{r}_2)d\mathbf{r}_3 d\mathbf{r}_4 + \ldots \qquad (14)$$

Here $B_\xi(\mathbf{r}_1, \mathbf{r}_2)$ is the correlation function of the random variable ξ, which is assumed to have a normal distribution.

Now, averaging Eq. (3), we find

$$\langle F_i(\mathbf{r})\rangle = G_{ij}^0(\mathbf{r},\mathbf{r}_0)n_j - k_0^2\varepsilon_0\int G_{ij}'(\mathbf{r},\mathbf{r}')\langle\xi F_j(\mathbf{r}')\rangle d\mathbf{r}'. \qquad (15)$$

Comparing this equation with Eq. (13), in accord with (6) and (9), we have

$$\xi_{ij}^{\text{eff}}(\mathbf{r}) = -Q_{ij}(\mathbf{r})/k_0^2\varepsilon_0;$$
$$\xi_{ij}^{\text{eff}}(\omega,\mathbf{k}) = -\frac{1}{k_0^2\varepsilon_0}\int Q_{ij}(\mathbf{r})e^{-i\mathbf{kr}}d\mathbf{r}. \qquad (16)$$

In the following calculations we limit ourselves only to the first term of the series of Eq. (14).[2)] As estimates by means of consideration of the next term of the expansion (23) show, all the formulas will be valid if

$$|\langle\xi^2\rangle(k_0\sqrt{\varepsilon_0}l)^3| \ll 1, \qquad (17)$$

where l is the scale of the function $B_\xi(r) = \langle\xi^2\rangle\Gamma_\xi(r/l)$.

Substituting the first term of the series (14) in (16), in carrying out integration over the angles, we get

$$\xi_{\text{eff}}^{l}(\omega,k) = -2\langle\xi^2\rangle q(p,p_0);$$

$$\xi_{\text{eff}}^{tr}(\omega,k) = \frac{p_0^2\langle\xi^2\rangle}{p}\int_0^\infty \Gamma_\xi(x)e^{ip_0x}\sin px\,dx + \langle\xi^2\rangle q(p,p_0),$$

$$q(p,p_0) = \int_0^\infty \Gamma_\xi(x)e^{ip_0x}\left[\frac{p_0^2\cos px}{p^2} + i\frac{p_0}{p}\sin px - \right.$$

[2)]In this approximation the assumption of a normal distribution for the variable ξ is unnecessary if its correlation function is known. The form of the distribution for ξ can be significant only in the estimate of the limits of applicability of this approximation.

$$- \sin px\left(1+\frac{p_0^2}{p^2}\right)+3i\frac{p_0}{p}\frac{\cos px}{px}-3\frac{\cos px}{p^2x^2}$$
$$- 3i\frac{p_0}{p}\frac{\sin px}{p^2x^2}+3\frac{\sin px}{p^3x^3}\Big]x^{-1}\,dx, \tag{18}$$

where $p = kl$, $p_0 = k_0\sqrt{\epsilon_0}\,l$. The last formula, together with (12), permits us to calculate the effective dielectric permittivity of the inhomogeneous medium, for which it is also necessary to find ϵ_0 from Eq. (4).

2. THE GREEN'S FUNCTION

Having the expression for $\epsilon_{ij}^{\mathrm{eff}}(\omega, \mathbf{k})$, we can write down the averaged Green's function of Eq. (1) (or, what amounts to the same thing, the Green's function for the mean field):

$$g_{im}{}^E(\mathbf{k})=\left(\delta_{im}-\frac{k_ik_m}{k^2}\right)g_1{}^E(k)+\frac{k_ik_m}{k^2}g_2{}^E(k), \tag{19}$$

$$g_1{}^E(k)=\frac{1}{8\pi^3}\frac{1}{k_0^2\varepsilon^{tr}_{\mathrm{eff}}(\omega,k)-k^2},$$

$$g_2{}^E(k)=\frac{1}{8\pi^3}\frac{1}{k_0^2\varepsilon^{l}_{\mathrm{eff}}(\omega,k)}$$

As is well known, the character of the oscillations of the field is determined by the poles of the functions $g_1^E(k)$ and $g_2^E(k)$. Let us first consider the longitudinal vibrations, which are defined by the zeros of the function $\epsilon^l_{\mathrm{eff}}(\omega, \mathbf{k})$. As follows from (12) and (18), $\epsilon^l_{\mathrm{eff}}(\omega, \mathbf{k})$ contains two factors, one of which—ϵ_0—depends only on the frequency. The possibility of vanishing of the second factor, which also depends on the wave number k, would indicate the presence of longitudinal waves. However, as investigation shows, this factor does not vanish for any real value of k. The equation $\epsilon^l_{\mathrm{eff}}(\omega, \mathbf{k}) = 0$ can, however, have a solution in the region of complex values of k. Such solutions will correspond to damped waves.

