SOLUTION OF THE GRAVITATION EQUATIONS FOR A HOMOGENEOUS ANISOTROPIC MODEL

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Solutions of the Einstein equations are derived for a homogeneous axially-symmetrical model in two limiting cases, for dust-like matter and an ultrarelativistic gas. The solutions possess a singularity for a certain value of time but for the equation $\epsilon = 3p$ they can be conjugated analytically on both sides of the singularity. A solution for a nonrelativistic gas with a constant isentropic exponent is also derived.

THE character of the singularity at $t = 0$ in the solution of the Einstein gravitation equations has been the subject of an extensive literature [1-4]. We shall show in this paper that the model with homogeneous distribution of matter but with an anisotropic metric has a different singularity in time than the Friedmann-Lemaitre homogeneous isotropic model.

We shall seek mixed energy-momentum tensor components that depend only on the time, in a co-moving synchronous system for which the metric form is

$$ds^2 = dt^2 - e^{\nu(r,\,t)}dr^2 - e^{\omega(r,\,t)}d\varphi^2 - e^{\mu(r,\,t)}dz^2. \qquad (1)$$

In other words, nothing depends at all on the coordinate z. Further calculations show that a co-moving synchronous reference frame actually exists under these assumptions.

The three-index Christoffel symbols are ($r = x_1$, $\varphi = x_2$, $z = x_3$, $t = x_4$):

$$\Gamma_{11}^{1} = \nu'/2,\quad \Gamma_{12}^{2} = \omega'/2,\quad \Gamma_{13}^{3} = \mu'/2,\quad \Gamma_{41}^{1} = \dot\nu/2,$$

$$\Gamma_{42}^{2} = \dot\omega/2,\quad \Gamma_{43}^{3} = \dot\mu/2,\quad \Gamma_{11}^{4} = e^{\nu}\dot\nu/2,\quad \Gamma_{22}^{4} = e^{\omega}\dot\omega/2,$$

$$\Gamma_{33}^{4} = e^{\mu}\dot\mu/2,\quad \Gamma_{22}^{1} = -e^{\omega-\nu}\omega'/2,\quad \Gamma_{33}^{1} = e^{\mu-\nu}\mu'/2.$$

The remaining Christoffel symbols vanish under the assumptions made. The prime stands for $\partial/\partial r$, and the dot over a letter for $\partial/\partial t$.

The mixed components T_k^i in the co-moving reference frame are $T_1^1 = T_2^2 = T_3^3 = p$, $T_4^4 = -\epsilon$, and the remaining components vanish. Then of all the Einstein equations

$$R_i^k = 8\pi\varkappa(T_i^k - {}^1/_2\delta_i^k T) \qquad (2)$$

there remain only the following five independent equations (we shall henceforth set $8\pi\kappa$ equal to unity):

$$\varepsilon - p = {}^1/_2[2\ddot\nu + \dot\nu(\dot\nu + \dot\omega + \dot\mu)] - {}^1/_2 e^{-\nu}[2(\omega'' + \mu'') + \omega'^2 + \mu'^2 - \nu'(\omega' + \mu')], \qquad (2a)$$

$$\varepsilon - p = {}^1/_2[2\ddot\omega + \dot\omega(\dot\nu + \dot\omega + \dot\mu)] - {}^1/_2 e^{-\nu}[2\omega'' + \omega'(-\nu' + \omega' + \mu')], \qquad (2b)$$

$$\varepsilon - p = {}^1/_2[2\ddot\mu + \dot\mu(\dot\nu + \dot\omega + \dot\mu)] - {}^1/_2 e^{-\nu}[2\mu'' + \mu'(-\nu' + \omega' + \mu')], \qquad (2c)$$

$$-(\varepsilon + 3p) = {}^1/_2[2(\ddot\nu + \ddot\mu + \ddot\omega) + (\dot\nu^2 + \dot\mu^2 + \dot\omega^2)], \qquad (2d)$$

$$-{}^1/_2(\dot\omega' + \dot\mu') + {}^1/_4\dot\nu(\omega' + \mu') - {}^1/_4(\omega'\dot\omega + \mu'\dot\mu) = 0. \qquad (2e)$$

The last equation is the co-moving condition.

