BREMSSTRAHLUNG IN THE SCATTERING OF NEUTRINOS ON NUCLEI

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The bremsstrahlung process in the scattering of neutrinos on nuclei is considered. Expressions are obtained for the angular and spectral distributions of the quanta and for the total cross section in the case when the neutrino energy is smaller or much larger than the momentum at which the nuclear form factor begins to decrease appreciably.

IN a recent paper [1] Rosenberg has considered the bremsstrahlung process in the scattering of a neutrino in the Coulomb field, calculated the matrix element for the process and gave an estimate for the total cross section which is valid for neutrino energies ϵ_1 below the value q_0 of the momentum transfer to the nucleus at which the form factor of the nucleus begins to decrease appreciably. We shall calculate the spectral and angular distributions of the quanta, the cross section of the process, and also the energy of the emitted quanta $\epsilon = \int \omega d\sigma_\omega$ for the case $\epsilon_1 < q_0$ and, with logarithmic accuracy, for the case $\epsilon_1 \gg q_0$.

1. CASE $\epsilon_1 < q_0$

The process is described by two graphs, one of which is shown in Fig. 1; the second graph differs from the first only in the directions of the momenta of the virtual electrons. Here $\varkappa_2$, $\varkappa_1$ (ϵ_2, ϵ_1) are the final and initial momenta (energies) of the neutrino, k (ω) is the momentum (energy) of the emitted quantum, q is the momentum transfer to the nucleus, and p is the momentum of the virtual electron.

Rosenberg's expression for the cross section has the form[1)]

$$d\sigma = (2\pi)^{-9} G^2 Z^2 e^6 f^2(q^2) |F(q, k)|^2 \times \omega(1 - zz_0)\delta(\varepsilon_1 - \omega - \varepsilon_2) d^3k d^3\varkappa_2, \qquad (1)$$

$$F(q,k) = 4\int_0^1 du_1 \int_0^{1-u_1} du_2 \frac{u_1(1-u_1-u_2)}{q^2u_1(1-u_1) + 2qku_1u_2 + m_e^2},$$

$$z_0 = \cos(\widehat{\mathbf{k}, \varkappa_1}), \quad z = \cos(\widehat{\mathbf{k}, \varkappa_2}), \quad z' = \cos(\widehat{\varkappa_1, \varkappa_2}), \qquad (2)$$

m_e is the electron mass and $f(q^2)$ is the nuclear form factor, which we replace by unity in this section, since we are concerned with the case $\epsilon_1 < q_0$. Therefore, we have from (1) for $\epsilon_1 < q_0$

$$\frac{d^2\sigma}{d\omega dz_0} = \frac{2}{(2\pi)^8} G^2 Z^2 e^6 \omega^3 \varepsilon_2^2 \int_{-1}^{1} dz' \int_{z_-}^{z_+} |F(q,k)|^2 (1 - zz_0) \frac{dz}{(1 + 2zz'z_0 - z^2 - z'^2 - z_0^2)^{1/2}}. \qquad (3)$$

For $\epsilon_1 \gg m_e$ we can neglect m_e^2 in (2), so that $F(q, k)$ takes the form

$$F(q,k) = -\frac{1}{qk} + \frac{1}{qk}\left(1 + \frac{q^2}{2qk}\right)\ln\left(1 + \frac{2qk}{q^2}\right);$$

$$F(q,k) \approx \frac{1}{qk}\ln\frac{2qk}{eq^2} \quad \text{for} \quad qk \gg q^2,$$

$$F(q,k) \approx \frac{1}{q^2} \quad \text{for} \quad |qk| \ll q^2 \qquad (4)$$

(e is the base of the natural logarithm).

If $qk < 0$, then $|qk| \leq q^2/2$. Indeed,

$$q^2 + 2qk = (q+k)^2 = (\varkappa_1 - \varkappa_2)^2 = -2\varkappa_1\varkappa_2 \geqslant 0.$$

For $qk \to -q^2/2$ we have $F(q, k) = -1/qk$.