It follows from (12) and (18) that the equation $\epsilon^l_{\mathrm{eff}}(\omega, \mathbf{k})/\epsilon_0 = 0$ is equivalent to the equation

$$q(p, p_0) = 3/4\langle\xi^2\rangle. \tag{20}$$

The investigation of this equation in the general form is difficult. We shall first consider the case of small-scale fluctuations $|p| \ll 1$, $|p_0| \ll 1$. The function $q(p, p_0)$ is represented here by the series

$$q(p, p_0) = -{}^1\!/_3a_1p_0^2+{}^1\!/_{15}a_1p^2-{}^1\!/_3ia_2p_0^3+\ldots, \tag{21}$$

where a_1 and a_2 are numbers of the order of unity:

$$a_1=\int_0^\infty \Gamma_\xi(x)x\,dx, \qquad a_2=\int_0^\infty \Gamma_\xi(x)x^2\,dx.$$

Calculations for the different models of the medium (plasma, dielectrics describable by the Lorenz-Lorentz formula) show that the quantity $\langle\xi^2\rangle$ cannot appreciably exceed unity. It is then clear that Eq. (20) does not have solutions in the region $|p| \ll 1$. In the other limiting case of large scale inhomogeneities ($|p| \gg 1$) the dependence of $\epsilon^l_{\mathrm{eff}}(\omega, \mathbf{k})$ and $\epsilon^{\mathrm{tr}}_{\mathrm{eff}}(\omega, \mathbf{k})$ on the wave number k generally disappears. This is understandable since this case corresponds to the geometrical-optics approximation. The case of intermediate values of p will be considered below with a specific correlation function $\Gamma_\xi(x)$. It will be shown that Eq. (20) has only complex roots.

Thus the mechanism of spatial dispersion brought about by the random inhomogeneities of the medium does not lead to the appearance of longitudinal waves. This statement is applicable, in particular, to cold plasma, in which longitudinal oscillations exist. The inhomogeneities of the electron concentration, as shown below, lead only to a displacement of the frequency of plasma oscillations.[3)]

Let us now consider in more detail the case with correlation function $\Gamma_\xi(x) = e^{-x}$. In this case, the function $q(p, p_0)$ can be calculated and is equal to

$$q(p, p_0)=\frac{1+ip_0+2p^2/3}{2p^2}+\frac{1+p_0^2+p^2}{2p^3}\operatorname{arc\,tg}[p(1-ip_0)^{-1}]. \tag{22*}$$

To simplify the calculations, we expand $q(p, p_0)$ in a series in p_0, keeping it in mind that by virtue of (17), for arbitrary $\langle\xi^2\rangle$, the inequality $p_0 \ll 1$ must hold. With accuracy up to terms of order p_0^2, we have

$$q(p,p_0)\approx {}^1\!/_2p^{-2}\{1+{}^2\!/_3p^2-(1+p^2)p^{-1}\operatorname{arc\,tg}p - p_0^2[p^{-1}\operatorname{arc\,tg}p-(1+p^2)^{-1}]\}; \tag{23}$$

as has already been pointed out, when $p = p^*$ the equation $q(p, p_0) = 3/4\langle\xi^2\rangle$ does not have any solutions. It is easy to see that Eq. (23) takes on real values on the imaginary axis $p = iv$. In this case

$$q(iv,p_0)=-\frac{1}{2v^2}\left\{1-\frac{2}{3}v^2-\frac{(1-v^2)}{2v}\ln\left|\frac{1+v}{1-v}\right| + p_0^2\left[(1-v^2)^{-1}-\frac{1}{2v}\ln\left|\frac{1+v}{1-v}\right|\right]\right\}. \tag{24}$$

[3)]In the case of a heated randomly inhomogeneous plasma, as calculations show, there is in addition to the Landau damping for longitudinal waves also damping associated with the transfer of energy from the regular field to the random vibrations of the field.

*arc tg = $\tan^{-1}$.

