NONLINEAR INTERACTION OF WAVES IN A PLASMA

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The nonlinear equations for slowly varying amplitudes and phases of the electric field strength are considered. An expression is obtained for the effective damping decrement, in which the change in the spectrum brought about by the dependence of the distribution function on the slow coordinates and time is taken into account. It is shown that in the approximations of three, four, etc. waves, the plasmon equation is replaced by a system of equations for a larger set of spectral functions, the number of which depends on the number of interacting waves. Stationary solutions of the spectral-function equations are presented.

A system of kinetic equations was obtained in the works of Lenard,[1] Balescu,[2] Silin,[3] and others for the distribution function of charged particles in a spatially homogeneous plasma. This set of equations differs from the Landau equations in that more accurate account is taken of the plasma polarization.

If waves are excited in a plasma to an appreciable extent, the relaxation times of which are comparable with the relaxation times of the distribution functions f_a, then it is not possible to obtain closed equations for f_a (kinetic equations).[4,5]

For states close to equilibrium, one can use the set of equations for the functions f_a and the spectral function of the electric field intensity $(\delta \mathbf{E} \cdot \delta \mathbf{E})_{\omega,\mathbf{k},t}$.[5]

Recently, interest has been strongly developed in the investigation of nonlinear electromagnetic processes in plasma. In the works of Kadomtsev and Petviashvili,[6] Silin,[7] Iordanskiĭ and Kulikovskiĭ[8], more complete equations were obtained for plasmons by different methods in the case of a spatially homogeneous plasma, with account of nonlinear interaction. The works of Vedenov and Velikhov,[9,10] Drummond and Pines,[11] Karpman,[12] Shapiro,[13] and others were devoted to the so-called quasilinear approximation for a set of equations with a self consistent field (the Khokhlov system of equations). These two approximations are in essence different limiting cases.[4,5]

In references[6-13] the following approximations were made in some form or other:

1. The phases of the random oscillations are rapidly changing, i.e., they are established much more rapidly than the amplitudes of the function f_a. As a consequence, only the equation for the intensity (the equation for plasmons) is used.

2. The quadruple and triple correlations are entirely expressed in terms of the pair correlations. In the same way, it is assumed that the nonequilibrium state of the plasma, with account of interaction of the waves, is completely determined by the initial values of the functions f_a and the spectral function of pair correlation of the field.

We consider in the present paper the nonequilibrium turbulent state of the plasma, in which the mean fields are equal to zero. In the account of the nonlinear interaction of the waves, it is assumed that the small quantities are not the fields but the tensors that determine the nonlinear contribution to the induction vector. In this connection, the approach is similar to that which is used in nonlinear optics (see the book of Akhmanov and Khokhlov[14]). In the first section, equations are obtained for the random complex amplitudes of the field with account of change in the spectrum. These equations allow us to find the equations for the real amplitudes and phases. Then the stationary solution of the equations for the correlation functions is considered in the cases of two-, three-, four-wave interactions. The nonstationary equations for the correlation functions are considered. It is shown that the equation for the spectral function of pair correlation can be obtained only in the case of two-wave interactions. In other cases, a set of equations is obtained for a larger number of correlation functions. Under the condition of quasi-stationarity of the correlations of the field, closed equations, which take into account the nonlinear interaction of the waves, are obtained for the functions f_a (kinetic equations).

As in the previous researches,[4,5] we shall determine the microscopic state of the plasma by specifying the microscopic phase densities

$$N_a(\mathbf{q}, \mathbf{p}, t) = \sum_i \delta(\mathbf{q} - \mathbf{q}_{ia}(t))\,\delta(\mathbf{p} - \mathbf{p}_{ia}(t))$$

for each component a and the microfields E^M and H^M. If there are no external fields, then the mean fields $\overline{E^M} = \overline{H^M} = 0$. The equations for the functions

$$f_a, \quad \delta N_a = N_a - n_a f_a,$$

$$\delta\mathbf{E} = \mathbf{E}^M - \overline{\mathbf{E}^M}, \quad \delta\mathbf{B} = \mathbf{H}^M - \overline{\mathbf{H}^M}$$

have in the nonrelativistic case the form

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\frac{\partial f_a}{\partial \mathbf{q}} = -e_a\frac{\partial}{\partial \mathbf{p}}\overline{\delta N_a \delta\mathbf{E}}, \tag{1}$$

$$\frac{\partial \delta N_a}{\partial t} + \mathbf{v}\frac{\partial \delta N_a}{\partial \mathbf{q}} + e_a n_a \delta\mathbf{E}\frac{\partial f_a}{\partial \mathbf{p}} = -e_a\frac{\partial}{\partial \mathbf{p}}\delta(\delta N_a \delta\mathbf{E}), \tag{2}$$

$$\mathrm{rot}\,\delta\mathbf{B} = \frac{1}{c}\frac{\partial \delta\mathbf{E}}{\partial t} + 4\pi\sum_a e_a \int \mathbf{v}\,\delta N_a\, d\mathbf{p}, \qquad \mathrm{div}\,\delta\mathbf{B} = 0, \tag{3*}$$

$$\mathrm{rot}\,\delta\mathbf{E} = -\frac{1}{c}\frac{\partial \delta\mathbf{B}}{\partial t}, \qquad \mathrm{div}\,\delta\mathbf{E} = 4\pi\sum_a e_a \int \delta N_a\, d\mathbf{p}. \tag{4}$$

