REGGE POLE TRAJECTORIES FOR THE NONRELATIVISTIC TWO-CHANNEL PROBLEM

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The behavior of the Regge trajectories for the nonrelativistic two-channel problem with Yukawa potentials is investigated. At high energies two trajectories collide at each negative integer point, corresponding to the number of channels. The dependence of the trajectories on the energy is investigated for the case of weak coupling. Qualitatively it remains the same as in the single-channel problem. With increasing number of trajectories the number of points of collision, i.e., the number of branch points in the complex energy plane, also increases. All results are easily generalized to the many-channel problem.

1. INTRODUCTION

In the last years a great number of papers has been devoted to the trajectories of Regge poles. The explicit form of these trajectories could be obtained only in a few cases. In particular, the Regge trajectories for the scattering from a Yukawa potential in the weak coupling case were found.[1,2] Here it turned out that in the complex energy plane the trajectories have real as well as complex branch points, corresponding to the coincidence of a pair of poles. In this plane all trajectories are different branches of a single analytic function. It is of interest to investigate the Regge trajectories in a more complicated case, where the reaction can go through several channels. Such a case is obviously more suited as an analogue of the relativistic situation than the single-channel problem.

Below we shall consider the behavior of the Regge poles for the scattering from Yukawa potentials with two possible reaction channels. All considerations are based on the method applied earlier in [1,2]. The generalization to the many-channel case is obvious. In Sec. 2 we obtain the equation for the Regge trajectories for weak coupling. In Sec. 3 we consider the behavior of the trajectories for not too small negative k_1^2 and k_2^2. The further motion of the poles for small k_1^2 or k_2^2 is described in Sec. 4. There it is also shown how the trajectories behave for positive k_1^2 and k_2^2. The analytic properties of the trajectories are described in Sec. 5. In Sec. 6 we discuss the appearance of the so-called N poles.

2. EQUATION OF THE TRAJECTORIES

Let us consider the two-channel scattering problem with Yukawa potentials

$$V_{ik} = g_{ik} r^{-1} \exp(-\mu_{ik} r)$$

$$(i, k = 1, 2;\ \mu_{12} = \mu_{21},\ g_{12} = g_{21})$$

with very small coupling constants g_{ik}. The radial wave function has two components, the Schrödinger equation for which is conveniently written down in integral form:

$$\psi_l^{(1)}(r) = C j_l(k_1 r) + \frac{1}{k_1 \cos \pi l} \int_0^r [j_l(k_1 r) j_{-l-1}(k_1 r') - j_l(k_1 r') j_{-l-1}(k_1 r)] \times [U_{11}(r') \psi_l^{(1)}(r') + U_{12}(r') \psi_l^{(2)}(r')]\, dr', \tag{1}$$

$$\psi_l^{(2)}(r) = D j_l(k_2 r) + \frac{1}{k_2 \cos \pi l} \int_0^r [j_l(k_2 r) j_{-l-1}(k_2 r') - j_l(k_2 r') j_{-l-1}(k_2 r)] \times [U_{22}(r') \psi_l^{(2)}(r') + U_{21}(r') \psi_l^{(1)}(r')]\, dr';$$

where

$$U_{ik} = 2m_i V_{ik} = 2m_i g_{ik} r^{-1} \exp[-\mu_{ik} r] \quad (i, k = 1, 2),$$
$$j_l(x) = (\pi x / 2)^{1/2} J_{l+1/2}(x),$$

and $J_{l+1/2}(x)$ is the Bessel function. The momenta k_1 and k_2 in the first and second channels are related through $k_1^2 = k_2^2 + \Delta^2$. In the following we shall assume that Δ^2 is of the order μ^2.

Let us now consider the scattering problem, assuming first that only the first channel contains an incident wave. Then the constant D is determined by the condition that there be no incident wave in the second channel, and is equal to

$$D=\frac{1}{ik_2}\int_0^\infty h_l^{(1)}(k_2r)[U_{22}(r)\psi_l^{(2)}(r)+U_{21}(r)\psi_l^{(1)}(r)]\,dr, \quad (2)$$

where

$$h_l^{(1,2)}(x)=(\pi x/2)^{1/2}H_{l+1/2}^{(1,2)}(x);$$

$H_{l+1/2}^{(1,2)}(x)$ are the Hankel functions of the first and second kind. The constant C is determined by the normalization condition. We can set it equal to unity without loss of generality.