Let us consider the function

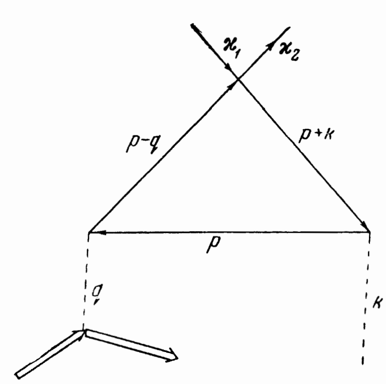

FIG. 1

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1)Formula (25) of [1] contains a superfluous factor 1/2.

$$\tilde{F}(q,k) = (q^2+qk)^{-1}$$
$$= [2\varepsilon_1\varepsilon_2(1-z') + \omega\varepsilon_1(1-z_0) - \omega\varepsilon_2(1-z)]^{-1}. \quad (4')$$

It is equal to F(q, k) for $qk = -q^2/2$ and $|qk| \ll q^2$ and differs little from F(q, k) when $|qk| \sim q^2$. On the strip $-q^2/2 \le qk \le q^2$ the maximal difference between the two functions is 22%, and for $qk = 10q^2$ we have $F \approx 2\tilde{F}$. For $qk \gg q^2$ the following relation holds: $F(q,k) \approx \tilde{F}(q,k) \times \ln(2qk/eq^2)$. It will be shown in the Appendix that the contribution from states for which $qk \gg q^2$ is small compared with the contribution from states for which $|qk| \sim q^2$, if we are interested in the total cross section and the spectral distribution $d\sigma/d\omega$. This holds also as far as the quantity $d^2\sigma/d\omega dz_0$ is concerned, with the exception of the region

$$\varepsilon_2 \ll \varepsilon_1, \quad \frac{\varepsilon_2^2}{\varepsilon_1^2}\frac{1}{1-z_0} \gtrsim 1.$$

Therefore, F(q, k) in (3) can be replaced approximately by $\tilde{F}(q,k)$. This allows us to carry out the integration completely, whereas the integrals with F(q, k) cannot be expressed in terms of elementary functions. As a result we obtain

$$\frac{d^2\sigma}{d\omega dz_0} = \frac{G^2Z^2e^6}{(2\pi)^7}\left\{\frac{\varepsilon_2^2/\omega^2}{\varepsilon_2/\varepsilon_1 + {}^1/_2(1-z_0)}\right.$$
$$\times\left[1 + z_0\frac{1+\omega/2\varepsilon_1 - (z_0-\omega/2\varepsilon_1)\omega/2\varepsilon_2}{1+\omega^2/4\varepsilon_1^2 - z_0\omega/\varepsilon_1}\right.$$
$$\left.\left.+\frac{1}{4}z_0\left(z_0 - \frac{\omega}{2\varepsilon_1}\right)L\right\}\frac{\omega^3}{\varepsilon_1^2}, \quad (5)$$

$$L = \left(1 + \frac{\omega^2}{4\varepsilon_1^2} - \frac{\omega}{\varepsilon_1}z_0\right)^{-3/2}$$
$$\times \ln\frac{[1-\omega/2\varepsilon_1 + (1-z_0)\omega/2\varepsilon_2 + (1+\omega^2/4\varepsilon_1^2 - z_0\omega/\varepsilon_1)^{1/2}]^2}{(\omega^2/2\varepsilon_2^2)(1-z_0)[\varepsilon_2/\varepsilon_1 + {}^1/_2(1-z_0)]}, \quad (6)$$

$$\frac{d\sigma}{d\omega} = \frac{G^2Z^2e^6}{(2\pi)^7}\varepsilon_1\left\{-\frac{4}{3}\frac{\omega\varepsilon_2}{\varepsilon_1^2} + \frac{8}{3}\ln\frac{\varepsilon_2}{\varepsilon_1} + \frac{1}{3}\frac{\omega^3}{\varepsilon_1^3}\ln\frac{\varepsilon_2}{\omega}\right.$$
$$\left.+\frac{\omega}{\varepsilon_1}\left(8\frac{\varepsilon_2}{\varepsilon_1} + 3\frac{\omega^2}{\varepsilon_1^2}\right)\ln\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_2}\right\}, \quad (7)$$