On the right hand side of Eq. (2),

$$\delta(\delta N_a \delta\mathbf{E}) = \delta N_a \delta\mathbf{E} - \overline{\delta N_a \delta\mathbf{E}}. \tag{5}$$

1. NONLINEAR EQUATIONS FOR THE RANDOM DEVIATIONS δE

We consider the case of a Coulomb plasma. We express the random deviation δN_a in terms of δE by means of Eq. (2). We represent δN_a in the form

$$\delta N_a = \delta N_a^{\,l} + \delta N_a^{\,nl}. \tag{1.1}$$

The first term on the right hand side depends linearly on δE, while the second is nonlinear. Using (2), we write down the stationary expression for δN_a^l in the form

$$\delta N_a^{\,l}(\mathbf{q}, \mathbf{p}, t) = -e_a n_a \int_0^\infty \delta\mathbf{E}(\mathbf{q} - \mathbf{v}\tau, t - \tau) \times \left(\frac{\partial f_a(\mathbf{p}, t - \tau, \mathbf{q})}{\partial \mathbf{p}}\right)_{\mathbf{q} = \mathbf{q} - \mathbf{v}\tau} d\tau. \tag{1.2}$$

The initial value of δN_a here is not written out since it is important only for the determination of the spontaneous term, which does not contain the spectral function. It is unimportant for large departures from the equilibrium state.

To account for the nonstationarity and inhomogeneity of the random process in a plasma, we assume that δN_a and δE depend on fast and slow variables as functions of time and coordinates; for example,

$$\delta\mathbf{E} = \delta\mathbf{E}(\mu t, \mu\mathbf{q}, t, \mathbf{q}).$$

We shall expand in a Fourier integral in the fast variable:

$$\delta\mathbf{E}(\mu t, \mu\mathbf{q}, t, \mathbf{q}) = \frac{1}{(2\pi)^4} \times \int \delta\mathbf{E}(\mu t, \mu\mathbf{q}, \omega, \mathbf{k})\, e^{-i\omega t + i\mathbf{k}\mathbf{q}}\, d\omega\, d\mathbf{k}. \tag{1.3}$$

We shall see below that, in the zeroth approximation in μ, the function $\delta E(\mu t, \mu q, \omega, k)$ differs from zero only for ω and k which satisfy the equation

$$\varepsilon^0(\omega, \mathbf{k}, \mu t, \mu\mathbf{q}) = 0, \qquad \varepsilon_{ij}^{\,0} = k_i k_j k^{-2} \varepsilon^0. \tag{1.4}$$

Here ϵ_{ij}^0 is the real part of the dielectric permittivity tensor of the Coulomb plasma in the linear approximation. The dielectric permittivity depends on the slow variables through the distribution functions $f_a(\mu t, \mu q, p)$. If we take this dependence into account, then Eq. (1.4) will connect the two functions $\omega = \omega(\mu t, \mu q)$ and $k = k(\mu t, \mu q)$. In order to take into account the dependence of ω and k on μt and μq, we use in place of (1.3) the equation

$$\delta E(\mu t, \mu\mathbf{q}, t, \mathbf{q}) = \frac{1}{(2\pi)^4} \times \int \delta E(\mu t, \mu\mathbf{q}, \omega, \mathbf{k})\, e^{i\Psi(t, \mathbf{q})}\, d\omega\, d\mathbf{k}, \tag{1.5}$$

which serves as a definition of the complex amplitudes. Here

$$-\partial\Psi/\partial t = \omega(\mu t, \mu\mathbf{q}), \qquad \partial\Psi/\partial\mathbf{q} = \mathbf{k}(\mu t, \mu\mathbf{q}). \tag{1.6}$$

In the determination of the equations for the functions $\delta E(\mu t, \mu q, \omega, k)$, we regard the arguments ω, k in them, by definition, as constant parameters.

We substitute the expansion (1.5) in Eq. (1.2), expand in a series in τ, and keep all terms of order μ. Equating the coefficients of the expansion, we obtain the following expression:

$$\delta N_a^{\,l}(\mu t, \mu\mathbf{q}, \omega, \mathbf{k}) = \gamma_j^{\,a}\delta E_j(\mu t, \mu\mathbf{q}, \omega, \mathbf{k}) + e_a n_a\Big\{\frac{\partial}{\partial\omega}\frac{1}{\omega - kv}\frac{\partial}{\partial t}\Big(\delta\mathbf{E}\frac{\partial f_a}{\partial\mathbf{p}}\Big) + \frac{1}{2}\Big[\Big(\frac{\partial\omega}{\partial t}\frac{\partial}{\partial\omega} + \frac{\partial\mathbf{k}}{\partial t}\frac{\partial}{\partial\mathbf{k}}\Big)\frac{\partial}{\partial\omega}\frac{1}{\omega - \mathbf{k}\mathbf{v}}\Big] \times \Big(\delta\mathbf{E}\frac{\partial f_a}{\partial\mathbf{p}}\Big) - \frac{\partial}{\partial\mathbf{k}}\frac{1}{\omega - \mathbf{k}\mathbf{v}}\frac{\partial}{\partial\mathbf{q}}\Big(\delta\mathbf{E}\frac{\partial f_a}{\partial\mathbf{p}}\Big) - \frac{1}{2}\Big[\Big(\frac{\partial\omega}{\partial q_i}\frac{\partial}{\partial\omega} + \frac{\partial k_j}{\partial q_i}\frac{\partial}{\partial k_j}\Big)\frac{\partial}{\partial k_i}\frac{1}{\omega - \mathbf{k}\mathbf{v}}\Big]\Big(\delta\mathbf{E}\frac{\partial f_a}{\partial\mathbf{p}}\Big)\Big\}. \tag{1.7}$$