The left sides of (2a)—(2d) depend, by definition, only on the time. The right side can be made dependent on the time only by assuming that $\nu = \nu(t)$ and $\mu = \mu(t)$. Then (2e) takes the form

$$e^{\omega/2} = e^{\nu/2}h(r) + F(t). \qquad (3)$$

The general integral of this equation is

$$\dot\omega' = \dot\nu\omega'/2 - \omega'\dot\omega/2. \qquad (4)$$

We assume, however, that φ is an angle coordinate. Therefore, in order for the line $r = 0$ not to be singular, we must stipulate $\exp[\omega(0, t)] = 0$. This leads to $F(t) = 0$ and $h(0) = 0$. We then get from (4) $\dot\omega = \dot\nu$. If we substitute this in the system (2a)—(2d), then Eq. (2b) will be merely a duplicate of (2a) and can be omitted henceforth.

To find $h(r)$ we need only (2a), in which $h(r)$ enters in the form $\exp[-(\nu + \omega)/2]h''(r)$. This expression should depend only on the time. Substituting in it $e^{-\omega/2}$ from (4) and taking account of the fact that $F(t) = 0$, we get an equation for hr:

$$h''(r)\,/\,h(r) = \text{const} = \pm\alpha^2. \quad (5)$$

It follows therefore that there are three types of solutions: $h = \sinh \alpha r$ (model that is open in r), $h = r$ (model that is quasi-Euclidean in r), and $h = \sin \alpha r$ (model closed in r). Let us consider first the second case, since it is the simplest. The system (2a), (2c), (2d) is rewritten in the form

$$2(\varepsilon - p) = 2\ddot{\nu} + 2\dot{\nu}^2 + \dot{\mu}\dot{\nu}, \quad (6a)$$
$$2(\varepsilon - p) = 2\ddot{\mu} + 2\dot{\mu}\dot{\nu} + \dot{\mu}^2, \quad (6b)$$
$$-2(\varepsilon + 3p) = 4\ddot{\nu} + 2\ddot{\mu} + 2\dot{\nu}^2 + \dot{\mu}^2. \quad (6c)$$

Let us consider two limiting cases: $p = 0$ (dustlike matter) and $\epsilon = 3p$ (ultrarelativistic gas). For the dust we obtain (we are omitting the inessential scale constants):

$$e^{\nu} = t^{4/3}, \quad (7)$$
$$e^{\mu} = t^{-2/3}(t - A)^2, \quad (8)$$
$$\varepsilon = 4\,/\,3t(t - A). \quad (9)$$

This is a particular case of a more general equation obtained by Shucking and Heckman [5]. Namely, writing the metric in the form

$$ds^2 = dt^2 - \sum_{\lambda=1}^{3} R_\lambda^2(t)\,dx_\lambda^2,$$

these authors obtain:

$$R_\lambda = t^{1/3\{1+2\sin[\gamma+2\pi(\lambda-1)/3]\}}(t - A)^{1/3\{1-2\sin[\gamma+2\pi(\lambda-1)/3]\}}.$$

Substituting here $\gamma = \pi/6$, we arrive at (7)—(9). This latter case is special in a certain sense, since the Shucking and Heckman solution for arbitrary γ can be continued beyond the point $t = A$, inasmuch as the factor $t - A$ is contained in the metric coefficients with an irrational exponent.