$$\sigma = \frac{G^2Z^2e^6}{(2\pi)^7}\varepsilon_1^2\left[\frac{20}{3}\ln 2 - \frac{13}{3}\right] \approx \frac{0.29}{(2\pi)^7}G^2Z^2e^6\varepsilon_1^2$$
$$= \left(\frac{\varepsilon_1}{M}\right)^2 Z^2\cdot 0.26\cdot 10^{-46}\ \mathrm{cm}^2. \quad (8)$$

We note that Rosenberg's estimate of the cross section is 13 times smaller than (8).

For $\omega \ll \epsilon_1$ we have

$$\frac{d^2\sigma}{d\omega dz_0} = \frac{1}{(2\pi)^7}G^2Z^2e^6\left[\frac{2(1+z_0)}{3-z_0} + \frac{z_0^2}{4}\frac{\omega^2}{\varepsilon_1^2}\ln\frac{1}{1-z_0}\right]\omega, \quad (9)$$

$$\frac{d\sigma}{d\omega} = \frac{1.54}{(2\pi)^7}G^2Z^2e^6\omega. \quad (9')$$

For $\epsilon_2 = \epsilon_1 - \omega \ll \epsilon_1$ we have

$$\frac{d^2\sigma}{d\omega dz_0} = \frac{1}{(2\pi)^7}G^2Z^2e^6\varepsilon_1$$
$$\times\left\{z_0\ln\left[1 + \frac{2\varepsilon_2}{\varepsilon_1(1-z_0)}\right] + 2\frac{\varepsilon_2}{\varepsilon_1}\frac{1+z_0-z_0^2\varepsilon_1/\varepsilon_2}{2+(1-z_0)\varepsilon_1/\varepsilon_2}\right\}, \quad (10)$$

$$\frac{d\sigma}{d\omega} = \frac{2}{(2\pi)^7}G^2Z^2e^6\varepsilon_2. \quad (10')$$

2. CASE $\epsilon_1 \gg q_0$

The kinematics and the notation to be used are explained in Fig. 2. If we assume a form factor $f(g^2)$ with threshold cut-off at $q^2 = q_0^2$, then there exists a limiting angle into which the photon can be emitted. If ω and $\epsilon_2 = \epsilon_1 - \omega$ are much larger than q_0, then

$$(1-z_0)_{max} = q_0\varepsilon_2/\omega\varepsilon_1. \quad (11)$$

Let us consider two angular regions in detail:

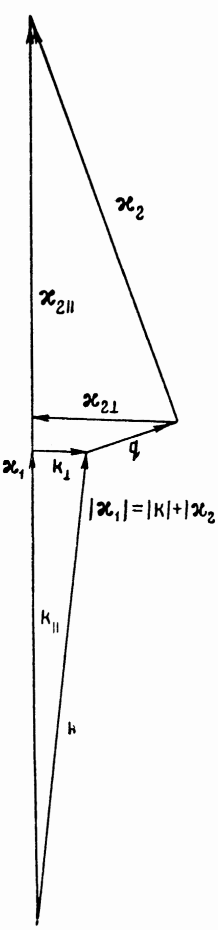

FIG. 2

$$(1 - z_0) \ll q_0^2 / 2\omega^2, \tag{12}$$

$$(1 - z_0) \gg q_0^2 / 2\omega^2. \tag{13}$$

In region (12) we have $q^2 = \kappa_{2\perp}^2 = 2\epsilon_2^2(1 - z')$ for q^2 close to q_0^2. In this region we can therefore replace $f^2(q^2)$ by unity in (1) and integrate over $(1 - z')$ from 0 to $q_0^2/2\epsilon_2^2$. In this case the proof for the smallness of the contribution to $d^2\sigma/d\omega dz_0$ from the region where qk/q^2 is sufficiently large still holds, and $F(q, k)$ in (3) can be replaced by $\widetilde{F}(q, k)$. For $\omega \sim \epsilon_2 \sim \epsilon_1$ the integration leads to the result

