SPIN KINEMATICS IN LOBACHEVSKIĬ SPACE

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The transformation properties of spin functions of particles (i.e., the "spin kinematics") is studied in the space of relative particle velocities, which is a Lobachevskiĭ space. The Q-representation for the spin functions is introduced, in which the matrices of the operators of the rotation group coincide with the matrices of the operators of the stationary subgroup of the particle velocity. The discrete transformations are discussed in the Q-representation. As an example, the technique is applied to the construction of the amplitudes and of the final state density matrix for Compton scattering. The problem of the connection between observable quantities for elastic scattering and annihilation reactions is considered.

1. INTRODUCTION

OVER the past few years a series of papers has appeared[1-4], in which attention is called to the usefulness of describing kinematics of relativistic particles in terms of Lobachevskian geometry. For the sake of geometrical interpretation it is necessary to consider only particles with a definite momentum, which excludes the possibility of considering interactions. However, in the framework of S-matrix theory, only asymptotic states (i.e., states of real particles with definite momenta) are used. Therefore it seems natural to interpret the quantities that occur in the theory by means of Lobachevskian geometry. These advantages become especially evident in the treatment of particles with spin. The usual covariant formalism leads to extremely complicated and clumsy equations which are devoid of intuitive content. Even though a certain amount of automatism is involved, computations based on the use of this formalism are sufficiently tiresome. At the same time, after writing the scattering matrix in covariant form, one goes over for simplicity to a non-covariant notation in a certain reference system. The equations obtained in this manner are no less clumsy and lack the intuitive clarity which the nonrelativistic formulas exhibit. This is connected with the fact that one cannot separate the coordinate and spin functions in the covariant description of spin. The spin and orbital angular momenta transform jointly under Lorentz transformations, and only their sum is a constant of the motion.

The description in the framework of velocity space permits to bring the relativistic equations to a form as simple and as intuitive as the nonrelativistic ones. All quantities are expressed in terms of the relative velocities of the particles and of the angles between them, quantities which are invariant with respect to Lorentz transformations. The resulting equations look like the nonrelativistic ones, and relativity manifests itself through the noneuclidean character of the velocity space, which has the geometry of a space of constant negative curvature.

2. THE SPACE OF RELATIVE VELOCITIES

As is well known, kinematics is the geometry of velocities. Therefore it is natural that the phenomena of relativistic kinematics have an intuitive interpretation in the space of relative velocities of the particles[2]. The representations of the Lorentz group, or more correctly, the representations of the stationary subgroup of the velocity four-vector of the particle ("little group") are connected with the "kinematics" of the spin of that particle. Therefore it seems natural to consider the wave function of a particle, which generates a representation of the stationary subgroup, as a function defined in the space of relative velocities.

The four-velocity satisfies the relation

$$u^2 = \mathbf{u}^2 - u_0^2 = -1. \qquad (1)$$

We associate with each velocity four-vector the point which its extremity defines on the hyperboloid (1). We thus obtain the three-dimensional space of relative velocities. The geometry of this space, which is Lobachevskian, has been treated in detail by Smorodinskiĭ[2].

The transformations of the proper Lorentz group, acting in the space of 4-velocities, transform the upper sheet of the hyperboloid into itself. They thus generate transformations of the points which represent the relative velocities of the particles, i.e. they generate a transitive group (group of motions) on the surface of the hyperboloid. The stationary subgroup of a point of Lobachevskiĭ space is then isomorphic to the rotation group.

For the sake of simplicity, in what follows we will not consider the three-dimensional hyperboloid, but the two-dimensional surface of the hyperboloid, i.e. a plane in Lobachevskiĭ space.

In the same manner as the Euclidean plane, a Lobachevskiĭ plane is determined by three of its points which are not collinear. The three 4-velocities corresponding to these points, determine the invariant normal vector to the plane. Let these points be denoted by 1, 2, 3 and the corresponding 4-velocities by u^1, u^2, u^3, respectively. Then the 4-vector

$$N_i = i\varepsilon_{iklm}u_k{}^1 u_l{}^2 u_m{}^3 \tag{2}$$

is the invariant normal vector. Assuming, for instance, $u^1 = u^0$, where u^0 is the 4-velocity of the rest system, we obtain

$$\mathbf{N} = [\mathbf{v}^3\mathbf{v}^2]\,\mathrm{sh}\,(u^0u^2)\,\mathrm{sh}\,(u^0u^3), \tag{3*}$$

where $\mathbf{v}$ is the unit vector along the relative velocity of the corresponding point.