It is curious, nonetheless, that the point $t = A$ is singular also in the solution (7)—(9), but in a somewhat different sense than in the more general solution where $\gamma \neq \pi/6$. The singularity is eliminated from the metric form completely if we go over to new coordinates*

$$\tau = (t - A)\,\mathrm{ch}\,z, \quad \zeta = (t - A)\,\mathrm{sh}\,z.$$

Then $dt^2 - (t - A)^2 dz^2$ goes over into the purely Galilean expression $d\tau^2 - d\zeta^2$. But the product $\epsilon\sqrt{-g}$ reverses sign at $t = A$. It is possible that this is a manifestation of a general property of relativity theory, that the energy is not positive definite in this theory, and that the transition through $t = A$ corresponds to a replacement of m by $-m$. This point is still unclear to us. We note that in the opposite extreme case, when $\epsilon = 3p$, solutions with positive or negative energies are strictly segregated and do not go over into one another.

As to the non-quasi-Euclidean solutions in the case of dustlike matter, the situation is not radically different from that already indicated. Namely, after some calculations we obtain ($\alpha^2 > 0$):

$$t = 2C\int_{Ce^{-\nu/2}}^{\infty}\frac{d\xi}{\xi^{3/2}(1+\xi)}, \quad (7')$$

$$e^{\mu/2} = (1 + Ce^{-\nu/2})^{1/2}\left(1 + C_1\int_{Ce^{-\nu/2}}^{\infty}\frac{d\xi}{\xi(1+\xi)^{3/2}}\right), \quad (8')$$

$$\varepsilon = \frac{1}{4}C_1 e^{-\nu/2}(1 + Ce^{-\nu/2})^{1/2}\left(1 + C_1\int_{Ce^{-\nu/2}}^{\infty}\frac{d\xi}{\xi(1+\xi)^{3/2}}\right)^{-1}. \quad (9')$$

Here, too, the product $\epsilon\sqrt{-g}$ reverses sign at some point.

For very large values of t, the broadening is uniform in all sides. Therefore, when extrapolating the present state of the universe to earlier stages, we must not lose sight of the fact that in these earlier stages the anisotropic solutions have a behavior different from that of the isotropic ones.

We now consider the quasi-Euclidean anisotropic solution for $\epsilon = 3p$. The system of fundamental equations takes the form

$$^4/_3\,\varepsilon = 2\ddot{\nu} + 2\dot{\nu}^2 + \dot{\nu}\dot{\mu}, \quad (10a)$$
$$^4/_3\,\varepsilon = 2\ddot{\mu} + 2\dot{\nu}\dot{\mu} + \dot{\mu}^2, \quad (10b)$$
$$-4\varepsilon = 4\ddot{\nu} + 2\ddot{\mu} + 2\dot{\nu}^2 + \dot{\mu}^2. \quad (10c)$$

Subtracting Eq. (10b) and Eq. (10a) multiplied by 4 from Eq. (10c), we obtain, apart from an inessential constant,

$$\mu = -6\ln\dot{\nu} - 5\nu. \quad (11)$$

Further, eliminating μ and ϵ, we get

$$2\dddot{\nu}\dot{\nu} - 8\ddot{\nu}^2 - 7\ddot{\nu}\dot{\nu}^2 - 3\dot{\nu}^4 = 0. \quad (12)$$

The first integral of (12) takes the form

$$(\ddot{\nu}/\dot{\nu}^2 + 1)^4(\ddot{\nu}/\dot{\nu}^2 + {}^3/_4)^{-3} = -C_2\dot{\nu}^2. \quad (13)$$

In order to clarify the character of the singularity near $t = 0$, we must put $\ddot{\nu}/\dot{\nu}^2 \gg 1$, which yields

$$\dot{\nu} = (3C_2 t)^{-1/3}. \quad (14)$$

For small t we get for ν itself

$$\nu = (3t)^{2/3}\,/\,2C_2^{1/3}. \quad (15)$$

Consequently, according to (11)

$$\mu = \ln t^2. \quad (16)$$

From (10a) we determine the energy density

*ch = cosh, sh = sinh.