In order to express δN_a^{nl} in terms of δE, we proceed in the following fashion. We assume that

*rot = curl.

the nonlinear term is of order of μ. On this basis in first approximation in μ, one does not have to take into account derivatives with respect to μt and $\mu\mathbf{q}$. By eliminating the function δN_a from the right hand side of Eq. (2) we can represent δN_a in the form of a series in $\delta\mathbf{E}$ with the help of the same equation.

We now write down the first two terms of the series for δN_a^{nl}:

$$\delta N_a^{\mathrm{nl}} = \frac{1}{(2\pi)^4}\int d\Omega'\, d\Omega''\, \delta(\Omega-\Omega'-\Omega'')\gamma_{jk}{}^a(\Omega,\Omega'') \times \delta(\delta E_j(\Omega')\delta E_k(\Omega'')) + \frac{1}{(2\pi)^8}\int d\Omega'\, d\Omega''\, d\Omega''' \times \delta(\Omega-\Omega'-\Omega''-\Omega''')\gamma^a_{jkl}(\Omega,\Omega''+\Omega''',\Omega''') \times \delta[\delta E_j(\Omega')\delta(\delta E_k(\Omega'')\delta E_l(\Omega'''))]. \quad (1.8)$$

Here Ω is the set ω, $\mathbf{k}$; $d\Omega = d\omega d\mathbf{k}$,

$$\gamma_j{}^a(\Omega) = -\frac{ie_a n_a}{\omega-\mathbf{kv}}\frac{\partial f_a}{\partial p_j}, \qquad f_a = f_a(\mu t, \mu\mathbf{q}, \mathbf{p}), \quad (1.9)$$

$$\gamma_{jk}{}^a(\Omega,\Omega') = -\frac{e_a{}^2 n_a}{\omega-kv}\frac{\partial}{\partial p_j}\frac{1}{\omega'-\mathbf{k}'\mathbf{v}}\frac{\partial f_a}{\partial p_k}, \quad (1.10)$$

$$\gamma^a_{jkl}(\Omega,\Omega',\Omega'') = \frac{ie_a{}^3 n_a}{\omega-\mathbf{kv}}\frac{\partial}{\partial p_j}\frac{1}{\omega'-\mathbf{k}'\mathbf{v}}\frac{\partial}{\partial p_k}\frac{1}{\omega''-\mathbf{k}''\mathbf{v}}\frac{\partial f_a}{\partial p_l}. \quad (1.11)$$

When account is taken of the transverse electromagnetic field in the expressions (1.7)—(1.11) we get $\delta\mathbf{E} \rightarrow \delta\mathbf{E} + c^{-1}[\mathbf{v}\times\delta\mathbf{B}]$.

In order to obtain an equation for the function $\delta\mathbf{E}$, we make use of the fact that, for example in the case of a Coulomb plasma, an equation can be obtained from (3) which has the following form for the Fourier components:

$$\left(\frac{\partial\delta\mathbf{D}}{\partial t}\right)_{\mu t\ \mu q,\,\omega,\,k} = -i\omega\delta\mathbf{E} + \frac{\partial\delta\mathbf{E}}{\partial\mu t} + 4\pi\sum_a e_a\int \mathbf{v}\delta N_a\, d\mathbf{p} = 0. \quad (1.12)$$

Eliminating δN_a here by means of (1.7) and (1.8), we obtain the desired equation. Before writing it out, however, we shall consider the following.

In the solution of the equations for $\delta\mathbf{E}$ or the equations for the spectral functions, we shall assume the small quantity to be not the field, but the functions χ_{ijk} and θ_{ijkl} [14] which characterize the role of the nonlinear interaction. We divide the real part of the dielectric permittivity into two parts:

$$\varepsilon' = \varepsilon^0 + \Delta\varepsilon', \qquad \Delta\varepsilon' = \frac{\partial\varepsilon^0}{\partial\omega}\Delta\omega + \frac{\partial\varepsilon^0}{\partial\mathbf{k}}\Delta\mathbf{k}.$$

Here ϵ^0 is the linear approximation for ϵ', and $\Delta\epsilon'$ is the change due to the nonlinear interaction. We shall assume that

$$\Delta\varepsilon' \sim \varepsilon'' \sim \chi_{ijk} \sim \theta_{ijkl} \sim \mu. \quad (1.13)$$

Of course, there can be several small parameters. This allows us to carry out a further simplification of the equations obtained below.