3. THE Q-REPRESENTATION

The isomorphism between the stationary subgroup G(u) of a point in Lobachevskiĭ space and the three-dimensional rotation group R^3 allows us to consider the representations of the group G(u) in the space of spin functions of a Dirac particle as representations of the rotation group. However, the bispinor u_+ which describes a positive frequency particle has four components, although, owing to the Dirac equation, only two components are independent. Therefore it is natural to try to select a base in the representation space of the group G(u) in which the spinor has only two nonvanishing components.

In the selected reference system (which will always be considered at rest) we associate to each rotation a transformation of the group G(u). Let $g_0(u)$ denote the transformation which transforms the vector u into the vector u^0; then the group G(u) is obtained from the rotation group in the following manner:

$$G(u) = g_0{}^{-1}(u)R^3g_0(u), \tag{4}$$

and the corresponding representation operators of these groups are connected by the relation $T(G) = L^{-1}(g_0)\,T(R)\,L(g_0)$.

In the representation space of the group G(u) we choose a base in which the operators representing the rotation group T(R) correspond to identical operators T(G). It is easy to see that in order to achieve this it is necessary to carry out a transformation in the representation space of the group G(u) corresponding to the Lorentz transformation $g_0(u)$. The Lorentz transformation of a bispinor under a transformation from the system in which the particle had the 4-velocity v, into the system in which the particle has the velocity u, has the form

$$L(u, v) = \left(\frac{1+\mathrm{ch}\,(uv)}{2}\right)^{1/2}\left(1 - \frac{\sigma_{ik}u_iv_k}{1+\mathrm{ch}\,(uv)}\right); \quad \sigma_{ik} = {}^1\!/\!_2(\gamma_i\gamma_k - \gamma_k\gamma_i). \tag{5*}$$

In order to emphasize the fact that a transformation of the base is carried out in the representation space belonging to one fixed reference system, and not a Lorentz transformation g_0, transforming the particle from the point u into the point u^0, we introduce the notation

$$L^{-1}(g_0) = Q(u, u^0) = \left(\frac{\varepsilon+m}{2m}\right)^{1/2}\left(1 - \frac{\boldsymbol{\alpha}\mathbf{p}}{\varepsilon+m}\right). \tag{6}$$

It is easy to verify that the spinor which corresponds to a positive frequency plane wave will have only two non-vanishing components in this Q-representation:

$$\varphi_Q(u) = Q(u, u^0)\,u_+(u) = \begin{pmatrix}\varphi\\ 0\end{pmatrix}. \tag{7}$$

This transformation can be given an intuitive geometric interpretation[1]. Let us consider the expectation value $(\varphi_Q^*\sigma\varphi_Q)$ of the spin operator, or of any other vector or tensor operator, in the Q-representation, and let us compare these with the same quantities for the particle at rest. It turns out that the quantities computed in the Q-representation in the system in which the particle has the 4-velocity u coincide with the corresponding quantities obtained by parallel translation from the rest system of the particle.

Thus we have associated to each rotation in the chosen reference system a rotation in the rest system of the particle. However, the operator which realizes the isomorphism between the rotation group and the stationary subgroup of the vector u naturally depends also on the vector u. Therefore

*$[\mathbf{v}^3\mathbf{v}^2] = \mathbf{v}^3 \times \mathbf{v}^2$, sh = sinh.

*ch = cosh.

a change from the system in which the particle has velocity u, to the system in which it has the velocity v, will also change the operator Q, i.e. each Lorentz system will have its own base in which the spinor u_+ becomes a two-component object. One may however consider that under a transformation from one reference system to another the spinor itself is subjected to a certain transformation, rather than transform the base.