$$\varepsilon = (C_2 t)^{-4/3}. \quad (17)$$

At first glance we see that the singularity of the metric form can be eliminated here in the first approximation by the same method as for dust matter at $t = A$, namely, by using the transformation $\tau = t \cosh z$, $\zeta = t \sinh z$. However, the expansion of the metric coefficients is in fractional powers of t, while the expression for the curvature contains derivatives up to second order inclusive. When $t = 0$ these derivatives become infinite. Consequently the product $\epsilon\sqrt{-g}$ also becomes infinite.[1)]

In contradistinction from the Friedmann solution, where negative t are meaningless (the formulas contain $\sqrt{t}$), the solution obtained here can be taken also for $t < 0$. The singularity which we obtained is similar in type to the cusp of a cycloid. We can see that (13) has also solutions of another type. Thus, there exists an expansion of $d\dot{\nu}^{-1}/dt$ near the point $3/4$. But for this solution the product $\epsilon\sqrt{-g}$ vanishes like $t^{1/3}$, in other words, the density of the matter tends to zero.

We now proceed to the general case $\alpha^2 \neq 0$, $\epsilon = 3p$. Replacing t by t/α, we obtain an equation for $\dot{\nu}$:

$$12\dddot{\nu}\dot{\nu} - 48\ddot{\nu}^2 - 42\ddot{\nu}\dot{\nu}^2 - 18\dot{\nu}^4 \pm 28\ddot{\nu}e^{-\nu}$$
$$\pm 21\dot{\nu}^2 e^{-\nu} - 4e^{-2\nu} = 0. \quad (18)$$

The upper signs correspond here to a model which is open in r, and the lower ones to the closed model. Making the substitution

$$\dot{\nu} \equiv e^{-\nu/2}u(\nu), \quad du/d\nu \equiv q(u), \quad (19)$$

we arrive at the first-order equation

$$12u^3 q\,dq/du - 36q^2u^2 - 18qu^3$$
$$- 3u^4 \pm 28qu \pm 7u^2 - 4 = 0. \quad (20)$$

In these variables, the sought functions are

$$\nu = \int \frac{du}{q}, \quad (21a)$$

$$\mu = 2\int \frac{du}{qu^2} - 2\int \frac{du}{q} - 6\ln u, \quad (21b)$$

$$\varepsilon = e^{-\nu}(1 - u^2 - 4qu), \quad (21c)$$

$$t = \int \frac{du}{qu} e^{\nu/2}. \quad (21d)$$

We now note the following important circumstance. The equation $1 - u^2 - 4qu = 0$ is a singular integral of (20). As can be seen from (21c), this solution segregates the region with positive and negative ϵ, so that a transition through $\epsilon = 0$ is impossible here. The point $t = 0$ corresponds to $u = \infty$, $q = \infty$ in the lower right quadrant of the (q, u), plane where Eq. (20) already corresponds to the quasi-Euclidean case investigated above. Therefore the general non-quasi-Euclidean solution has here, too, a qualitatively similar dependence on the time as in the case of dustlike matter.

We now consider the intermediate case, when the matter has non-zero pressure but the kinetic-energy density is still much lower than the rest-energy density. We assume that the substance goes through an isentropic process. Then the energy density and pressure are expressed in terms of the particle density in the following fashion:

$$\varepsilon = nm + bn^{1+\delta}, \quad p = \delta b n^{1+\delta}. \quad (22)$$

When $\delta = 2/3$ we have the case of a monatomic gas. With this value of the constants, this is simultaneously also the case of a cold Fermi gas. The equations can be integrated in quadratures for all values of δ.

Let us subtract (6b) from (6a), so as to exclude terms pertaining to the matter. The resultant equation admits of a first integral

$$\dot{\nu} - \dot{\mu} = Ce^{-\nu-\mu/2}. \quad (23)$$

Using (6b) and (6c) we can now express n and $n^{1+\delta}$ separately, and set up the equation

$$2\ddot{\nu} + 2\ddot{\mu} + \dot{\nu}^2 + \dot{\nu}\dot{\mu} + \dot{\mu}^2 = -b_1[2(1-\delta)\ddot{\nu}$$
$$+ 2(1+\delta)\ddot{\mu} + (1+3\delta)\dot{\nu}\dot{\mu} + (1-\delta)\dot{\nu}^2 + (1+\delta)\dot{\mu}^2]^{1+\delta}, \quad (24)$$

where the meaning of the constant b_1 is obvious. It is convenient in what follows to introduce a new unknown

$$\zeta = \nu + \mu/2, \quad (25)$$

so that $e^{\zeta} = \sqrt{-g}$.