Investigation of the function

$$a(v) = -\frac{1}{2v^2}\left[1 - \frac{2}{3}v^2 - \frac{1-v^2}{2v}\ln\left|\frac{1+v}{1-v}\right|\right]$$

shows that it is bounded

$$a(0) = -{}^2/_3, \qquad a(\pm 1) = -{}^1/_6; \qquad a(\pm\infty) = {}^1/_3$$

and does not exceed 2/3 in modulus. The function

$$b(v) = -\frac{1}{2v^2}\left\{(1-v^2)^{-1} - \frac{1}{2v}\ln\left|\frac{1+v}{1-v}\right|\right\}$$

goes to infinity when $v = \pm 1$. Therefore, Eq. (20), which can be written in the form

$$b(v) = p_0^{-2}[3/4\langle\xi^2\rangle - a(v)],$$

always has for $p_0 \ll 1$ a solution close to the points $v_0 \approx \pm 1$. Setting $a(\pm 1) = -1/6$ in the right hand side, and replacing $b(v_0)$ by $(-1/4)(1-v_0)$ (for $v_0 \approx +1$) or by $(-1/4)(1+v_0)$ (for $v_0 \approx -1$), it is easy to obtain the approximate values of the roots

$$p_{1,2} \approx \pm i\left[1 + \frac{3p_0^2\langle\xi^2\rangle}{9+2\langle\xi^2\rangle}\right]. \qquad (25)$$

The equation for the Green's function $G_{ij}(\mathbf{r})$ can be represented in the form (for isotropic fluctuations)

$$G_{ij}^E(\mathbf{r}) = \left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)G_1(r) + \frac{x_i x_j}{r^2}G_2(r);$$

$$G_1(r) = \frac{4\pi}{r}\int_0^\infty g_1^E(k)\left[\sin kr + \frac{\cos kr}{kr} - \frac{\sin kr}{k^2r^2}\right]kdk - \frac{4\pi}{r^2}\int_0^\infty g_2^E(k)\left[\cos kr - \frac{\sin kr}{kr}\right]dk,$$

$$G_2(r) = \frac{4\pi}{r}\int_0^\infty g_2^E(k)\left[\sin kr + \frac{2\cos kr}{kr} - \frac{2\sin kr}{k^2r^2}\right]kdk - \frac{8\pi}{r^2}\int_0 g_1^E(k)\left[\cos kr - \frac{\sin kr}{kr}\right]dk. \qquad (26)$$

In the region of large r (in the wave zone) one can keep in these expressions only terms proportional to r^{-1}, so that

$$G_2(r) \approx \frac{4\pi}{r}\int_0^\infty g_2^E(k)\sin kr\, kdk = \frac{1}{2\pi^2 p_0^2 r}\int_0^\infty \frac{1 + {}^2/_3\langle\xi^2\rangle q(p,p_0)}{1 - {}^4/_3\langle\xi^2\rangle q(p,p_0)}\sin\left(\frac{pr}{l}\right)pdp.$$

Separating the integer part from the fraction in the integrand and carrying out integration in this term, we obtain a term proportional to $\delta(r)/2\pi r^2 = \delta(\mathbf{r})$, which is equal to zero in the region of large r. Moreover, since the function $q(p, p_0)$ is even, we can write, setting $\rho = r/l$,

$$G_2(r) = \frac{3}{8\pi^2 p_0^2 ri}\int_{-\infty}^{\infty}\frac{e^{ip\rho}pdp}{1 - {}^4/_3\langle\xi^2\rangle q(p,p_0)}. \qquad (27)$$

Closing the contour of integration in the upper half plane and finding the residue at the point

$$p = i\left[1 + \frac{3p_0^2\langle\xi^2\rangle}{9+2\langle\xi^2\rangle}\right], \qquad (27a)$$

we find

$$G_2(r) = -\frac{\langle\xi^2\rangle}{(1+{}^2/_9\langle\xi^2\rangle)^2}\frac{1}{4\pi r}\exp\left[-\frac{r}{l}\left(1 + \frac{3p_0^2\langle\xi^2\rangle}{9+2\langle\xi^2\rangle}\right)\right]. \qquad (28)$$

Thus, the longitudinal wave (account of the next terms of the expansion of $q(p, p_0)$ in a series in p_0 and $G_2(r)$ yields a factor e^{iar}) is damped over the correlation radius of the fluctuations of ξ.