Defining the S matrix such that the element $S_l^{(pk)}$ describes the scattering in the p channel for an incident wave in the k channel, we may write

$$\psi_l^{(p)}(r)=\delta_{pk}\sin(kr-l\pi/2)+{}^1/_2 i(-i)^l(1-S_l^{(pk)})e^{ikr}.$$

Comparing this formula with the expression for $\psi_l^{(p)}(r)$ in (1) for large r and taking account of (2), we find

$$S_l^{(11)}=\left\{1-\frac{i}{k_1}\int_0^\infty h_l^{(2)}(k_1r)[U_{11}(r)\psi_l^{(1)}(r)+U_{12}(r)\psi_l^{(2)}(r)]\,dr\right\}$$
$$\times\left\{1+\frac{i}{k_1}\int_0^\infty h_l^{(1)}(k_2r)[U_{11}(r)\psi_l^{(1)}(r)+U_{12}(r)\psi_l^{(2)}(r)]\,dr\right\}^{-1},$$
$$S_l^{(21)}=\left\{\frac{1}{ik_2}\int_0^\infty j_l(k_2r)[U_{22}(r)\psi_l^{(2)}(r)+U_{21}(r)\psi_l^{(1)}(r)]\,dr\right\}$$
$$\times\left\{1+\frac{i}{k_1}\int_0^\infty h_l^{(1)}(k_1r)[U_{11}(r)\psi_l^{(1)}(r)+U_{12}(r)\psi_l^{(2)}(r)]\,dr\right\}^{-1}. \quad (3)$$

If the incident wave is in the second channel, we find by an analogous consideration

$$S_l^{(22)}=\left\{1-\frac{i}{k_2}\int_0^\infty h_l^{(2)}(k_2r)[U_{22}(r)\psi_l^{(2)}(r)+U_{21}(r)\psi_l^{(1)}(r)]\,dr\right\}$$
$$\times\left\{1+\frac{i}{k_2}\int_0^\infty h_l^{(1)}(k_2r)[U_{22}(r)\psi_l^{(2)}(r)+U_{21}(r)\psi_l^{(1)}(r)]\,dr\right\}^{-1},$$
$$S_l^{(12)}=\left\{\frac{1}{ik_1}\int_0^\infty j_l(k_1r)[U_{11}(r)\psi_l^{(1)}(r)+U_{12}(r)\psi_l^{(2)}(r)]\,dr\right\}$$
$$\times\left\{1+\frac{i}{k_2}\int_0^\infty h_l^{(1)}(k_2r)[U_{22}(r)\psi_l^{(2)}(r)+U_{21}(r)\psi_l^{(1)}(r)]\,dr\right\}^{-1}. \quad (3a)$$

In nonrelativistic quantum mechanics it is known that the only singularities of the S matrix in the complex l plane are poles whose positions change with energy. It is easy to see that such a pole only occurs when the denominator in the expression for $S_l^{(ik)}$ vanishes. It can also be shown that the various denominators in (3) vanish simultaneously.

The equation which determines the zero of the denominator of (3), i.e., the trajectory of the pole of the S matrix, will be written in the following form:

$$\frac{1}{k_1}\int_0^\infty j_l(k_1r)[U_{11}(r)\psi_l^{(1)}(r)+U_{12}(r)\psi_l^{(2)}(r)]\,dr$$
$$=\frac{ie^{i\pi l}}{k_1}\int_0^\infty j_{-l-1}(k_1r)[U_{11}(r)\psi_l^{(1)}(r)+U_{12}(r)\psi_l^{(2)}(r)]\,dr$$
$$+ie^{i\pi l}\cos\pi l. \quad (4)$$

The wave functions $\psi_l^{(1)}(r)$ and $\psi_l^{(2)}(r)$ can be found by integrating the system of equations (1). In the lowest order of perturbation theory

$$\psi_l^{(1)}\approx j_l(k_1r), \qquad \psi_l^{(2)}(r)\approx Dj_l(k_2r),$$

from where we obtain the following equation for the trajectories of the poles:

$$\left\{1-\frac{m_1g_{11}}{ik_1\cos\pi l}[e^{-i\pi l}Q_l(x_{11})-iR_l(1,1)]\right\}\left\{1-\frac{m_2g_{22}}{ik_2\cos\pi l}\right.$$
$$\left.\times[e^{-i\pi l}Q_l(x_{22})-iR_l(2,2)]\right\}$$
$$+\frac{m_1m_2}{k_1k_2}g_{12}^2\frac{1}{\cos^2\pi l}\{e^{-i\pi l}Q_l(x_{12})-iR_l(2,1)\}$$
$$\times\{e^{-i\pi l}Q_l(x_{12})-iR_l(1,2)\}=0. \quad (5)$$

Here

$$x_{ij}=\frac{1}{2}(k_i/k_j+k_j/k_i+\mu_{ij}^2/k_ik_j),$$
$$Q_l(x_{ij})=2\int_0^\infty j_l(k_ir)j_l(k_jr)r^{-1}\exp[-\mu_{ij}r]\,dr,$$
$$R_l(i,j)=2\int_0^\infty j_{-l-1}(k_ir)j_l(k_jr)r^{-1}\exp[-\mu_{ij}r]\,dr. \quad (6)$$

$Q_l(z)$ is the Legendre function of the second kind. Formulas (5) and (6) go over directly into the formulas of Azimov et al.[1] if $g_{12}=0$.

Formula (5) contains terms of different order of smallness. The terms proportional to g or g^2 give a contribution to (5) only if Q_l or R_l are sufficiently large. It is known that the singularities of Q_l are simple poles at the negative integer points in the complex l plane; R_l is an entire function of l. From this we see that (5) can be satisfied in a small neighborhood of the negative integer values of l; here it suffices to include only the terms with $Q_l(x_{ik})$ and the free term not containing small multipliers of the type g or g^2, and to neglect $R_l(i, k)$. For small k_1 or k_2, i.e., for large values of x_{ik}, Q_l behaves like k^{2l+1} and can be arbitrarily large for Re l < $-\frac{1}{2}$. Then (5) can be satisfied far away from the poles of Q_l. It is easy to see, moreover, that for $k_1^2 > 0$ one can also satisfy (5) in the region of large complex l.