In the zeroth approximation in μ, we get Eq. (1.4) from (1.12) with account of (1.7). Its expanded form is

$$1 + \sum_a \frac{4\pi e_a{}^2 n_a}{k^2}\mathrm{P}\int\frac{\mathbf{k}\partial f_a/\partial\mathbf{p}}{\omega-\mathbf{kv}}d\mathbf{p} = 0, \quad (1.4')$$

where P is the symbol for the principal value.

The equation of first approximation in μ can be written in the form

$$\frac{\partial\delta\mathbf{E}}{\partial t} + \mathbf{v}_{\mathrm{gr}}\frac{\partial\delta\mathbf{E}}{\partial\mathbf{q}} = -\hat{\gamma}_{\mathrm{eff}}\,\delta\mathbf{E} + i\omega\delta\mathbf{D}^{\mathrm{nl}}/\omega\frac{\partial\varepsilon^0}{\partial\omega}. \quad (1.14)$$

In this equation

$$\mathbf{v}_{\mathrm{gr}} = -\frac{\partial\varepsilon^0}{\partial\mathbf{k}}\Big/\frac{\partial\varepsilon^0}{\partial\omega}$$

is the group velocity:

$$\hat{\gamma}_{\mathrm{eff}}(\mu t, \mu\mathbf{q}, \omega, \mathbf{k}) = \gamma\delta_{ij} + \Gamma_{ij} \quad (1.15)$$

is the effective damping decrement. It consists of two parts: the ordinary damping decrement $\gamma = \epsilon''(\partial\epsilon^0/\partial\omega)^{-1}$ and an additional decrement (or increment) brought about by the dependence of ϵ^0 on the variables μt and $\mu\mathbf{q}$.

The following expression is obtained for Γ_{ij}:

$$\Gamma_{ij} = \left[\frac{\partial}{\partial t}\frac{\partial\omega\varepsilon_{ij}{}^0}{\partial\omega} + \frac{1}{2}\left(\frac{\partial\omega}{\partial t}\frac{\partial}{\partial\omega} + \frac{\partial\mathbf{k}}{\partial t}\frac{\partial}{\partial\mathbf{k}}\right) \times \frac{\partial\omega\varepsilon_{ij}{}^0}{\partial\omega} - \frac{\partial}{\partial\mathbf{q}}\frac{\partial\omega\varepsilon_{ij}{}^0}{\partial\mathbf{k}} - \frac{1}{2}\left(\frac{\partial\omega}{\partial\mathbf{q}}\frac{\partial}{\partial\omega} + \frac{\partial k_i}{\partial\mathbf{q}}\frac{\partial}{\partial k_i}\right) \times \frac{\partial\omega\varepsilon_{ij}{}^0}{\partial\mathbf{k}}\right]\Big/\omega\frac{\partial\varepsilon^0}{\partial\omega}. \quad (1.16)$$

δD^{nl} is the random deviation of the induction vector, brought about by the nonlinear interaction. It can be represented in the form

$$\delta D_i{}^{\text{нл}} = \frac{1}{(2\pi)^4}\int d\Omega'\, d\Omega''\, \delta(\Omega-\Omega'-\Omega'')\delta(\delta E_j(\Omega') \times \delta E_k(\Omega''))\chi_{ijk}(\Omega,\Omega'')\,\frac{1}{(2\pi)^8}\int d\Omega'\, d\Omega''\, d\Omega''' \times \delta(\Omega-\Omega'-\Omega''-\Omega''')\theta_{ijkl}(\Omega,\Omega''+\Omega''',\Omega''') \times \delta[\delta E_j(\Omega')\delta(\delta E_k(\Omega'')\delta E_l(\Omega'''))]. \quad (1.17)$$

The tensors χ_{ijk} and θ_{ijkl} are determined by the relations

$$\chi_{ijk} = \sum_a \frac{4\pi i e_a}{\omega} \int v_i \gamma_{jk}{}^a \, d\mathbf{p},$$

$$\theta_{ijkl} = \sum_a \frac{4\pi i e_a}{\omega} \int v_i \gamma_{jkl}^a \, d\mathbf{p}. \qquad (1.18)$$

If we do not take into account the effect of the change of the distribution functions f_a on the spectrum, then $\hat{\Gamma} = 0$. In this case Eq. (1.14) will correspond to the dynamic equation in the work of Karpman.[12] In order to show how important it is to take account of $\hat{\Gamma}$, we shall consider some examples.

First we note that the derivatives in the expression (1.16) are

$$\partial\omega/\partial t, \quad \partial\mathbf{k}/\partial t, \quad \partial\omega/\partial q, \quad \partial k_j/\partial q_i.$$

How shall we determine them? To this end, we turn to Eq. (1.14). With account of (1.6) it is the eikonal equation; therefore, all the derivatives entering into the expression for Γ can be found if the function $\Psi(\mathbf{q}, t)$ is known. However, in many cases, a knowledge of the function $\Psi(\mathbf{q}, t)$ itself is not obligatory. Actually, two equations follow from Eq. (1.4):

$$\frac{\partial\varepsilon}{\partial t} + \frac{\partial\varepsilon}{\partial\omega}\frac{\partial\omega}{\partial t} + \frac{\partial\varepsilon}{\partial\mathbf{k}}\frac{\partial\mathbf{k}}{\partial t} = 0,$$

$$\frac{\partial\varepsilon}{\partial\mathbf{q}} + \frac{\partial\varepsilon}{\partial\omega}\frac{\partial\omega}{\partial\mathbf{q}} + \frac{\partial\varepsilon}{\partial k_i}\frac{\partial k_i}{\partial\mathbf{q}} = 0, \qquad (1.19)$$

which allow us to find two relations between these derivatives.