$$d\sigma = \frac{G^2Z^2e^6}{(2\pi)^7}\left\{\frac{\omega\varepsilon_2}{\varepsilon_1}\left[1 + \frac{1 + \omega/2\varepsilon_1 - (1 - \omega/2\varepsilon_1)\omega/2\varepsilon_2}{(1 - \omega/2\varepsilon_1)^2}\right] + \frac{\omega^3}{(\varepsilon_1 + \varepsilon_2)^2}\ln\frac{(\varepsilon_1 + \varepsilon_2)^2 q_0^2}{2\omega^2\varepsilon_1\varepsilon_2(1 - z_0)}\right\} d\omega dz_0. \tag{14}$$

Let us now consider the region (13). Here varying q^2 from the smallest possible value (for fixed ω and z_0), which we denote by $\widetilde{q}^2$, to q_0^2 leads to insignificant change in qk, and we have

$$qk = (\varepsilon_1 / \varepsilon_2)\,\omega^2(1 - z_0). \tag{15}$$

In the region (13) qk is much larger than q^2, and we find from (4)

$$F(q, k) = \frac{1}{qk}\ln\frac{2qk}{eq^2} = \frac{\varepsilon_2}{\varepsilon_1\omega^2(1 - z_0)}\ln\frac{2\omega^2\varepsilon_1(1 - z_0)}{e\varepsilon_2 q^2}. \tag{16}$$

The quantity $1 - zz_0$ entering in (1) also remains almost constant [as long as (13) holds]:

$$1 - zz_0 = (1 - z_0)(\varepsilon_1^2 + \varepsilon_2^2) / \varepsilon_2^2. \tag{17}$$

Under the same condition we have

$$d\Omega_{\varkappa_2} = \pi\frac{d(q^2 - \widetilde{q}^2)}{\varepsilon_2^2} = \pi\frac{dq^2}{\varepsilon_2^2}, \tag{18}$$

where we must integrate over q^2 from $\widetilde{q}^2$ to q_0^2.

Substituting (15) to (18) in (1) and integrating over $d^3\kappa_2$, we obtain, keeping only the highest power of the logarithm,

$$d\sigma = \frac{G^2Z^2e^6}{(2\pi)^7}\frac{\varepsilon_1^2 + \varepsilon_2^2}{2\varepsilon_1^2\omega(1 - z_0)}\left[\ln\frac{2\omega^2\varepsilon_1(1 - z_0)}{e\varepsilon_2 q_0^2}\right]^2 \times (q_0^2 - \widetilde{q}^2)\, d\omega dz_0. \tag{19}$$

For $\omega^2(1 - z_0) \sim q_0^2$ and $\omega \sim \epsilon_2 \sim \epsilon_1$, formulas (19) and (14) join.

It is clear from (14) that the contribution to $d\sigma/d\omega$ from quanta for which $1 - z_0 \le q_0^2/2\omega^2$ contains no terms logarithmic in ω/q_0 and therefore, the main contribution to $d\sigma/d\omega$ for $\omega/q_0 \gg 1$ comes from quanta for which $(1 - z_0)_{max} \ge (1 - z_0) \ge q_0^2/2\omega^2$. Integrating (19) within these limits and keeping only the highest power of the logarithm in q_0, we have

$$\frac{d\sigma}{d\omega} = \frac{1}{3}\frac{G^2Z^2e^6}{(2\pi)^7}\frac{\varepsilon_1^2 + \varepsilon_2^2}{2\varepsilon_1^2\omega}q_0^2\left[\ln\frac{2\omega}{eq_0}\right]^3. \tag{20}$$

The total cross section and the energy of the emitted quanta $E = \int \omega d\sigma$ are of the form (keeping only the main terms)

$$\sigma = \frac{1}{12}\frac{G^2Z^2e^6}{(2\pi)^7}q_0^2\left[\ln\frac{2\varepsilon_1}{eq_0}\right]^4, \tag{21}$$

$$E = \frac{2}{9}\frac{G^2Z^2e^6}{(2\pi)^7}q_0^2\varepsilon_1\left[\ln\frac{2\varepsilon_1}{eq_0}\right]^3. \tag{22}$$