Let us now turn to the analysis of the motion of the poles as k_1^2 and k_2^2 are varied.

3. NEGATIVE k_1^2 AND k_2^2

For not too small negative k_1^2 and k_2^2 the argument of Q_l and R_l is of order unity. As already shown, the smallness of the factors g_{ik} can only be compensated by a large value of Q_l near negative integer l. In order to find the equation for the trajectory in this region, we replace $Q_l(x_{ik})$ in (1) by the pole term $(l + n + 1)^{-1} P_n(x_{ik})$ and discard the singularity-free functions R_l. Solving the quadratic equation for $l + n + 1$, we find the trajectory near the point $-n-1$:

$$l = -n-1+\frac{1}{2}\left[\frac{m_1g_{11}}{ik_1}P_n(x_{11})+\frac{m_2g_{22}}{ik_2}P_n(x_{22})\right]$$
$$\pm\frac{1}{2}\left\{\left[\frac{m_1g_{11}}{ik_1}P_n(x_{11})-\frac{m_2g_{22}}{ik_2}P_n(x_{22})\right]^2 + 4\frac{m_1m_2}{ik_1ik_2}g_{12}^2P_n^2(x_{12})\right\}^{1/2}. \quad (7)$$

For $k_1^2 = -\infty$ both solutions coincide. With increasing k_1^2 and $k_2^2 = k_1^2 + \Delta^2$ they oscillate around the negative integer points without meeting each other (at the points of coincidence the expression under the root sign must vanish; it is easy to see that this is in general impossible for real k_1^2 and k_2^2). Thus (7) defines two solutions, l^{I} and l^{II}, of which the first corresponds to the plus sign in front of the root, and the second, to the minus sign. The phase of the expression under the root sign is chosen such that the root is for $g_{12} \to 0$ equal to

$$\frac{m_1g_{11}}{ik_1}P_n(x_{11})-\frac{m_2g_{22}}{ik_2}P_n(x_{22}).$$

It is clear that in this limit solution l^{I} corresponds to the first, and l^{II} to the second channel.

In the following we restrict the discussion to the attractive case ($g_{ik} < 0$). The repulsive case can be treated by an obvious generalization. For attraction and the above choice of the phase, the solution l^{I} lies to the right of l^{II} at a point with even n. For odd n the situation is the reverse. When k_1^2 reaches the value $-\mu_{11}^2/4$, the argument of $P_n(x_{11})$ becomes equal to -1. It is seen from (7) that, as k_1^2 increases further, the solutions l^{I} begin to move toward each other monotonically along the real axis of l with a velocity which increases with k_1^2 and grows exponentially with n. In the following we shall call these solutions poles of the class I. The solutions l^{II}, which we shall call poles of the class II, all the while continue to oscillate around the negative integer points.

4. SMALL k_1^2 OR k_2^2. POSITIVE k^2

In order to follow the further motion of the poles, we must return to Eq. (5). As already noted earlier, it can now be satisfied far away from the poles of Q_l for sufficiently small k_1^2, such that $|k_1^2/\mu_{11}^2| \ll 1$. As in [1], the regions of applicability of the new equation to be obtained and of Eq. (7) ($l + n + 1 \ll 1$) overlap; thus we follow the trajectory of the poles continuously.

In the region where $|k_1^2/\mu_{11}^2| \ll 1$ we can use the asymptotic forms of Q_l and R_l with respect to the argument. For $k_1^2 \to 0$ the quantity k_2^2 is about equal to $-\Delta^2$, i.e., not small. Therefore

$$Q_l(x_{11}) \approx \sqrt{\pi}\frac{\Gamma(-l-1/2)}{\Gamma(-l)}\left(\frac{k_1^2}{\mu_{11}^2}\right)^{l+1}\operatorname{ctg}\pi l,$$

$$Q_l(x_{12}) \approx \sqrt{\pi}\frac{\Gamma(-l-1/2)}{\Gamma(-l)}\left(\frac{k_1k_2}{\mu_{12}^2+k_2^2}\right)^{l+1}\operatorname{ctg}\pi l,$$

$$R_l(1,1) \sim k_1, \quad R_l(1,2) \sim k_1^{-l}, \quad R_l(2,1) \sim k_1^{l+1}. \quad (8)^*$$

The equation for the trajectories of the poles of class I is found by substituting (8) in (5) and assuming that $g_{ik}k_1^{2(l+1)} \sim 1$, although g_{ik} is small. The equation for the poles of class II, located near the negative integer points for $k_1 \to 0$, is most simply obtained directly from (7) by expanding $P_n(x_{ik})$ in the argument. We find

$$\sqrt{\pi}\frac{\Gamma(-l-1/2)}{\Gamma(-l)}\frac{m_1g_{11}}{\mu_{11}}\left(\frac{k_1^2}{\mu_{11}^2}e^{-i\pi}\right)^{l+1/2} = \sin\pi l \quad \text{(class I)}$$

$$l^{\mathrm{II}} = -n-1+\frac{m_2g_{22}}{ik_2}P_n(x_{22}) - \frac{m_2}{ik_2}\frac{g_{12}^2}{g_{11}}\frac{\Gamma(n+1/2)}{\Gamma(n+1)\sqrt{\pi}}\left(\frac{(\mu_{12}^2+k_2^2)^2}{\mu_{11}^2k_2^2}\right)^n \quad \text{(class II)} \quad (9)$$