Similar calculations for $G_1(r)$ show that the spatial dispersion $\epsilon_{\text{eff}}^{tr}(\omega, \mathbf{k})$ is also unimportant when $r \gg l$. This result is valid in the general case, when the form of the correlation function is not specified. Therefore, when calculating $G_{im}^E(\mathbf{r})$ in expressions for $\epsilon_{\text{eff}}^{tr}$ and $\epsilon_{\text{eff}}^{l}$ we can set $k = 0$. Thus the Green's function of mean field has the form

$$G_{im}^E(\mathbf{r}) = -\left(\delta_{im} + \frac{1}{k_0^2\varepsilon_{\text{eff}}^{tr}(\omega,0)}\frac{\partial^2}{\partial x_i\partial x_m}\right) \times \frac{\exp[ik_0(\varepsilon_{\text{eff}}(\omega,0))^{1/2}r]}{4\pi r} \qquad (29)$$

Neglect of spatial dispersion imposes the following limitations on the value of r for which the formula (29) is valid, as estimates show:

$$l \lll r \lll 60\,\pi l / a_1\langle\xi^2\rangle(k_0 l)^3\varepsilon_0^{3/2}. \qquad (30)$$

Let us write down some relations pertaining to damping of the waves. We consider the case $k_0 l\sqrt{\epsilon_0} \ll 1$ (small-scale fluctuations). The value of $\langle\xi^2\rangle$ cannot exceed unity by much. Therefore, the terms $\langle\xi^2\rangle(k_0 l)^2\epsilon_0$ are also small. Hence, we get for $\epsilon_{\text{eff}}^{l}(\omega, 0)$ and $\epsilon_{\text{eff}}^{tr}(\omega, 0)$

$$\varepsilon_{\text{eff}}^{l}(\omega,0) \approx \varepsilon_{\text{eff}}^{tr}(\omega,0) \approx \varepsilon_0[1 + {}^2/_3\langle\xi^2\rangle(k_0 l)^2\varepsilon_0 + {}^2/_3 i\langle\xi^2\rangle(k_0 l)^3\varepsilon_0^{3/2} + \ldots]. \qquad (31)$$

For a specific calculation of the damping of the wave, it is necessary to know the values of ϵ_0 and $\langle\xi^2\rangle$. We first consider the case of small fluctuations, in which $|\epsilon - \langle\epsilon\rangle|/\langle\epsilon\rangle \ll 1$. From Eqs. (2)

and (4) we get in this case the formula given in[5]:

$$\varepsilon_0 \approx \langle\varepsilon\rangle - \frac{1}{3}\frac{\langle\Delta\varepsilon^2\rangle}{\langle\varepsilon\rangle} - \frac{4}{27}\frac{(\langle\Delta\varepsilon^2\rangle)^2}{\langle\varepsilon\rangle^3} + \ldots \quad (32)$$

In the same approximation,

$$\langle\xi^2\rangle \approx \langle\Delta\varepsilon^2\rangle / \langle\varepsilon\rangle^2. \quad (33)$$

Substituting (32) and (33) in (12) and (18), we get

$$\varepsilon^{l}_{\text{eff}}(\omega, k) = \langle\varepsilon\rangle - \frac{1}{3}\frac{\langle\Delta\varepsilon^2\rangle}{\langle\varepsilon\rangle} - \frac{2\langle\Delta\varepsilon^2\rangle}{\langle\varepsilon\rangle} q(kl, k_0 l\sqrt{\varepsilon_0}) \quad (34)$$

and a similar expression for $\epsilon^{\text{tr}}_{\text{eff}}(\omega, k)$. These formulas, which are valid for $\langle\Delta\epsilon^2\rangle \ll \langle\epsilon^2\rangle$ generalize the well known results of the research of Kaner,[1] obtained without account of spatial dispersion.

We consider these expressions as applied to an inhomogeneous plasma. In connection with the condition $\langle\Delta\epsilon^2\rangle/\langle\epsilon^2\rangle \ll 1$, we should consider the field at frequencies that are sufficiently far removed from the plasma frequency determined by the equation $\omega_0^2 = 4\pi e^2\langle N\rangle/m$. It is then evident, in particular, that it is not possible to consider the problem of plasma oscillations by means of Eq. (34).[4)] At frequencies close to resonance, the plasma is a medium with strong fluctuations of the dielectric permittivity even in the case in which the relative fluctuations of the electron density are small.