We shall consider special cases.

1. In the absence of spatial dispersion, we find from (1.19)

$$\frac{\partial\omega}{\partial t} = -\frac{\partial\varepsilon}{\partial t}\Big/\frac{\partial\varepsilon}{\partial\omega}, \qquad \frac{\partial\omega}{\partial\mathbf{q}} = -\frac{\partial\varepsilon}{\partial\mathbf{q}}\Big/\frac{\partial\varepsilon}{\partial\omega}. \qquad (1.20)$$

2. The values of k are given. From (1.19), we find the derivatives $\partial\omega/\partial t$ and $\partial\omega/\partial\mathbf{q}$.

3. The frequency ω is a constant, the vector k depends only on a single variable, for example, x. It then follows from (1.19) that

$$\frac{\partial k_x}{\partial x} = -\frac{\partial\varepsilon}{\partial x}\Big/\frac{\partial\varepsilon}{\partial k_x}. \qquad (1.21)$$

This case has been treated in detail in the literature (see the review of Rukhadze and Silin[15]).

We now consider examples of the determination of Γ. We begin with the case of a spatially homogeneous plasma in which $\Gamma_{ij} = k_i k_j k^{-2}\Gamma$.

A. $\epsilon^0 = 1 - \omega_L^2(\mu t)/\omega^2$. From (1.16), we find

$$\Gamma = \left[\frac{\partial}{\partial t}\frac{\partial\omega\varepsilon^0}{\partial\omega} + \frac{1}{2}\frac{\partial\omega}{\partial t}\frac{\partial^2\omega\varepsilon^0}{\partial\omega^2}\right]\Big/\omega\frac{\partial\varepsilon^0}{\partial\omega} \qquad (1.22)$$

$$= \frac{1}{2\omega_L}\frac{d\omega_L}{dt}.$$

B. In the case of ionic sound, we have

$$\varepsilon^0 = \frac{1}{a^2} - \frac{\omega_{Li}^2(\mu t)}{\omega^2}, \qquad a^2(\mu t) = r_{de}^2 k^2.$$

From (1.22) we find

$$\Gamma = \frac{1}{2}\frac{a^3}{\omega_{Li}}\frac{d}{dt}\left(\frac{\omega_{Li}}{a^3}\right). \qquad (1.23)$$

We represent δE in the form

$$\delta\mathbf{E} = |\delta\mathbf{E}(\mu t, \mu\mathbf{q}, \omega, \mathbf{k})| \, e^{i\varphi(\mu t, \mu q, \omega, k)}, \qquad (1.24)$$

where $|\delta E|$ and φ are the amplitude and phase at the given values of ω and k. For example B, we get the following equation for the amplitudes and phases from (1.14) in the linear approximation:

$$\frac{d}{dt}|\delta\mathbf{E}| = -\frac{\omega_{Li}a^3}{2}\varepsilon''|\delta\mathbf{E}| - \frac{a^3}{2\omega_{Li}}\frac{d}{dt}\left(\frac{\omega_{Li}}{a^3}\right)|\delta\mathbf{E}|,$$

$$\frac{d\varphi}{dt} = 0. \qquad (1.25)$$

Similar equations for the example A have the form

$$\frac{d}{dt}|\delta\mathbf{E}| = -\frac{\omega_L}{2}\varepsilon''|\delta\mathbf{E}| - \frac{1}{2\omega_L}\frac{d\omega_L}{dt}|\delta\mathbf{E}|, \quad \frac{d\varphi}{dt} = 0. \qquad (1.25')$$

The corresponding adiabatic invariants are the expressions

$$\text{A. } \omega_L|\delta\mathbf{E}|^2; \qquad \text{Б. } \omega_{Li}a^{-3}|\delta\mathbf{E}|^2. \qquad (1.26)$$

It follows from (1.25) and (1.25′) that the phases do not change in the linear approximation.

C. We now consider the case of spatially inhomogeneous plasma:

$$\omega = \text{const}, \quad k \| x, \quad k = k(x),$$

$$\varepsilon^0 = 1 - \frac{\omega_L^2(x)}{\omega^2} - \frac{3r_d^2(x)\,\omega_L^4(x)}{\omega^4}k^2.$$

It follows from (1.14) that the following quantity enters in the equation for the function δE:

$$\frac{\Gamma}{v_{gr}} = \left[\frac{\partial}{\partial x}\frac{\partial\varepsilon^0}{\partial k} + \frac{1}{2}\frac{\partial k}{\partial x}\frac{\partial^2\varepsilon^0}{\partial k^2}\right]\Big/\frac{\partial\varepsilon^0}{\partial k}.$$

Using the second expression of (1.21), we get the following equations for the amplitude and phase in the linear approximation:

$$\frac{d|\delta E|}{dx} = -\left(\frac{\varepsilon''}{\partial\varepsilon'/\partial k} + \left(2a^3\sqrt{1 - \frac{\omega_L^2}{\omega^2}}\right)^{-1}\right.$$

$$\left.\times\frac{d}{dx}\left(a^3\sqrt{1 - \frac{\omega_L^2}{\omega^2}}\right)\right)|\delta E|, \qquad d\varphi/dx = 0,$$

where $a^2 = 3r_d^2\omega_L^4/\omega^4$. It then follows that the expression $(1 - \omega_L^2/\omega^2)^{1/2}a^3|\delta E|^2$ is adiabatically invariant.