The first of Eqs. (9), describing the motion of the poles of class I, agrees exactly with the equation obtained in [1] for the single-channel problem. Hence, for $k_1 \to 0$ the presence of a second channel does not affect the motion of the poles of class I in lowest order of g_{ik}. For the poles of class II, on the other hand, the presence of the first channel leads to the appearance of the addi-

*ctg = cot.

tional last term. As shown in [1], it follows from (9) that as $|k_1^2|$ is decreased, the poles of class I collide, go off into the complex plane, and reach the point $l = -1/2$ for $k_1 \to 0$ along curves resembling circular arcs in the upper and lower half-planes. Further, for $k_1^2 > 0$, the poles again recede into the upper and lower half-planes, preserving the sign of Im l. In the upper half-plane they move to the right of Re $l = -1/2$ to a distance of the order $m_1 g_{11}/\mu_{11}$ and then go off into the second quadrant of the l plane along nearly straight line trajectories. In the lower half-plane the poles move along trajectories close to a circular arc of half the radius as for $k_1^2 < 0$, intersect the real axis at the negative half-odd integer points, and drop into the nearest left integer points for $k_1^2 \to \infty$. The zeroth pole returns through the upper half-plane. The trajectories of all poles of class I except the zeroth pole in the attractive case, are open since their limiting values for $k_1^2 = \pm\infty$ are not identical.

Let us now turn to the discussion of the trajectories for small $|k_2^2|$ when $k_1^2 \sim \Delta^2$. The equations for the trajectories of the poles of classes I and II are found exactly in the same way as in the case of small $|k_1^2|$. The result has the form $(k_1^2 > 0)$

$$l^{\mathrm{I}} = -n - 1 + \frac{m_1 g_{11}}{ik_1} P_n(x_{11}) - \frac{m_1 g_{11}}{ik_1} \frac{g_{12}^2}{g_{11} g_{22}} \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)} \left(\frac{(\mu_{21}^2 + k_1^2)^2}{\mu_{22}^2 k_1^2}\right)^n,$$

$$\sqrt{\pi}\,\frac{\Gamma(-l-1/2)}{\Gamma(-l)} \frac{m_2 g_{22}}{\mu_{22}} \left(\frac{k_2^2}{\mu_{22}^2} e^{-i\pi}\right)^{l+1/2} = \sin \pi l. \tag{10}$$

The first equation describes the poles of class I, moving in the neighborhood of the negative integers through the upper (if $g_{12}^2/g_{11}g_{22} \ll 1$) or lower (if $g_{12}^2/g_{11}g_{22} \gg 1$) half-planes. In the latter case (despite the attraction in all channels) the presence of the second channel leads to a picture which is characteristic for the single-channel problem with repulsion.

The second equation of (10) describes the motion of the poles of class II. If we want to make this equation more precise, we can easily convince ourselves [for example, by considering (7), which is valid if k_2 is not too small] that the trajectories of class II become complex for $k_1^2 > 0$, moving into the upper half-plane by the small distance $\sim g_{12}^2/g_{22}$. This corresponds to the presence of a branch point of the scattering amplitude at $k_1 = 0$, and hence to a branch point of the Regge pole at the same point. If we disregard this complication, the trajectories of class II behave for small k_2, according to (10), exactly as the trajectories of class I for $k_1 \to 0$.

Let us now consider the trajectories of the poles of classes I and II in the upper k_1^2 plane for large complex l. For this purpose we turn again to (5). Here we can use the asymptotic form of Q_l in the index l (for large l we have $e^{i\pi l} R_l \sim l^{-1}$, so that R_l can again be neglected). Replacing $\cos \pi l$ by an increasing exponential, we obtain

$$\left\{1 + 2\frac{m_1 g_{11}}{k_1}\left[\frac{\pi}{-2l \operatorname{sh} \alpha_{11}}\right]^{1/2} \exp\left[-\left(l+\frac{1}{2}\right)\alpha_{11}\right]\right\} \times \left\{1 + 2\frac{m_2 g_{22}}{k_2}\left[\frac{\pi}{-2l \operatorname{sh} \alpha_{22}}\right]^{1/2} \exp\left[-\left(l+\frac{1}{2}\right)\alpha_{22}\right]\right\} - 4\frac{m_1 m_2}{k_1 k_2} g_{12}^2 \frac{\pi}{-2l \operatorname{sh} \alpha_{12}} \exp\left[-2\left(l+\frac{1}{2}\right)\alpha_{12}\right] = 0, \tag{11*}$$

where

$$\operatorname{ch} \alpha_{11} = 1 + \frac{\mu_{11}^2}{2k_1^2}; \quad \operatorname{ch} \alpha_{22} = \frac{\mu_{22}^2}{2k_2^2} + 1; \quad \operatorname{ch} \alpha_{12} = \frac{(k_1^2 + k_2^2 + \mu_{12}^2)}{2k_1 k_2}.$$

Using (11), we can follow the motion of the poles of classes I and II continuously up to large k_1^2. Their trajectories are distorted in comparison with the trajectories of the single-channel problem due to the presence of the term with g_{12} in (11). For $g_{12} \to 0$ Eq. (11) goes over directly into the corresponding formula of [1].