We consider now the function in the Q-representation in the reference systems in which the particle has the velocities u and v. From Eqs. (5)—(7) it follows

$$\varphi_Q(v) = Q(v, u^0)L(v, u)Q^{-1}(u, u^0)\varphi_Q(u)$$
$$= \exp\{i\sigma\mathbf{n}\Omega/2\}\varphi_Q(u), \qquad (8)$$

where $\mathbf{n}$ is the unit vector perpendicular to the Lobachevskiĭ plane and

$$\cos\frac{\Omega}{2} = [1 + \operatorname{ch}(vu^0) + \operatorname{ch}(uu^0) + \operatorname{ch}(uv)] \times \left[4\operatorname{ch}\frac{(vu^0)}{2}\operatorname{ch}\frac{(uu^0)}{2}\operatorname{ch}\frac{(uv)}{2}\right]^{-1}. \qquad (9)$$

Comparing Eq. (9) with the expression for the area of a triangle in the Lobachevskiĭ plane[5], we find that the quantity Ω coincides with the area of the triangle with vertices in the points v, u, and u^0.

This result can be interpreted as follows. With each rotation in the point u^0 are associated rotations in the points u and v. If one establishes a direct correspondence between the rotations in the points u and v, one can see that these latter rotations differ by a rotation around the normal to the Lobachevskiĭ plane by an angle equal to the area of the triangle formed by the points u, v, u^0 and in a direction opposite to the one determined by the orientation of the contour of the triangle. This rotation is due to the curvature of the velocity space, which manifests itself in the fact that the motions in this space (i.e., the Lorentz transformations) do not commute with rotations.

One usually considers the Foldy-Wouthuysen transformation[6] which also reduces a spinor to two-component form in a given reference system. However, in this representation the spinor differs from the one in the Q-representation by a noninvariant multiplier: $\varphi_{FW} = (\epsilon/m)^{1/2}\varphi_Q$ and therefore it becomes impossible to attribute a simple geometrical meaning to the transformations of the operators.

The Q-representation can be generalized in a natural manner to vector particles. Let the particle have 4-velocity v and an amplitude f satisfying the condition (vf) = 0. We define the operator Q by means of the following equation:

$$Q_{\alpha\beta} = \delta_{\alpha\beta} + (u^0 + v)_\alpha (u^0 + v)_\beta / (1 - u^0 v).$$

Under Lorentz transformation from one reference system to another the vector $\mathbf{f}^Q$ transforms as:

$$f_\alpha{}^Q(v) = T_{\alpha\beta}(v, u) f_\beta{}^Q(u).$$

The expression for $T_{\alpha\beta}$ can be expressed in terms of the spin operator:

$$T_{\alpha\beta} = \delta_{\alpha\beta}\cos\Omega + in_\gamma\varepsilon_{\gamma\alpha\beta}\sin\Omega = \exp\{i\mathbf{s}\mathbf{n}\,\Omega\}. \qquad (10)$$

One could derive similar transformation formulas for the wave functions in the Q-representation of particles with higher spins, but we will not give them here.

4. DISCRETE TRANSFORMATIONS IN THE Q-REPRESENTATION

To include in the present description of particles with spin in the Q-representation the operations of space reflection, time reversal, and also charge conjugation, it is necessary to consider not only the positive-frequency solutions of the relativistic equations, but also the negative-frequency solutions. We consider only the case of spinor particles, which is the most interesting one for applications.

For a geometric interpretation of the negative-frequency solutions it is natural to make use of the lower shell of the hyperboloid (1). We can associate with negative frequency solutions a velocity 4-vector which is opposite to the one for positive frequencies, i.e., it has its end-point on the lower shell of the hyperboloid. However, real particles cannot have a negative time-component of the 4-velocity, therefore one should associate with such solutions Dirac holes and represent these by points on the lower shell of the hyperboloid.

Thus, in the Q-representation, a general solution of the Dirac equation can be written in the form

$$\psi_Q = \begin{pmatrix}\varphi\\0\end{pmatrix}e^{ipx} + \begin{pmatrix}0\\\chi\end{pmatrix}e^{-ipx}. \qquad (11)$$

The two-component spinors φ and χ are linear combinations of the canonical components of the bispinor, i.e., of the dotted spinor $\eta_{\dot\lambda}$ and the undotted spinor ξ^λ:

$$\varphi = (\xi^\lambda + \eta_{\dot\lambda})/\sqrt{2}, \qquad \chi = (\xi^\lambda - \eta_{\dot\lambda})/\sqrt{2}.$$

Under space reflections the quantities φ and χ transform independently and the operator p has the form

$$P\psi_Q = iU_P\gamma_4\psi_Q, \qquad (12)$$

where $U_P = (\mathbf{\Sigma}\cdot\mathbf{n})$ is the unitary operator determined by the condition that the normal components

$(\sigma \cdot n)$ of the matrix σ commute with U_p, whereas the other components anticommute.

