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POLES OF THE VERTEX FUNCTION AND ORTHOGONALIZATION OF ONE-PARTICLE STATES

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It is shown that the poles of the vertex function, which appear when two elementary particles a_1 and a_2 are present in the same channel a, disappear when the wave functions of the particles a_1 and a_2 are made orthogonal to each other.

In the study of the analytic properties of the vertex function $\Gamma(k_a^2, m_b^2, m_c^2) \equiv \Gamma(k^2)$ of three spinless bosons a, b, c as a function of k^2 it is most important to know whether $\Gamma(k^2)$ may have poles in the complex k^2-plane. It is usually assumed that the product of $\Gamma(k^2)$ and the Green's function $G(k^2)$ of particle a has no poles other than the poles of the Green's function $G(k^2)$. Therefore the function $\Gamma(k^2)$ may have poles only at the zeros of $G(k^2)$. In the case when the Green's function $G(k^2)$ has only one pole at $k^2 = m_a^2$ it can be seen from the Kallen-Lehmann representation that it can have no more than one zero and that that zero, if it exists, must lie on the real axis in the interval $m_a^2 < k^2 < (m_b + m_c)^2$. Consequently in this case $\Gamma(k^2)$ can have at most one pole.

Assuming that $\Gamma(k^2)$ has no poles we have obtained previously[1] an upper bound on the coupling constant for the interaction of particles a, b, c. Goebel and Sakita[2] and Braun[3] have expressed the opinion that the assumption of absence of poles in $\Gamma(k^2)$ is incorrect. The assertions of these authors were in a substantial manner based on the considerations of an example when in channel a there occurs beside the particle a_1 an additional discrete state a_2 (more precisely, such a state exists when the interaction causing the transitions $a_1 \longleftrightarrow a_2$ is turned off, i.e., for example, in the primary Lagrangian). In this example the authors of [2,3] find poles in the vertex function. In their proof, however, a very important circumstance was not taken into account. If two states exist in one channel, then the wave functions for these states should be orthogonal, i.e., the physical (renormalized) operators should be so defined that transitions from one state to the other (on the mass shell of each state) are absent (see, for example, [4]). In this paper we discuss the problem with orthogonalization taken into account and show that after orthogonalization the pole in the vertex function is absent.

Let us consider first the case when there are in channel a two elementary particles a_1 and a_2. We do not assume a specific form for the interaction Lagrangian. The equations for the Green's functions of these particles will be written in the form

$$\sum_{k=1}^{2}[(k^2-\mu_{i0}^2)\delta_{ik} - M_{ik}(k^2)]G_{kl}(k^2) = \delta_{il}, \qquad i, l = 1, 2. \tag{1}$$

Here μ_{i0} is the bare mass of the i-th particle, $M_{ii}(k^2)$ and $G_{ii}(k^2)$ are the mass operator and Green's function of the i-th particle $M_{ik}(k^2)$ and $G_{ik}(k^2)$ $(i \neq k)$ are the mass operator and Green's function for the transition $i \to k$. It is obvious that $M_{ik} = M_{ki}$, $G_{ik} = G_{ki}$. Eliminating from Eq. (1) the functions G_{12} and G_{21} we easily obtain the equations for G_{11} and G_{22}:

$$\left[k^2-\mu_{10}^2-M_{11}(k^2)-\frac{M_{12}^2(k^2)}{k^2-\mu_{20}^2-M_{22}(k^2)}\right]G_{11}(k^2)=1,$$

$$\left[k^2-\mu_{20}^2-M_{22}(k^2)-\frac{M_{12}^2(k^2)}{k^2-\mu_{10}^2-M_{11}(k^2)}\right]G_{22}(k^2)=1. \tag{2}$$

It is seen from (2) that $G_{11}(k^2)$ has a zero for k^2 satisfying the equation $k^2 = \mu_{20}^2 + M_{22}(k^2)$, and $G_{22}(k^2)$ has one for $k^2 = \mu_{10}^2 + M_{11}(k^2)$, provided that at these values $M_{12}(k^2)$ does not also vanish. Corresponding to the zeros of the Green's functions $G_{11}(k^2)$ and $G_{22}(k^2)$ the vertex functions $\Gamma_1(k^2)$ and $\Gamma_2(k^2)$ have poles. The presence of poles in $\Gamma_1(k^2)$ and $\Gamma_2(k^2)$ can also be seen directly by writing Γ_1 and Γ_2 in the form

$$g_1\Gamma_1(k^2) = R_1(k^2) + R_2(k^2)M_{12}(k^2)\,/\,(k^2-\mu_{20}^2-M_{22}(k^2)),$$

$$g_2\Gamma_2(k^2) = R_2(k^2) + R_1(k^2)M_{12}(k^2)\,/\,(k^2-\mu_{10}^2-M_{11}(k^2)), \tag{3}$$

where $R_1(R_2)$ represents the contribution of the totality of those diagrams representing $\Gamma_1(\Gamma_2)$ which cannot be broken up into two parts by cutting just one line of particle $a_2(a_1)$.