5. ANALYTIC PROPERTIES OF THE TRAJECTORIES

Up to this point we have restricted the discussion to the trajectories of the poles for real k_1^2. In this section we shall consider the analytic properties of the Regge pole trajectories in the complex k_1^2 plane. In this plane there are two branch points: at $k_1^2 = 0$ and $k_1^2 = \Delta^2$, corresponding to the threshold for the two channels. The so-called physical cuts are drawn from these points, as usual, along the positive real axis. Following [2], we shall study the trajectories as $\kappa_1^2 = |k_1^2|$ changes from zero to infinity along a ray with the fixed phase angle φ: $k_1^2 = \kappa_1^2 e^{i(\pi-\varphi)}$.

We now introduce a few definitions. According to Secs. 3 and 4 and the results obtained earlier, [1] the trajectories of the poles of class I are for $k_1^2 \to 0$ described by the equation

$$l = -\frac{1}{2} \pm 2\pi i p \left[\ln \frac{\mu_{11}^2}{k_1^2} + i\pi\right]^{-1}, \quad p = 1, 2, 3 \ldots \tag{12}$$

*sh = sinh, ch = cosh.

The plus and minus signs refer to the so-called upper and lower trajectories receding at the point $l = -\frac{1}{2}$ into the upper and lower half-planes, respectively. The number p may serve as a numbering index for these trajectories. The poles of class II for $k_2^2 \to 0$ are described by an equation which is obtained from (12) by replacing μ_{11} by μ_{22} and k_1 by k_2. The division of the trajectories into upper and lower trajectories is carried out in the same fashion as in the case of the poles of class I.

For sufficiently small $|k_1^2|$ the upper and lower trajectories of class I collide, as was shown in [2]. Analogously, the upper and lower trajectories of class II collide for small $|k_2^2|$. Let these be called collisions of the single-channel or O type. They correspond in the complex k_1^2 plane to root type branch points, whose position is determined by the values of the phase and the modulus of $k_1^2 = \kappa_1^2 e^{i(\pi-\varphi)}$ at which such a collision occurs. The branch points corresponding to collisions of the upper and lower trajectories of classes I and II will also be called branch points of the O type. Their position is determined by the simultaneous solution of (5) and the equation obtained from (5) by differentiation with respect to l. It is convenient simply to set the logarithmic derivative of (5) with respect to l equal to zero:

$$\left\{ i\pi + \pi \operatorname{ctg} \pi l + \frac{m_1 g_{11}}{ik_1} \frac{e^{-i\pi l}}{\sin \pi l} \frac{d}{dl} [Q_l(x_{11}) \operatorname{tg} \pi l] \right\} \times \left\{ 1 + \frac{m_1 g_{11}}{ik_1} \frac{e^{-i\pi l}}{\sin \pi l} [Q_l(x_{11}) \operatorname{tg} \pi l] \right\}^{-1} + \left\{ i\pi + \pi \operatorname{ctg} \pi l + \frac{m_2 g_{22}}{ik_2} \frac{e^{-i\pi l}}{\sin \pi l} \frac{d}{dl} [Q_l(x_{22}) \operatorname{tg} \pi l] \right\} \times \left\{ 1 + \frac{m_2 g_{22}}{ik_2} \frac{e^{-i\pi l}}{\sin \pi l} [Q_l(x_{22}) \operatorname{tg} \pi l] \right\}^{-1} = 2 \frac{d}{dl} \ln [\operatorname{tg} \pi l \cdot Q_l(x_{12})]. \quad (13)^*$$

This determines the value of k_1^2 at which the collision occurs.

If $|k_1^2/\mu_{11}^2| \ll 1$ (collisions of poles of class I), then, according to Sec. 4, we consider only the first term on the left-hand side of (13). Then (13) goes over into the corresponding equation of [2]. There its solution was found. If $|k_2^2/\mu_{22}^2| \ll 1$ the equation for the collision of the poles of class II also coincides with the corresponding formula of [2]. Thus the collisions of the O type for class I as well as class II are described by almost the same formulas as in the single-channel problem.