For constant pressure, when $r_d^2 \omega_L^4$ = const, the expression $(1 - \omega_L^2/\omega^2)^{1/2} |\delta E|^2$ is adiabatically invariant.

2. STATIONARY AND HOMOGENEOUS CORRELATIONS OF THE ELECTRIC FIELD

In a stationary and homogeneous case,

$$\delta \mathbf{E}(\mu t, \mu \mathbf{q}, \omega, \mathbf{k}) = |\delta \mathbf{E}(\omega, \mathbf{k})| \exp\{-i\Delta\omega t + i\Delta \mathbf{k}\mathbf{q}\}, \quad (2.1)$$

where $\Delta\omega$ and $\Delta \mathbf{k}$ are the changes of ω and $\mathbf{k}$ brought about by the nonlinear interaction. We substitute this expression in Eq. (1.14). From the condition of stationarity and homogeneity it follows that the vector $\delta \mathbf{D}^{nl}$ has the form

$$\delta \mathbf{D}^{nl}(\mu t, \mu \mathbf{q}, \omega, \mathbf{k}) = \delta \mathbf{D}^{nl}(\omega, \mathbf{k}) \exp\{-i\Delta\omega t + i\Delta \mathbf{k}\mathbf{q}\}; \quad (2.2)$$

$\delta \mathbf{D}^{nl}(\omega, \mathbf{k})$ is a complex function. Taking this into account, we get from Eq. (1.14)

$$(-i\Delta\varepsilon' + \varepsilon''_{\text{eff}}) |\delta \mathbf{E}| = i\delta \mathbf{D}^{nl}(\omega, \mathbf{k}). \quad (2.3)$$

Here,

$$\Delta\varepsilon' = (\Delta\omega - v_{gr}\Delta \mathbf{k}) \partial\varepsilon^0/\partial\omega, \quad \varepsilon''_{\text{eff}} = \gamma_{\text{eff}}\, \partial\varepsilon^0/\partial\omega. \quad (2.4)$$

If we multiply Eq. (2.3) by $|\delta E|$ and average, then we obtain an equation for the spectral function. However, this equation will not be closed, because of the nonlinearity of the function $\delta \mathbf{D}^{nl}$. For an approximate solution of this set of intermeshing equations we make use of the following circumstance. In the zeroth approximation in μ, the function $\delta \mathbf{E}(\omega, \mathbf{k})$ differs from zero only for values of ω and $\mathbf{k}$ which are related by Eq. (1.4). On this basis, it is natural to seek a solution for the function $\delta \mathbf{E}(\omega, \mathbf{k})$ in the form

$$\delta \mathbf{E}(\omega, \mathbf{k}) = \sum_{1 \leqslant \alpha \leqslant n} \delta(\omega - \omega_\alpha)\delta(\mathbf{k} - \mathbf{k}_\alpha)\mathbf{E}_\alpha;$$

$$\sum_\alpha \omega_\alpha = 0, \qquad \sum_\alpha \mathbf{k}_\alpha = 0, \quad (2.5)$$

where n is the number of characteristic waves. The approximation which is used here consists in the fact that the interaction of a small number of waves is taken into account. We shall consider two special cases: the four-wave and the three-wave interactions.

Four-wave interaction will be considered for the case of an isotropic plasma when $f = f_a(p)$. If there is no spatial dispersion, then it follows from Eqs. (1.10), and (1.18) that the tensor χ_{ijk} is equal to zero. The term containing this tensor can be neglected also in the case of a weak spatial dispersion. Assuming this condition for simplicity, we omit the term with χ_{ijk} in Eq. (1.17).

We substitute the expression (2.1) in Eq. (1.14) with account of (2.2). As a result, the following conditions should be satisfied:

$$\Delta\omega_\alpha = \Delta\omega_\beta + \Delta\omega_\gamma + \Delta\omega_\delta, \quad \Delta \mathbf{k}_\alpha = \Delta \mathbf{k}_\beta + \Delta \mathbf{k}_\gamma + \Delta \mathbf{k}_\delta.$$

As a result, we get a set of equations for the functions $|E_\alpha|$ (we shall omit the sign of the modulus):

$$(\Delta\varepsilon'(\Omega_\alpha) + i\varepsilon''(\Omega_\alpha))E_\alpha = -\theta_{\alpha\beta\gamma\delta}E_\beta E_\gamma E_\delta. \quad (2.6)$$

Summation over repeated Greek indices is not carried out. In this equation,

$$\theta_{\alpha\beta\gamma\delta} = \{k_{i\alpha}{}^0 k_{j\beta}{}^0 k_{k\gamma}{}^0 k_{l\delta}{}^0 \theta_{ijkl}(\Omega_\alpha, \Omega_\gamma + \Omega_\delta, \Omega_\delta)\}_{\beta\gamma\delta},$$

$$k^0 = \mathbf{k}/|k| \quad (2.7)$$

(the brackets $\{\ \}_{\beta\gamma\delta}$ denote symmetrization with respect to the indices $\beta\gamma\delta$).