Consequently, the not orthogonalized Green's functions G_{11} and G_{22} have zeros, and the not orthogonalized vertex functions Γ_1 and Γ_2 have poles. (Orthogonalization of one-particle states has been carried out previously in a number of papers.[4]) Let us now orthogonalize the states 1 and 2. It is necessary that on the mass shell of particles a_1 and a_2 the transition matrix element $M_{12}(k^2)$ vanish, i.e., $M_{12}(\mu_1^2) = M_{12}(\mu_2^2) = 0$ (μ_1 and μ_2 stand for the physical masses). In order to accomplish this we write the mass operator in the form

$$M_{ik}(k^2) = A_{ik} + B_{ik}k^2 + \widetilde{M}_{ik}(k^2),$$

where

$$A_{ik} = \frac{\mu_2^2 M_{ik}(\mu_1^2) - \mu_1^2 M_{ik}(\mu_2^2)}{\mu_2^2 - \mu_1^2},$$

$$B_{ik} = \frac{M_{ik}(\mu_2^2) - M_{ik}(\mu_1^2)}{\mu_2^2 - \mu_1^2}$$

It is clear that $\widetilde{M}_{ik}(\mu_1^2) = \widetilde{M}_{ik}(\mu_2^2) = 0$. Equation (1) for the Green's functions becomes

$$\sum_k [k^2(\delta_{ik} - B_{ik}) - (\mu_{0i}^2\delta_{ik} + A_{ik}) - \widetilde{M}_{ik}(k^2)] G_{kl}(k^2) = \delta_{il}.$$

Let us introduce now in place of the φ operators of the fields a_1 and a_2 new operators φ' by the linear transformation

$$\varphi_i' = \sum_k \alpha_{ik}\varphi_k$$

and we choose the parameters in this transformation such that the tensors $\mu_{0i}^2\delta_{ik} + A_{ik}$ and $\delta_{ik} - B_{ik}$ become diagonal, with the tensor $\delta_{ik} - B_{ik}$ equal to the unit tensor δ_{ik}. This can be done because the quadratic forms

$$\sum_{i,k} \varphi_i(\delta_{ik} - B_{ik})\varphi_k, \qquad \sum_{i,k} \varphi_i(\mu_{0i}^2\delta_{ik} + A_{ik})\varphi_k$$

are positive definite. After orthogonalization the equations for the Green's functions become

$$\sum_k [(k^2 - \mu_i^2)\delta_{ik} - \widetilde{M}_{ik}'(k^2)] G_{kl}'(k^2) = \delta_{il},$$

and, consequently, the transformed Green's functions $G_{ik}'(k^2)$ and vertex parts $\Gamma_i'(k^2)$ are expressed in terms of the transformed mass operators $\widetilde{M}_{ik}'$ by means of the same formulas (2), (3) (with the bare masses replaced by the physical ones).

The renormalized Green's function $G_{ii}^R(k^2)$ differs from $G_{ii}'(k^2)$ by a factor:

$$G_{ii}'(k^2) = Z_{2i}G_{ii}^R(k^2), \quad Z_{2i}^{-1} = 1 - d\widetilde{M}_{ii}'(k^2) / dk^2|_{k^2=\mu_i^2}.$$

From (2) and (3) it is directly apparent that as a result of orthogonalization the renormalized Green's function $G_{11}^R(k^2)$ does not have a zero at the point $k^2 = \mu_2^2$, at which the expression $k^2 - \mu_2^2 + \widetilde{M}_{22}'(k^2)$ appearing in the denominator in (2) vanishes, and correspondingly $\Gamma_1'(k^2)$ does not have a pole at this point.

Analogously $G_{22}^R(k^2)$ has no zero (and Γ_2' has no pole) at the point $k^2 = \mu_1^2$. Except for the zero at $k^2 = \mu_2^2$ the expression $Q = k^2 - \mu_2^2 - \widetilde{M}_{22}'(k^2)$ has no other zeros. To see this we note first that it follows from the Kallen-Lehmann representation and (2) that $Q(k^2)$ has at most two zeros. Next, the derivative $Q'(\mu_2^2) > 0$, hence if $Q(k^2)$ has just one more zero at the point $k^2 = k_0^2$ it must be true that

$$Q'(k_0^2) < 0, \quad dG_{11}^R(k^2) / dk^2|_{k^2=k_0^2} > 0$$

which is in contradiction with the sign of the derivative of the Green's function at its zero, as determined from the Kallen-Lehmann representation.

Thus we have shown that in the case when there are in channel a two elementary particles a_1 and a_2, the poles in the vertex functions Γ_1 and Γ_2 due to the transitions $a_1 \longleftrightarrow a_2$ disappear when these states are made orthogonal. It seems to us that this conclusion remains valid when one of these states (or both) are not elementary particles but bound states, since also in that case both physical states should be so defined that the transition matrix element vanishes on both mass shells, so that the vanishing of the denominator in expression (3) for Γ is compensated as before by the vanishing of the numerator.

It follows from what has been said above, in particular, that the results obtained previously[1] on the assumption of absence of poles in the vertex function are sufficiently general.

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