In contrast to the single-channel problem, there is in our case the possibility of collisions of trajectories from different classes having the same limiting position for $k^2 = -\infty$. These we call collisions of the many-channel or M type. The position of the collision points of this type is most easily found from (7), which describes the behavior of the trajectories of classes I and II for sufficiently large $|k_1^2|$ near the negative integer points $l = -n - 1$. Each collision, evidently, corresponds to the vanishing of the expression under the root sign. Using the fact that $k_2^2 = k_1^2 - \Delta^2$, we can write the expression under the root sign in the form of a sum of two terms

$$(k_1^2)^{-2n-1} (k_2^2)^{-2n-1} [f_{4n+1}(k_1^2) + k_1 k_2 f_{4n}(k_1^2)],$$

where $f_{4n+1}(k_1^2)$ and $f_{4n}(k_1^2)$ are polynomials of the degree $4n + 1$ and $4n$, respectively. The expression under the root sign vanishes for $f^2_{4n+1}(k_1^2) = k_1^2(k_1^2 - \Delta^2) f_{4n}^2(k_1^2)$, i.e., at the points given by the roots of a polynomial of degree $8n + 2$. These roots define the position of $4n + 1$ pairs of complex conjugate branch points of the M type in the complex k_1^2 plane. The cuts originating in these points can be drawn in the same way as for the branch points of the O type. If we classify the trajectories by the number p for $k_1^2 \to 0$ or $k_2^2 \to 0$, we must draw the cuts to infinity.[2]

If we classify the trajectories by their limiting position $l = -n - 1$ for $k_1^2 \to \infty$, then we must draw the cuts from the M type branch points and from the 0 point branch points belonging to class I to the point $k_1^2 = 0$. The cuts from the 0 type branch points of class II are conveniently drawn to the point $k_1^2 = \Delta^2$.[2]

Since the two trajectories corresponding to the two signs in front of the root in (7) have the same limiting position $l = -n - 1$, they require the introduction of two Riemann sheets. The $4n + 1$ pairs of complex conjugate M type branch points are located on both sheets. The cuts leading from these points to the point $k_1^2 = 0$ divide each sheet into $8n + 2$ sectors which we shall designate as sector l^{I} or l^{II} according to the class to which the trajectory in that sector belongs. [We recall that the trajectories of class I lead into the point $l = -\frac{1}{2}$ for $k_1 \to 0$ and so do the trajectories of class II for $k_2 \to 0$, so that the two functions obtained by analytic continuation of the solutions (7) with a plus and a minus sign describe the behavior of the poles of classes I and II, respectively.] Evidently, the sector $l^{I}(l^{II})$ on one sheet corresponds to the sector $l^{II}(l^{I})$ on the other.

The O type branch points correspond to collisions of trajectories of the same class but having

*tg = tan.

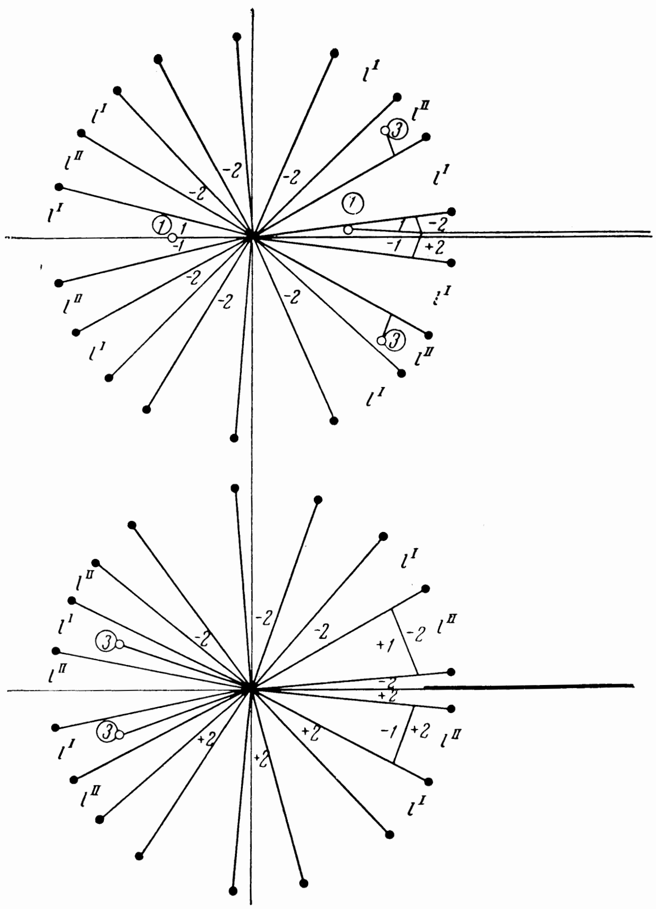

FIG. 1. Schematic illustration of the complex k_1^2 plane for two trajectories with n = 2. The empty dots denote O-type branch points, the full dots denote M-type branch points. The cuts from these points lead into the point $k_1^2 = 0$ (the point $k_1^2 = \Delta^2$ for the O-type branch points of class II). The sectors l^{I} and l^{II} are indicated. Also given are the values $\pm p$ which define the behavior of the trajectories of class I (II) for $k_1^2 \to 0$ ($k_2^2 \to 0$). The numbers n of the trajectories which give rise to an O-type branch point when they collide with another trajectory are enclosed in circles.

different limiting values for $k_1^2 \to \infty$. They are located on only one of the sheets, namely, on the one where they are in the sector l^{I} or l^{II} with the index of their class. The total number of O type branch points is $2(2n - 1)$ (double the number of the single-channel problem; this is connected with the presence of two trajectories with the same limiting position for $k_1^2 \to \infty$).