The threshold fall-off of $f(q^2)$ is an idealization of the real situation, and one must specify what has to be substituted for q_0 in the realistic case. Taking account of (15) to (18), we therefore write (1) in the following form:

$$\frac{d^3\sigma}{d\omega dz_0 dq^2} = \frac{G^2Z^2e^6}{(2\pi)^7}\frac{\varepsilon_1^2 + \varepsilon_2^2}{2\varepsilon_1^2\omega(1 - z_0)}f^2(q^2)\left[\ln\frac{2\omega^2\varepsilon_1(1 - z_0)}{e\varepsilon_2 q^2}\right]^2. \tag{23}$$

Let us integrate (23) first over $(1 - z_0)$ (as the upper limit we must take $q\epsilon_2/\omega\epsilon_1$) and then over q^2:

$$\frac{d\sigma}{d\omega} = \frac{1}{3}\frac{G^2Z^2e^6}{(2\pi)^7}\frac{\varepsilon_1^2 + \varepsilon_2^2}{2\varepsilon_1^2\omega}\int\left[\ln\frac{2\omega}{eq}\right]^3 f^2(q^2)\, dq^2. \tag{24}$$

It is clear from (24) that the main term of $d\sigma/d\omega$ is given by (20) for realistic form factors [2] if we take

$$q_0^2 = \int f^2(q^2)\, dq^2. \tag{25}$$

However, the corrections to the main term will be different for the different form factors. Thus, for a form factor with threshold cut-off the correction to the main term (20) is equal to $(3/2) \times [\ln(2\omega/eq_0)]^2$. In the realistic case it will be less. For $f^2(q^2) = 1 - q^2/2q_0^2$, which is closer to reality, the correction is $1.21[\ln(2\omega/q_0)]^2$.

Badalyan and Chou Kuang-chao [3] have made an estimate of the cross section for the creation of a lepton pair in the scattering of a neutrino in the Coulomb field without account of the nuclear form factor. The total cross section is $\sim \epsilon_1^2$, as was to be expected (logarithmic factors are of no interest here). Kozhushner and Shabalin and Czyz and Walecka [4] have made a calculation taking into account the form factor, and found that the cross section is $\sim q_0\epsilon_1$. In our case the inclusion of the form factor led to an even stronger cut-off of the cross section ($\sim q_0^2$). In other words, in our case the role of the small momentum transfers to the nucleus is smaller than in the papers just mentioned. Another aspect of this fact is the circumstance that in the work of Kozhushner et al. [4] the calculation could have been carried out by the

Weizsäcker-Williams method (method of equivalent photons),[5,6] whereas our problem cannot be solved by this method, since our graph for the elastic scattering of a real quantum by a neutrino leads to a vanishing matrix element.[7]

In the consideration of light nuclei (e.g., the proton we have the particular situation that q_0 is not very different from the mass of the nucleus M and therefore the condition $\epsilon_1 \gg q_0$ means that $\epsilon_1 \gtrsim M$. Since the graphs of Fig. 1 give contributions only for small momentum transfers to the nucleus and q is almost pure space-like, all formulas remain valid if the energy of the neutrino ϵ_1 is comparable to, or larger than, the mass of the nucleus. But for such energies of the incoming particle the radiation from the nucleus becomes important.[6] However, this radiation is distributed over a much larger region of angles than we are considering. One may therefore expect that formula (14) and, in some angular region, formula (19) are valid when $\epsilon_1 > M$.

Another peculiarity of the case of light nuclei is that the magnetic moment of the nucleus makes an appreciable contribution to the graphs considered by us. This contribution amounts to $\sim q_0/M$ of the main term.