From (2.6), we obtain a closed set of equations for the pair and quadruple correlations:

$$(\Delta\varepsilon'(\Omega_\alpha) + i\varepsilon''(\Omega_\alpha))(EE)_\alpha = -\theta_{\alpha\beta\gamma\delta}(EEEE)_{\alpha\beta\gamma\delta},$$

$$(\Delta\varepsilon'(\Omega_\alpha) + i\varepsilon''(\Omega_\alpha))(EEEE)_{\alpha\beta\gamma\delta}$$

$$= -\theta_{\alpha\beta\gamma\delta}(EE)_\beta(EE)_\gamma(EE)_\delta. \quad (2.8)$$

The quadruple correlations cannot be broken up into pair correlations if all the Ω_α are different.

Equating the real and imaginary parts, we obtain a set of four equations of which one is the consequence of the other three. Solving these equations, we find the desired functions:

$$(EEEE)_{\alpha\beta\gamma\delta} = \left(\frac{\varepsilon''(\Omega_\alpha)\varepsilon''(\Omega_\beta)\varepsilon''(\Omega_\gamma)\varepsilon''(\Omega_\delta)}{\theta''_{\alpha\beta\gamma\delta}\,\theta''_{\beta\gamma\delta\alpha}\,\theta''_{\gamma\delta\alpha\beta}\,\theta''_{\delta\alpha\beta\gamma}}\right)^{1/2}, \quad (2.9)$$

$$(EE)_\alpha = -\frac{\theta''_{\alpha\beta\gamma\delta}}{\varepsilon''(\Omega_\alpha)}(EEEE)_{\alpha\beta\gamma\delta}, \quad \Delta\varepsilon' = \frac{\theta'_{\alpha\beta\gamma\delta}}{\theta''_{\alpha\beta\gamma\delta}}\varepsilon''(\Omega_\alpha). \quad (2.10)$$

The conditions for which there exists a stationary solution that differs from zero are evident from these solutions.

A particular case of the four-wave interaction is the two-wave interaction. Only in this case do the closed equations for the pair correlations follow immediately from Eqs. (2.6).

We now consider the case of the three-wave interaction. We keep the first nonvanishing term in the expansion of (1.17). In place of (2.6), we obtain the set of equations

$$(\Delta\varepsilon'(\Omega_\alpha) + i\varepsilon''(\Omega_\alpha))E_\alpha = -\chi_{\alpha\beta\gamma}E_\beta E_\gamma. \quad (2.11)$$

Here

$$\chi_{\alpha\beta\gamma} = \{k_{i\alpha}{}^0 k_{j\beta}{}^0 k_{k\gamma}{}^0 \chi_{ijk}(\Omega_\alpha, \Omega_\gamma)\}_{\beta\gamma}. \quad (2.12)$$

Using these equations, we obtain a closed set of equations for the functions $(EE)_\alpha$, $(EEE)_{\alpha\beta\gamma}$. The solution of this set of equations has the form

$$(EEE)_{\alpha\beta\gamma} = -\frac{\varepsilon''(\Omega_\alpha)\,\varepsilon''(\Omega_\beta)\,\varepsilon''(\Omega_\gamma)}{\chi''_{\alpha\beta\gamma}\chi''_{\beta\gamma\alpha}\chi''_{\gamma\alpha\beta}}, \quad (2.13)$$

$$(EE)_\alpha = -\frac{\chi''_{\alpha\beta\gamma}}{\varepsilon''(\Omega_\alpha)}(EEE)_{\alpha\beta\gamma}, \quad \Delta\varepsilon'(\Omega_\alpha) = \frac{\chi'_{\alpha\beta\gamma}}{\chi''_{\alpha\beta\gamma}}\,\varepsilon''(\Omega_\alpha). \quad (2.14)$$

3. NONSTATIONARY EQUATIONS FOR THE SPECTRAL FUNCTIONS OF THE FIELD IN A SPATIALLY HOMOGENEOUS PLASMA

The stationary equations in the general case are complicated since, in addition to the correlation amplitudes, there also appear phase correlations and mutual correlations of amplitudes and phases of the random waves. We consider here the special case in which the values of $\Delta\epsilon'$ determined by Eqs. (2.10) and (2.14) are much smaller than ϵ''. Then, in first approximation in μ, we can disregard the slow change in phase. With this condition for the four-wave interaction [again we omit the term with χ_{ijk} in (1.17)] we obtain the following set of equations:

$$\frac{\partial}{\partial t}(EE)_\alpha = -2\gamma_{\text{eff}}(\Omega_\alpha)(EE)_\alpha - 2D_{\alpha\beta\gamma\delta}(EEEE)_{\alpha\beta\gamma\delta},$$

$$D_{\alpha\beta\gamma\delta} = \frac{\theta''_{\alpha\beta\gamma\delta}}{\partial\varepsilon^0/\partial\omega_\alpha}; \quad (3.1)$$

$$\frac{\partial}{\partial t}(EEEE)_{\alpha\beta\gamma\delta} = -(\gamma_{\text{eff}}(\Omega_\alpha) + \gamma_{\text{eff}}(\Omega_\beta) + \gamma_{\text{eff}}(\Omega_\gamma) + \gamma_{\text{eff}}(\Omega_\delta))(EEEE)_{\alpha\beta\gamma\delta} - \{D_{\alpha\beta\gamma\delta}(EE)_\beta(EE)_\gamma(EE)_\delta + D_{\beta\gamma\delta\alpha}(EE)_\gamma(EE)_\delta(EE)_\alpha + D_{\gamma\delta\alpha\beta}(EE)_\delta(EE)_\alpha(EE)_\beta + D_{\delta\alpha\beta\gamma}(EE)_\alpha(EE)_\beta(EE)_\gamma\}. \quad (3.2)$$