We now consider the operation which corresponds to the nonrelativistic "time reversal" T_W, or, following Wigner, to the operation of "reversal of motion." In nonrelativistic theories this operation is defined by an antiunitary operator (cf. e.g.[7]); in the case of the Lorentz group a similar condition is implied by the supplementary physical requirement that the energy of the particle be positive[8]. The antiunitary operator can be represented in the form

$$T_w = iU_T H, \tag{13}$$

where H is the operator of Hermitean conjugation and $U_T = (\mathbf{\Sigma} \cdot \mathbf{v})$ is the unitary operator defined through its commutation relations with the components of σ, analogously to the case of space reflection.

The change in sign of the fourth component of the 4-momentum is compensated in this case by taking complex conjugates of observables and by changing the state in which a particle is absorbed into the state in which the same particle is emitted. In other words, the operation T_W transforms the two shells of the hyperboloid into themselves, changing only the positions of points into their opposites on each hyperboloid shell, and thus changing the directions of relative velocities.

However, relativistic theory admits of another possibility of getting rid of the negative energy states that appear. This possibility is connected with particle-antiparticle conjugation. With the notation

$$\varphi_c = (\eta^\lambda + \xi_{\dot\lambda}) / \sqrt{2}, \qquad \chi_c = (\eta^\lambda - \xi_{\dot\lambda}) / \sqrt{2},$$

Eq. (13) for the operator T_W can be rewritten in the form

$$T_w \psi = iH(\mathbf{\Sigma v})\psi = iT(\mathbf{\Sigma v})\gamma_5 C\psi_c,$$

where T is the transposition operator.

Defining the charge conjugation operator C as

$$C\psi = iT\gamma_4\Sigma_2\psi, \tag{14}$$

it is easy to conclude from (13) and (14) that the "strong time reversal" operator T_S will have the form

$$T_s = CT_w = i(\mathbf{\Sigma v})\gamma_5\psi_c. \tag{15}$$

We also consider the operation of "strong space-time reflection" (CPT) $x \to -x'$. The corresponding operator has the form

$$J\psi_c = CT_w P\psi_c = -(\mathbf{\Sigma n})(\mathbf{\Sigma v})\gamma_5\gamma_4\psi_c. \tag{16}$$

For the following it is useful to write explicit expressions for the action of these operators on positive frequency two-component wave functions:

$$P\varphi = i(\boldsymbol{\sigma}\mathbf{n})\varphi, \qquad T_w\varphi = iH(\boldsymbol{\sigma}\mathbf{v})\varphi = i\varphi^+(\boldsymbol{\sigma}\mathbf{v}),$$
$$T_s\varphi = -i(\boldsymbol{\sigma}\mathbf{v})\chi_c, \qquad J\varphi = i(\boldsymbol{\sigma}[\mathbf{vn}])\chi_c. \tag{17}$$

5. MASSLESS PARTICLES

For particles of zero rest mass (we consider only the photon) the 4-velocity as defined by $u = p/m$ is infinite and Eq. (1) is no longer valid. However, for a geometrical interpretation in a Lobachevskiĭ space one can associate these vectors with the ensemble of infinitely distant points. The 4-vectors corresponding to such infinitely distant points are situated on the light cone.

Thus, massless particles are included in a natural manner in the scheme under consideration. However, the stationary subgroup (little group) of a generator of the cone differs essentially from the stationary subgroup of the hyperboloid. As is well known this subgroup is isomorphic to the group of motions of the Euclidean plane orthogonal to the particle 3-momentum (two-dimensional Euclidean group).

Since the stationary subgroup does not contain transformations which change the direction of the 3-momentum, eigenfunctions of the rotation matrix around the momentum vector belonging to given eigenvalues form invariant subspaces. Therefore the spin of a massless particle is "rigidly" coupled with its direction of motion and its projection on the direction of motion—the helicity of the particle—is a relativistic invariant.