Expressing now μ in terms of ζ, we rewrite (24) and simultaneously lower its order by means of the substitutions

1)As indicated by E. Lifshitz and Khalatnikov[3], the geodetics in the synchronous co-moving system are time lines, and consequently dustlike matter can condense to an infinite density simply because the individual particles do not repel each other and come together in one point. In an ultrarelativistic gas this could be prevented by the pressure which deflects the particle motion from the geodetic lines. In the present problem, however, we took the case of a coordinate-independent pressure, so that the pressure gradient is equal to zero. Consequently, there is no force to move the gas particle away from the geodetic lines.

$$\dot{\zeta} = w, \qquad Ce^{-\zeta} = x, \qquad \dot{v} = {}^1/_3(2w + x),$$
$$\ddot{v} = -{}^1/_3 w(2x dw / dx + x),$$

reducing (24) to the form

$$-\frac{8}{3} wx \frac{dw}{dx} + \frac{4}{3} w^2 + \frac{x^2}{3} = -b_1 \Big[-\frac{8}{3} wx \frac{dw}{dx} + \frac{4}{3}(1+\delta) w^2 + \frac{1}{3}(1-\delta) x^2\Big]^{1+\delta}.$$

Further lowering of the order is attained by means of the substitution

$$w^2 = x^2(1 + Av / x^2), \qquad A = {}^3/_4(1+\delta)^{1+1/\delta} b_1^{1/\delta}.$$

If we again go over from x to ζ, we obtain in final form

$$\frac{dv}{d\zeta} + v = -\Big[\frac{1}{1+\delta}\frac{dv}{d\zeta} + v\Big]^{1+\delta}. \qquad (27)$$

The integral of this equation is

$$\zeta = -\ln C_1\Big(v + \frac{k}{1+\delta}\Big), \qquad k = \frac{dv}{d\zeta}. \qquad (28)$$

It is convenient to express the solution in parametric form, starting from the definition $k = -sv$. The limits of variation of s will be $1 + \delta \geq s \geq 1$. Then

$$v = (s-1)^{1/\delta}\Big(1 - \frac{s}{1+\delta}\Big)^{-1/\delta},$$
$$e^{\zeta} = C_1^{-1}\Big(v + \frac{k}{1+\delta}\Big)^{-1} = \Big(1 - \frac{s}{1+\delta}\Big)^{1/\delta - 1}(s-1)^{-1/\delta}. \qquad (29)$$

The expression for the particle density is obtained in terms of the same parameter s, since

$$n \sim v + k / (1+\delta). \qquad (30)$$

Thus, the product $ne^{\zeta} = n\sqrt{-g}$ remains constant, which is merely an expression for the conservation of number of particles. This requirement is contained in the very equation relating p and ϵ. To the contrary, the ultrarelativistic case does not incorporate this requirement, so that the product $\epsilon\sqrt{-g}$ does not remain constant and ϵ becomes infinite at a faster rate than the vanishing of $\sqrt{-g}$.

It is convenient to choose as the time origin the point $s = 1 + \delta$, where the particle density is infinite. Near this point we have

$$t = \Big(1 - \frac{s}{1+\delta}\Big)^{1/2\delta}. \qquad (31)$$

For arbitrary values of δ we cannot, of course, go over from t to $-t$ (although, for example, this can be done when $\delta = 2/3$). But when the particle density tends to infinity, the equation of state goes over into $\epsilon = 3p$, which again permits the passage through zero time.

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