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APPENDIX

We show that the contribution to the integral (3) from the region where qk/q^2 is sufficiently large, is small, except in the case when $\epsilon_2 \gg \epsilon_1$, $(\epsilon_2^2/\epsilon_1^2)/(1 - z_0) \gtrsim 1$. We prove this by substituting $\tilde{F}(q, k)$ for $F(q, k)$ in (3). The proof remains valid for $F(q, k)$, since these functions are close to each other for $kq \sim q^2$ and differ only by a factor $\ln(2qk/eq^2)$ in the region $qk \gg q^2$.

It is clear that if the assertion holds for $\omega \sim \epsilon_1$, then it is a fortiori valid for $\omega \ll \epsilon_1$. Let us therefore consider only $\omega \sim \epsilon_2 \sim \epsilon_1$ and $\epsilon_2 \ll \epsilon_1$. For these values of ω the function $\tilde{F}(q, k)$ [formula (4')] varies smoothly with varying z (from z_+ to z_-) and arbitrary fixed z_0 and z' (for $\omega \ll \epsilon_1$ this assertion would be false). This circumstance allows us to consider, instead of (3), the integral

$$\int_{-1}^{1} A(z')\,dz' \int_{z_-}^{z_+} \frac{dz}{(1 + 2zz'z_0 - z_0^2 - z^2 - z'^2)^{1/2}}, \qquad (A.1)$$

for an estimate of the relative contribution of the various regions of z to the integral (3) (for fixed z', z_0).

Let us consider the case $\omega \sim \epsilon_2$. For simplicity we take $\epsilon_2 = \omega = \epsilon_1/2$. As z changes (for fixed z', z_0), the quantity qk/q^2 takes different values. The states with $qk/q^2 \gg 1$ give the largest contribution in the integration of (A.1) over z, if $1 - z' = 1 - z_0$ and $1 - z_0 \to 0$. Indeed, in this case

$$q^2/qk = 4 - (1 - z)/(1 - z_0), \qquad (A.2)$$

and the limits of integration over z are

$$1 - z_- = 4(1 - z_0)\left(1 - \frac{1 - z_0}{2}\right), \qquad 1 - z_+ = 0, \qquad (A.3)$$

so that $qk/q^2 \to 1/2(1 - z_0)$ for $1 - z \to 1 - z_-$.

From (A.2) we obtain the following condition: if $(1 - z) \geq (\leq) 4(1 - z_0)(1 - 1/N)$, then $qk/q^2 \geq (\leq) N$. We have from this that for fixed $1 - z' = 1 - z_0$ and $1 - z_0 \to 0$ the ratio of the contribution to the integral (A.1) from the region where $qk/q^2 > N$ over the contribution from the region where $qk/q^2 < N$ is of the order $N^{-1/2}$.

If z' is different from z_0, then the relative contribution of the region where $qk/q^2 > N$ can only be less, and therefore the role of the regions where $qk/q^2 > N$ in the integration over z' in (A.1) can only become less important in comparison with our estimate. We arrive at the same result if $(1 - z_0)$ is increased.

If $\epsilon_2 \ll \epsilon_1$, we consider two regions of angles:

$$1)\ 1 - z_0 \gg \varepsilon_2^2/\varepsilon_1^2, \qquad 2)\ 1 - z_0 \lesssim \varepsilon_2^2/\varepsilon_1^2.$$

In the first region, evidently $q^2 = 2qk$. In the second region we obtain through a simple analysis that, after averaging over z and z' $qk/q^2 \sim \epsilon_1/\epsilon_2$, and therefore $d^2\sigma/d\omega dz_0$ is, in this region of angles, indeed larger than expression (10) by a factor $[\ln(\epsilon_1/\epsilon_2)]^2$. But it is easy to see from (10) that the contribution from this region to $d\sigma/d\omega$ is small and the result (10') is true.

It should be noted that, for very small $(1 - z_0)$, the coefficient in front of the logarithmic term in $d^2\sigma/d\omega dz_0$ will be larger than indicated in (5), even if $\omega \sim \epsilon_2$, owing to the difference between $F(q, k)$ and $\tilde{F}(q, k)$ by a logarithmic factor for $qk \gg q^2$. However, the character of the increase of $d^2\sigma/d\omega dz_0$ for $(1 - z_0) \to 0$ is correctly given by formula (5).

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