Before proceeding to consider an inhomogeneous plasma, we shall pause briefly for the calculation of the permittivity ϵ_0 for dielectrics describable by the Lorenz-Lorentz formula:

$$(\varepsilon - 1) / (\varepsilon + 2) = \gamma, \quad (35)$$

where γ is a quantity proportional to density. Inasmuch as the values $1 < \epsilon < \infty$ are possible, the limits $0 < \gamma < 1$ are possible for γ. It is then clear that it is not possible to regard the fluctuations in γ as Gaussian (formally, this distribution yields $\langle\epsilon\rangle = \infty$).

In order not to violate the condition $0 < \gamma < 1$, we set $\gamma = \alpha^2/(\alpha^2 + a^2)$, where α is a random variable distributed according to the normal law $\langle\alpha\rangle = 0$, $\langle\alpha^2\rangle = \sigma_\alpha^2$, while the quantities a^2 and σ_α^2 are expressed in terms of $\langle\epsilon\rangle$ and $\langle\Delta\epsilon^2\rangle$:

$$\langle\varepsilon\rangle = 1 + 3\sigma_\alpha^2 / a^2, \qquad \langle\Delta\varepsilon^2\rangle = 18\sigma_\alpha^4 / a^4. \quad (36)$$

Substituting $\epsilon = 3/(1 - \gamma) - 2$ in Eq. (14), we get

$$\varepsilon_0 = (\langle\varepsilon\rangle - 1)x^2 - 1/2,$$

where x is the root of the equation

$$1 - \Phi(x) = \frac{4Nx}{\sqrt{\pi}(6Nx^2 - 1)} e^{-x^2} \quad (37)$$

and $\Phi(x)$ is the probability integral, with $N = \sigma_\alpha^2/a^2$. Solving this equation, for example, in the limiting case $N \to 0$ (small fluctuations), we get Eq. (32). As $N \to \infty$, the root of Eq. (37) approaches the value $x = 0.74$. We get for ϵ_0:

$$\varepsilon_0 \sim 0.55\langle\varepsilon\rangle - 1.05. \quad (38)$$

For an increase in the fluctuations of ϵ, the value of ϵ_0 at first decreases [see (32)], and then begins to increase, remaining positive.

3. THE EFFECTIVE DIELECTRIC PERMITTIVITY OF THE PLASMA

The calculation of $\epsilon_{ij}^{\text{eff}}(\omega, k)$ in a cold, inhomogeneous plasma reduces to the determination of the values of ϵ_0 and $\langle\xi^2\rangle$. We set $\epsilon(\omega) = 1 - \alpha N$, where $\alpha = 4\pi e^2/m\omega^2$. We assume that the electron density distribution obeys the law

$$W(N) = \begin{cases} (2\pi\sigma^2)^{-1/2}\exp\left[\dfrac{-(N - \langle N\rangle)^2}{2\sigma^2}\right], & N \geqslant 0 \\ 0, & N < 0 \end{cases}. \quad (39)$$

We assume that the fluctuations of the electron density are small ($\langle\Delta N^2\rangle^{1/2} \ll \langle N\rangle$). This condition allows us to assume that $\langle N\rangle$ does not depend on $\sigma^2 = \langle\Delta N^2\rangle$. Averaging ξ, we get with the help of (39) the following equation for the determination of ϵ_0:[5)]

$$I_+\left(\frac{\langle\varepsilon\rangle + 2\varepsilon_0}{\sigma\alpha}\right) = \frac{\sqrt{2\pi}}{3}\frac{\sigma\alpha}{\varepsilon_0} \quad (40)$$

$$\langle\varepsilon\rangle = 1 - \alpha\langle N\rangle, \qquad I_+(z) = \int_{-\infty}^{\infty}\frac{e^{-x^2/2}}{z - x}dx. \quad (41)$$

We introduce the notation $\langle\epsilon\rangle/\sigma\alpha = x$, $2\epsilon_0/\sigma\alpha = \beta$. We analyze the equation near the plasma frequency ω_0, when one can assume that $x \ll 1$. Expanding $I_+(\beta + x)$ in powers of x, and restricting ourselves to the linear term in x, we get the equation

$$I_+(\beta) \approx \frac{2\sqrt{2\pi}}{3\beta}\left(1 - \frac{\beta x}{2}\right). \quad (42)$$

Equation (42) can be solved by successive approximations, assuming that $|\beta x| \ll 1$. Here, the zeroth approximation in $|\beta x|$ has a purely imaginary root $\beta_0 \approx 1.03i$,[7] and the next approximation makes a real contribution to β_0.