The whole situation is illustrated schematically in Fig. 1, where the two sheets for the trajectories with n = 2 (limiting value $l = -3$ when $|k_1^2| \to \infty$) are shown. Among the O type branch points there is one real point as in the single-channel problem. This is a branch point of class I located in the sector l^{I}. The analogous point for class II in sector l^{II} is shifted into the upper half-plane by the small amount $\sim g_{12}^2/g_{22}$, i.e., is "almost real" (cf. Sec. 4). Between these two branch points we find $4n + 1$ M type branch points, around which one proceeds from l^{I} to l^{II} and vice versa. Since the number $4n + 1$ is odd, the sector l^{I} is on the same sheet as the real branch point and the sector l^{II} is together with the "almost real" branch point on the other sheet. The picture is complicated by the fact that the cuts from the O type branch points of class II intersect the cuts from the M type branch points. Matters simplify a little if we draw all cuts to the point $k_1^2 = 0$. This, however, involves such a deformation of the cuts from the O type branch points of class II that part of the branch points are shifted from one sheet to the other.

If the cuts from the O type branch point of class II with index 3 (Fig. 1; the index is enclosed by a circle) are drawn to the point $k_1^2 = 0$, the "almost real" branch point with index 1 is shifted to the sheet of the trajectories with n = 3. At the same time, the "almost real" branch point, which earlier was on the sheet of the trajectories with n = 3, now falls on the sheet of the trajectories with n = 2. This is the point with index 4 which now corresponds to the collision of the second trajectory with the fourth for "almost real" k_2^2. The whole situation is illustrated schematically in Fig. 2.

6. N-POLES

According to Sec. 3, the poles of class I move to the point $l = -\frac{1}{2}$ for $|k_1^2| \to 0$. All the while, the poles of class II still remain close to the negative integer points. When $|k_2^2| \to 0$ the reverse situation obtains: the poles of class I are close to the negative integer points, and the poles of class II move to the point $l = -\frac{1}{2}$. The trajectories of the poles moving toward the point $l = -\frac{1}{2}$ have almost the same form as in the single-channel problem. This picture corresponds to arbitrarily small g_{ik}.

Earlier [1,2] it was shown that increasing the coupling constant leads to the appearance of poles of a different type whose limiting position for $k_1^2 = 0$ is to the left of the axis $\mathrm{Re}\, l = -\frac{1}{2}$. These poles were called N-poles, and the corresponding trajectories N trajectories. According to [1,2] the N poles appear already in second order of perturbation theory.

In the two-channel problem we have an analogous situation. The equation for the N-poles can be found in the same way as in [2]. Let us consider the case of small $|k_1^2|$ such that $|k_1^2/\mu_{11}^2| \ll 1$,

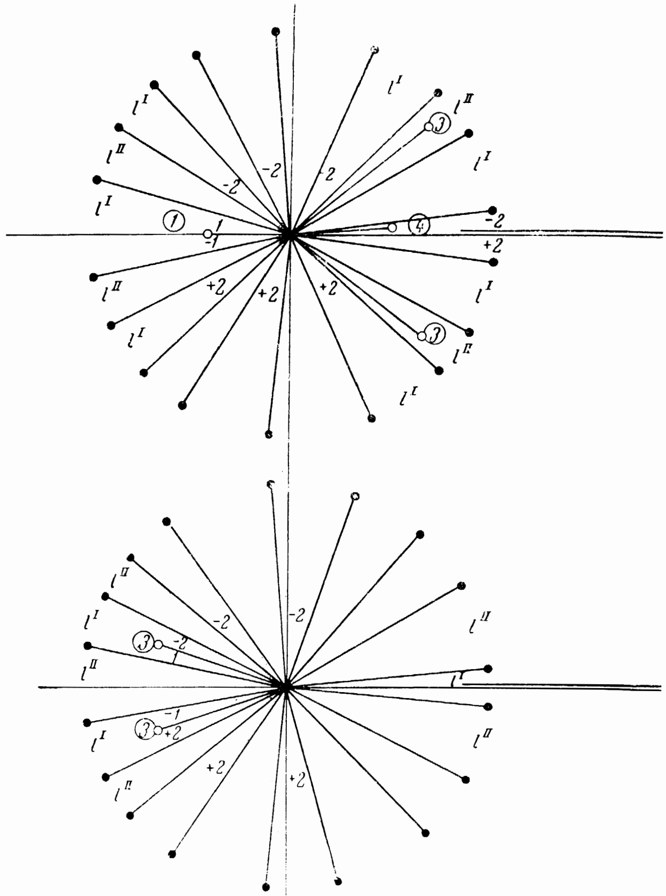

FIG. 2. Schematic illustration of the complex k_1^2 plane for the trajectories with n = 2, when all cuts lead into the point $k_1^2 = 0$. The notation is the same as in Fig. 1.