The stationary solution of these equations is identical with (2.9) and (2.10).

The corresponding equations for the three interacting waves have the form

$$\frac{\partial}{\partial t}(EE)_\alpha = -2\gamma_{\text{eff}}(EE)_\alpha - 2\Gamma_{\alpha\beta\gamma}(EEE)_{\alpha\beta\gamma},$$

$$\Gamma_{\alpha\beta\gamma} = \chi''_{\alpha\beta\gamma} \Big/ \frac{\partial\varepsilon^0}{\partial\omega_\alpha}; \quad (3.3)$$

$$\frac{\partial}{\partial t}(EEE)_{\alpha\beta\gamma} = -(\gamma_{\text{eff}}(\Omega_\alpha) + \gamma_{\text{eff}}(\Omega_\beta) + \gamma_{\text{eff}}(\Omega_\gamma)) \times (EEE)_{\alpha\beta\gamma} - (\Gamma_{\alpha\beta\gamma}(EE)_\beta(EE)_\gamma + \Gamma_{\beta\gamma\alpha}(EE)_\gamma \times (EE)_\alpha + \Gamma_{\gamma\alpha\beta}(EE)_\alpha(EE)_\beta). \quad (3.4)$$

Thus a closed equation for the plasmons (pair correlations of the field) exists only for two-wave interactions.

4. KINETIC EQUATIONS FOR THE DISTRIBUTION FUNCTIONS WITH ACCOUNT OF NONLINEAR INTERACTION OF THE WAVES

If the distribution functions f_a are not given, then the equation for the spectral functions of the field is not closed, since the tensors ϵ_{ij}, χ_{ijk}, and θ_{ijkl} depend on the functions f_a.

In the case of three-wave interaction, a closed set of equations is obtained for the functions f_a, $(EE)_\alpha$, $(EEE)_{\alpha\beta\gamma}$, in the case of four-wave interaction—for the functions f_a, $(EE)_\alpha$, $(EEEE)_{\alpha\beta\gamma\delta}$ etc.[5].

From the results in these sets of equations, there follow the laws of conservation of the number of particles, of the total energy of the particles, and of the total momentum. For example, the law of conservation of energy can be written in the form

$$\sum_a n_a \int \frac{p^2}{2m_a} f_a d\mathbf{p} + \frac{1}{8\pi} \sum_{1\leqslant\alpha\leqslant n} (EE)_\alpha.$$

For the values of ω_α and k_α, for which the time of establishing the spectral functions of the wave is much shorter than the time of establishing the distribution function f_a, one can use the quasistationary solution of Eqs. (3.1)—(3.4). They differ from the solutions (2.9), (2.10), (2.13), and (2.14) by replacing ϵ'' with ϵ''_{eff}. Eliminating the spectral functions, we obtain a closed set of equations for the distribution functions f_a with account of the nonlinear interaction of the plasma waves.

Like the collision integral J_a^{col}, the integral J_a^{i}, which takes into account the contribution of the waves, possesses the properties

$$\sum_a \int \varphi_a(\mathbf{p}) J_a{}^{\text{i}}\, d\mathbf{p} = 0 \text{ for } \varphi_a(\mathbf{p}) = 1,\ \mathbf{p},\ p^2/2m_a.$$

This means that the total momentum of the particles and the total kinetic energy are conserved in the quasistationary approximation for the field.

At the same time, the structure of the integral J_a^{i} is quite different when account is taken of nonlinear interaction. The collision integral is proportional to the fourth power of the charge while the integral J_a^{i} does not depend explicitly on the charge. The dependence on the charge enters in only implicitly through the value of the roots of Eq. (1.4). Actually, it follows from (2.9), (2.10), (2.13), (2.14) and from the expressions for the tensors ϵ_{ij}, χ_{ijk}, and θ_{ijkl} that, in the quasistationary case,

$$(EE)_\alpha \sim e^{-2}, \quad (EEE)_{\alpha\beta\gamma} \sim e^{-3}, \quad (EEEE)_{\alpha\beta\gamma\delta} \sim e^{-4}$$

Upon elimination of the spectral functions the value of the charge for $e_+ = |e|$ does not enter explicitly into the expression for J^i_a. As a consequence, the relaxation time of the function f_a does not depend explicitly on the charge; therefore, the smallness of the relaxation time can be attributed only to the small number of charged particles which effectively interact with the plasma waves.

In a similar way, one can obtain the corresponding equations of quasi-linear approximation for the set of equations of Vlasov.

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