It is convenient to describe the photon by means of invariant helicity amplitudes. There are only two such amplitudes, so that formally the photon appears in this description as a two-component particle. We define in an arbitrary Lorentz system the vectors

$$\zeta_+ = -(\mathbf{m} - i\mathbf{n}) / \sqrt{2}, \qquad \zeta_- = (\mathbf{m} + i\mathbf{n}) / \sqrt{2}, \tag{18}$$

where $\mathbf{n}$ is the invariant normal and $\mathbf{m} = \mathbf{v} \times \mathbf{n}$ (where $\mathbf{v}$ is the unit vector along the direction of motion of the photon). Then the components of the vector amplitude along these vectors will be eigenfunctions of the helicity operator (spin projection along the direction of motion of the photon) corresponding to the eigenvalues 1 and −1, i.e., they will be helicity amplitudes. Constructing the two-component quantity

$$f = \begin{pmatrix} (\mathbf{e}\zeta_+) \\ (\mathbf{e}\zeta_-) \end{pmatrix}, \tag{19}$$

out of these amplitudes, a quantity which formally resembles a spinor, one can elegantly rewrite the equations involving photon amplitudes by means of Pauli matrices, and the form of the resulting equation will be relativistically invariant.

Under space reflection ζ_+ transforms into ζ_- and the vector amplitude e changes sign. Thus, the operator p has the form

$$p\begin{pmatrix}(\mathbf{e}\zeta_+)\\(\mathbf{e}\zeta_-)\end{pmatrix} = -\begin{pmatrix}(\mathbf{e}\zeta_-)\\(\mathbf{e}\zeta_+)\end{pmatrix} = -\sigma_1 f. \qquad (20)$$

Similarly, under time reversal (since the photon is neutral, strong and weak time reversal are indistinguishable)

$$Tf = \sigma_1 f^*. \qquad (21)$$

6. COMPTON SCATTERING IN THE Q-REPRESENTATION

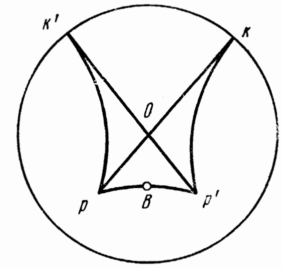
We consider the scattering of a photon on a spinor particle. The figure represents the kinematical diagram for this process. In the Poincaré model the Lobachevskiĭ plane is represented by the interior of the unit circle while the circumference of the unit circle corresponds to the infinitely distant points. The "straight lines" are circle arcs, orthogonal to the unit circle. The points k and k′, p and p′ in the diagram represent the initial and final photons and spinor particles, respectively. The point O corresponds to the rest system; for the given reaction it coincides with the center of mass system of the reaction. Lorentz transformations generate an automorphism of the unit circle, such that distances measured along the "straight lines" between two points are preserved, as well as the angles between two such "lines." Therefore each point in the circle can be considered a reference system, measuring distances and angles from this point.

The Compton scattering amplitude is written in the form *

$$T = (2\pi)^4 \frac{m}{2V^2(\varepsilon\varepsilon'\omega\omega')^{1/2}} U(s, t, u),$$

where V is the normalization volume, ϵ, ϵ', ω, and ω' are the energies of the particles and the photons, in the initial and final states respectively, and m is the mass of the particles. The invariant amplitude U(s, t, u) is determined by six invariant functions of the kinematic variables.

In order to write out the matrix U(s, t, u) we first consider the spinor amplitudes. We select the reference system in the point B of the figure. This point corresponds to the so-called Breit system, i.e., the reference system in which the particle is scattered backwards. In this system one can construct four independent scalars out of the spinor amplitude ψ of the final particle and the amplitude φ of the initial particle (we will call a scalar a quantity which does not change under rotations, whereas a Lorentz scalar will be called an invariant), namely:



$$\psi^*\varphi, \quad |\mathbf{N}|\psi^*(\sigma\mathbf{n})\varphi, \quad |\mathbf{v}|\psi^*(\sigma\mathbf{v})\varphi, \quad |[\mathbf{vn}]|\psi^*(\sigma[\mathbf{vn}])\varphi. \qquad (22)$$

The invariant multipliers in front of the spinor "blocks" have their origin in the normalization of the base vectors.

The four spinor "blocks" in (22) behave differently under space reflection and weak time reversal. The first and second "blocks" are scalars, whereas the third and fourth "blocks" are pseudoscalars. Under time reversal the third "block" changes its sign, the other three remain unchanged. According to the ζ rules established above, the "blocks" which have been constructed in the Breit system can be transformed into any other system, and thus one obtains the amplitudes in that system.