[4)]Therefore the corresponding conclusions of Kalashnikov and Ryazanov[6], on the magnitude of the shift of the plasma frequency because of inhomogeneities, are in error.

[5)]The properties of the integral $I_+(z)$ were considered in [7].

As a result we get for ϵ_0 the following expression:

$$\varepsilon_0 \approx 0.56\langle\varepsilon\rangle + 0.52 i\alpha\sigma, \quad (43)$$

which is valid only at frequencies close to the plasma frequency ($\langle\epsilon\rangle \ll \sigma/\langle N\rangle$).

By expanding the formulas for ϵ_0, we can make explicit the characteristics of the mean field at a frequency close to the plasma frequency. For example, if $k_0 l\sqrt{\epsilon_0} \ll 1$, then

$$\varepsilon_{\text{eff}}^{tr}(\omega, 0) \approx \varepsilon_0 \approx 0.52 i\alpha \langle\Delta N^2\rangle^{1/2}.$$

This expression shows that in this case the mean field attenuates like $e^{-r/\Lambda}$, where

$$\Lambda \approx \frac{2c}{\omega_0} \Big/ \left\{\frac{[\langle\Delta N^2\rangle]^{1/2}}{\langle N\rangle}\right\}^{1/2}.$$

We explained above that the mechanism of spatial dispersion brought about by the presence of inhomogeneities of the medium does not lead to the appearance of longitudinal waves. It remains for us to clear up the problem of longitudinal oscillations of an inhomogeneous plasma. As follows from Eq. (12), $\epsilon_{\text{eff}}^{l}(\omega, 0)$ can vanish as a result of the factor $\epsilon_0(\omega)$. Equation (4) for the determination of ϵ_0 formally excludes the equality $\epsilon_0(\omega) = 0$. However, as Eq. (43) shows, the real part of $\epsilon_0(\omega)$ vanishes for a plasma at the resonance frequency ω_0, while the imaginary part is small. Thus the root of Eq. (4) can be sufficiently small so as to allow one to come as close as desired to the resonance frequency. In this connection, it is clear that the expression (43) can be used for the approximate determination of the shift in the resonance frequency and the damping decrement of the plasma oscillations.[6)] Setting (43) equal to zero, we get (with account of the collisions in the plasma):

$$\omega \approx \omega_0 - \frac{i}{2}\left[\nu_{\text{eff}} + 0.93\frac{[\langle\Delta N^2\rangle]^{1/2}}{\langle N\rangle}\omega_0\right]. \quad (44)$$

[6)]Similar phenomena occur in a reasonant LC circuit without damping, in which either L or C fluctuates. These fluctuations lead to a certain effective damping of the free oscillations, associated with a transfer of the energy of the regular oscillations to the energy of the fluctuation oscillations in the circuit.

Thus the presence of inhomogeneities leads to an increase in the damping decrement of plasma oscillations, which can be interpreted as an increase in the effective collision frequency. We write down an expression for $\langle\xi^2\rangle$ in the case of a plasma. With the help of the distribution (39), we obtain for $\sigma \ll \langle N\rangle$:

$$\langle\xi\rangle^2 = \frac{9}{\sqrt{2\pi}}\left[\sqrt{2\pi} - \frac{9\sqrt{2\pi}\,\varepsilon_0^2}{(\sigma\alpha)^2} + \frac{9\varepsilon_0^2}{(\sigma\alpha)^2}\gamma I_+(\gamma) - \frac{6\varepsilon_0}{\sigma\alpha} I_+(\gamma)\right];$$
$$\gamma = (\langle\varepsilon\rangle + 2\varepsilon_0)/\sigma\alpha. \quad (45)$$

This expression, which takes into account the relation (40), is described in the following simple form:

$$\langle\xi^2\rangle = 9[3\varepsilon_0(\langle\varepsilon\rangle - \varepsilon_0) / (\sigma\alpha)^2 - 1]. \quad (46)$$

Close to resonance, it follows from (43) and (46) that

$$\langle\xi^2\rangle \approx -1.7 - 1.6 i\langle\varepsilon\rangle / \alpha\sigma.$$

We thank M. A. Miller, A. V. Gaponov, and V. L. Ginzburg for discussion of the research and valuable comments.

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Translated by R. T. Beyer
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