$k_2^2 \approx -\Delta^2 \sim -\mu^2$. Restricting our attention to the corrections of first order to (5) [the terms of the form $g_{ik} Q_l(x_{ik})$ near negative integer l are considered of order unity], we find

$$\xi_l\left(\frac{k_1^2}{\mu_{11}^2}e^{-i\pi}\right)^{l+1/2}\Big\{1+\frac{m_1 g_{11}}{\mu_{11}(l+1/2)}[2^{-(2l+1)}-1+I_l] + \frac{m_2 g_{22}}{ik_2}\Big[a_l\Big(1+\frac{m_1 g_{11}}{\mu_{11}(l+1/2)}[2^{-2l-1}-1]\Big) - \frac{g_{12}^2}{g_{11}g_{22}}\left(\frac{\mu_{11}^2\Delta^2}{\mu_{12}^2(\mu_{12}^2-\Delta^2)}\right)^{l+1} b_l\Big(1-\frac{m_2 g_{22}}{k_2}c_l\Big)\Big]\Big\} = -\sin\pi l\Big[1-\frac{m_1 g_{11}}{\mu_{11}(l+1/2)}\Big]\Big[1+\frac{m_2 g_{22}}{ik_2}a_l\Big], \qquad (14)$$

where

$$\xi_l = \frac{m_1 g_{11}}{\mu_{11}}\frac{\sqrt{\pi}\,\Gamma(-l-1/2)}{\Gamma(-l)},$$

$$a_l = \frac{1}{\cos\pi l}[e^{-i\pi l}Q_l(x_{22}) - iR_l(2,2)],$$

$$b_l = \frac{\Gamma(-l-1/2)}{\sqrt{\pi}}[\Gamma(l+1) - e^{-i\pi(l+1/2)}\Lambda_l],$$

$$c_l = \frac{1}{\cos\pi l}\Big[R_l(2,2) - \sqrt{\pi}\,\Lambda_l\frac{Q_l(x_{22})}{\Gamma(l+1)}\Big],$$

$$I_l = \frac{\mu_{11}\mu_{12}^{l+3/2}}{\Gamma(l+1)}\int_0^\infty r^{2l+1}e^{-\mu_{11}r}dr\int_0^r\Big[\Big(\frac{x}{r}\Big)^{2l+1}-1\Big] \times e^{-\mu_{12}x}J_{l+1/2}(k_2x)x^{-l-1/2}dx,$$

$$\Lambda_l = \Big[\frac{1}{4}\Big(1-\frac{\mu_{12}^2}{\Delta^2}\Big)\Big]^{(2l+1)/4}P^{l+1/2}_{l+1/2}\Big[\Big(1-\frac{\Delta^2}{\mu_{12}^2}\Big)^{-1/2}\Big],$$

and $P^{l+1/2}_{l+1/2}(z)$ is the associated Legendre polynomial. This equation goes over directly into (5) in the limit of small $|k_1^2|$, if we neglect the correction terms. If we set $g_{12} = g_{22} = 0$, then (14) goes over into the equation for the single-channel problem.

For $k_1^2 \to 0$ the N-poles move to the left of the axis Re $l = -1/2$. The large quantity $(k_1^2/\mu_{11}^2)^{l+1/2}$ is compensated by the vanishing of the curly bracket in (14). This is possible since the small constants g_{ik} are multiplied by quantities of the type $x^{-(2l+1)}$ which may become sufficiently large for $x > 1$ and Re $(-l) \gg 1$.

Far away from the negative integers $\Gamma(l+1)$ and $Q_l(x_{22})$ have no singularities. Then the condition of the vanishing of the curly bracket in (14) leads to an equation which is considerably more complicated than in the case of the single-channel problem. Near the negative integer points a new complication arises which is connected with the fact that the solution of the class II lies in their neighborhood, for which the smallness of g_{ik} is compensated by the singular nature of $Q_l(x_{22})$ or $\Gamma(l+1)$. An explicit solution of the resulting equations is only possible in special cases. It can, however, be shown that increasing g_{ik} leads to collisions of N trajectories and trajectories of the classes I and II in analogy to the collisions described in [2].

CONCLUSION

From the results derived above we can draw the following conclusions about the type of behavior of the Regge trajectories for the two- (or many-) channel problem:

1. For $|k^2| \to \infty$ there are two poles at each negative integer point of the real axis in the l plane. In the case of s channels, the number of poles at each negative integer point for $k^2 \to \infty$ is equal to s.

2. For the s-channel problem the set of poles divides into s classes according to their type of behavior for small momenta. For $k_i^2 \rightarrow 0$ the poles of the i th class go to the point $l = -\frac{1}{2}$ (i = 1, 2, . . . , s).

3. The number of branch points in the complex k_1^2 plane increases. Additional branch points appear owing to the collisions of trajectories from different classes but with the same limiting value for $|k^2| \rightarrow \infty$.

4. The general character of the motion of the poles is not changed as compared with the single-channel problem, although the explicit form of the trajectories is somewhat different. For trajectories of the class i the difference with respect to the single-channel case practically disappears for $k_i^2 \rightarrow 0$.

[1] Azimov, Ansel'm, and Shekhter, JETP 44, 361 (1963), Soviet Phys. JETP **17**, 246 (1963).
[2] Azimov, Ansel'm, and Shekhter, JETP 44, 1078 (1963), Soviet Phys. JETP 17, 726 (1963).

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269