Let us now consider the photon amplitudes. As has been shown in Sec. 5, it is convenient to describe photons by means of helicity amplitudes. In a reaction in which two photons participate, one can construct out of the helicity amplitudes of these photons four different combinations which can be written compactly in terms of the photon "spinors" f, defined in Eq. (19):

$$f'^*f = (\mathbf{e}\zeta_+)'^*(\mathbf{e}\zeta_+) + (\mathbf{e}\zeta_-)'^*(\mathbf{e}\zeta_-),$$
$$f'^*\tau_1 f = (\mathbf{e}\zeta_+)'^*(\mathbf{e}\zeta_-) + (\mathbf{e}\zeta_-)(\mathbf{e}\zeta_+),$$
$$f'^*\tau_2 f = -i[(\mathbf{e}\zeta_+)'^*(\mathbf{e}\zeta_-) - (\mathbf{e}\zeta_-)'^*(\mathbf{e}\zeta_+)],$$
$$f'^*\tau_3 f = (\mathbf{e}\zeta_+)'^*(\mathbf{e}\zeta_+) - (\mathbf{e}\zeta_-)'^*(\mathbf{e}\zeta_-). \qquad (23)$$

It is easy to verify that the first two of these combinations are invariant with respect to space reflection whereas the last two change signs. Under time reversal only the third combination changes sign.

Out of the spinor "blocks" (22) and the photon "spinors" (23) one can construct only six combinations which are invariant under weak time re-

*Here u is the third Mandelstam variable and not a 4-velocity! (Transl. note) versal. Multiplying each such combination by an invariant function of the kinematic variables, we obtain the final result, valid in any reference frame

$$U = f'^{*}(R(s, t, u) + \mathbf{R}(s, t, u)\boldsymbol{\tau})f; \quad (24)$$

$$2R = \psi^{*}\exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_2/2\}(\Phi_1 + |\mathbf{N}|\Phi_2 i(\boldsymbol{\sigma}\mathbf{n}))$$
$$\times \exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_1/2\}\varphi \operatorname{ch} a,$$
$$2R_x = \psi^{*}\exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_2/2\}(\Phi_3 + |\mathbf{N}|\Phi_4 i(\boldsymbol{\sigma}\mathbf{n}))$$
$$\times \exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_1/2\}\varphi \operatorname{ch} a,$$
$$2iR_y = \Phi_5\psi^{*}\exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_2/2\}(\boldsymbol{\sigma}\mathbf{v})\exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_1/2\}\varphi \operatorname{sh} a,$$
$$2R_z = |\mathbf{N}|\Phi_6\psi^{*}\exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_2/2\}(\boldsymbol{\sigma}[\mathbf{v}\mathbf{n}])\exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_1/2\}\varphi \operatorname{sh} a, \quad (25)$$

where a is the length of the arc between the points B and p on the hyperboloid, i.e., cosh a = ϵ/m where ϵ is the energy of one of the particles in the Breit system.

The choice of the Breit system as a basic one was determined only by considerations of convenience. One could have started from any reference system and defined different sets of invariant amplitudes, in analogy to a different choice of independent invariants in the ordinary formulation of the scattering matrix.

Utilizing the amplitude written in the Q-representation it is easy to go over to the annihilation channel. As is well known, the scattering and annihilation reactions are connected by means of the T_S transformation, applied only to the final state particle and the initial state photon, and not to all particles simultaneously. Making use of the expressions (17) and (21) for the operator T_S and taking into account the fact that the variables s and t exchange their roles, we derive from (24) and (25) the amplitude for the annihilation of two particles:

$$\bar{U} = 2f'^{*}(\bar{R} + \bar{\mathbf{R}}\boldsymbol{\tau})\tau_1 f_c^{*};$$
$$2\bar{R} = -i\chi^{*}\exp\{i\boldsymbol{\sigma}\mathbf{n}\bar{\Omega}_2/2\}(\boldsymbol{\sigma}\mathbf{v})(\bar{\Phi}_1 + |\bar{\mathbf{N}}|\bar{\Phi}_2 i(\boldsymbol{\sigma}\mathbf{n}))$$
$$\times \exp\{i\boldsymbol{\sigma}\mathbf{n}\bar{\Omega}_1/2\}\varphi\,\overline{\operatorname{ch} a},$$
$$2\bar{R}_x = -i\chi^{*}\exp\{i\boldsymbol{\sigma}\mathbf{n}\bar{\Omega}_2/2\}(\boldsymbol{\sigma}\mathbf{v})(\bar{\Phi}_3 + |\bar{\mathbf{N}}|\bar{\Phi}_4 i(\boldsymbol{\sigma}\mathbf{n}))$$
$$\times \exp\{i\boldsymbol{\sigma}\mathbf{n}\bar{\Omega}_1/2\}\varphi\,\overline{\operatorname{ch} a},$$
$$2i\bar{R}_y = -i\chi^{*}\exp\{i\boldsymbol{\sigma}\mathbf{n}\bar{\Omega}_2/2\}(\boldsymbol{\sigma}\mathbf{v})\bar{\Phi}_5\exp\{i\boldsymbol{\sigma}\mathbf{n}\bar{\Omega}_1/2\}\varphi\,\overline{\operatorname{sh} a},$$
$$2\bar{R}_z = -i|\bar{\mathbf{N}}|\Phi_6\chi^{*}\exp\{i\boldsymbol{\sigma}\mathbf{n}\bar{\Omega}_2/2\}(\boldsymbol{\sigma}[\mathbf{v}\mathbf{n}])$$
$$\times \exp\{i\boldsymbol{\sigma}\mathbf{n}\bar{\Omega}_1/2\}\varphi\,\overline{\operatorname{sh} a}. \quad (26)$$

A bar above an invariant quantity denotes the usual change of variables for a crossed reaction.

The Compton scattering amplitudes derived above allow for a simple calculation of the final state density matrix. Denoting the initial state density matrices for the particle and the photon respectively by

$$\rho_e = \varphi\varphi^{*} = (1 + \boldsymbol{\zeta}\boldsymbol{\sigma})/2, \qquad \rho_\gamma = (1 + \boldsymbol{\xi}\boldsymbol{\tau})/2, \quad (27)$$

where $\boldsymbol{\zeta}$ is the particle polarization vector and $\boldsymbol{\xi}$ are the Stokes parameters, one can write the density matrix for the final state in the form

$$\rho = \exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_2/2\}M\exp\{i\boldsymbol{\sigma}\mathbf{n}\Omega_1/2\}\rho_e\rho_\gamma$$
$$\times \exp\{-i\boldsymbol{\sigma}\mathbf{n}\Omega_1/2\}M^{+}\exp\{-i\boldsymbol{\sigma}\mathbf{n}\Omega_2/2\}. \quad (28)$$

In this form the kinematic and dynamic effects of the reaction are explicitly separated. We will not write out the expressions for concrete quantities (cross sections, polarizations etc.), since the simpler ones have already been computed by many authors[9,10] and complete expressions have been obtained by Frolov[11] who operated with covariant quantities, although an example of such calculations would again exhibit the advantage of the description of processes in the Q-representation.

The matrix (28) is simply related to the analogous matrix for the annihilation channel. The antiparticle density matrix, has in terms of the spinor χ the form

$$\bar{\rho} = \overline{\chi\chi^{*}} = (1 - \bar{\boldsymbol{\zeta}}\boldsymbol{\sigma})/2,$$

where $\bar{\boldsymbol{\zeta}}$ is the polarization of the antiparticle. Making use of the transformation properties of spinor and photon amplitudes under the T_S transformation, it is easy to verify that the final state density matrix for the annihilation reaction has exactly the same structure as the matrix for elastic scattering, if one carries out the following substitutions among the photon variables:

$$\xi_y, \xi_z \to -\tau_y, -\tau_z, \qquad \xi_x \to \tau_x$$

and among the particle variables:

$$(\boldsymbol{\sigma}\mathbf{v}) \to -(\bar{\boldsymbol{\zeta}}\mathbf{v}), \qquad (\boldsymbol{\sigma}\mathbf{n}) \to (\bar{\boldsymbol{\zeta}}\mathbf{n}), \qquad (\boldsymbol{\sigma}[\mathbf{v}\mathbf{n}]) \to (\bar{\boldsymbol{\zeta}}[\mathbf{v}\mathbf{n}]),$$

together with a corresponding permutation of the kinematic variables in the invariant functions.

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313