GROWTH OF FLUCTUATIONS IN AN UNSTABLE SYSTEM. II

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Submitted to JETP editor January 16, 1964; resubmitted June 8, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 1291-1302 (October, 1964)

Growth of sound fluctuations in a piezoelectric semiconductor is considered for the case of sound instability due to the effect of a stationary electric field. An equation is obtained which makes it possible to determine the intensity of the fluctuations in a much wider frequency range than the analogous equation of the previous paper.[9] The possibility of applying the method of description of growing fluctuations to consideration of other cases of instability is considered.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

LAST year, a sound instability was discovered in piezoelectric semiconductors[1] and was investigated in a number of works[2-7]. The instability consists in the fact that in constant electric field E in which the drift velocity of the conduction electrons V exceeds the phase velocity of the sound, the sound absorption is replaced by its amplification; in this case the noise (i.e., small sound fluctuations) begins to grow and can increase to a very high level.

The sound instability in the simplest cases is convective.[1)] This means that at a given point of space the intensity of sound fluctuations is stationary in time but on the other hand it reveals a spatial growth "along the current." The growth is limited only by the finite dimensions of the semiconductor or by the nonlinear scattering of the individual growing vibrations on one another. In this paper we consider the case in which the dimensions of the semiconductor limit the growing fluctuations to such a level that nonlinear effects do not yet play a role.

We shall be interested in fluctuations at an instant of time which can be characterized by the quantity $U_{\mathbf{Q}}(\mathbf{R}, t)$ introduced previously in [9][2)]; this quantity has the meaning of a mean-square displacement amplitude in a traveling sound wave with wave vector **Q**. This function is connected with the more useful quantity $N_{\mathbf{Q}}$—the number of phonons in a state with wave vector **Q**—by the simple relation $N_{\mathbf{Q}} = \rho V_0 \omega_{\mathbf{Q}} U_{\mathbf{Q}}/\hbar$, where ρ is the density of the crystal, V_0 is its volume, $\omega_{\mathbf{Q}}$ is the frequency of the sound vibrations.

In our case of classical fluctuations it is natural to use the quantity $U_{\mathbf{Q}}$, and not $N_{\mathbf{Q}}$. We shall show that if $\omega_{\mathbf{Q}}$ does not depend on the time, then the function $U_{\mathbf{Q}}(\mathbf{R})$ satisfies an equation of the form

$$\frac{\partial U_{\mathbf{Q}}}{\partial t} + \frac{\partial \omega_{\mathbf{Q}}}{\partial \mathbf{Q}} \frac{\partial U_{\mathbf{Q}}}{\partial \mathbf{R}} + \gamma U_{\mathbf{Q}} = \left[\frac{\partial U_{\mathbf{Q}}}{\partial t}\right]_T, \qquad (1.1)$$

where γ is the damping coefficient of the sound vibrations, which is negative in the region of instability. The left hand side of this equation has the usual form. The role of the right hand side in the kinetic equation for phonons is played by the operator of phonon-phonon and phonon-electron collisions.

However, in the given case, we cannot describe the right hand side of (1.1) in terms of this operator, since we are interested in low frequency sound fluctuations whose period $2\pi/\omega_{\mathbf{Q}}$ is much greater than the electron relaxation time τ_e, or than the relaxation time τ_p of "thermal" phonons (that is, phonons with energy $\approx T$, where T is the lattice temperature in energy units). Therefore, the interaction of the low frequency vibrations with electrons and thermal phonons cannot be considered as an elementary act, and it is necessary to apply the phenomenological theory of hydrogynamic fluctuations.

The theory of hydrodynamic fluctuations in the state of thermodynamic equilibrium was introduced by Landau and Lifshitz.[10] The equations of the theory of elasticity are obtained in it, and on the right hand sides of these equations there are random forces, while correlations are found between these random forces. The prescription for application of these equations is that in order to express

1)The concept of convective instability was introduced in the book of Landau and Lifshitz.[8]

2)Referred to hereinafter as I.

their solution—the random deformation—in terms of a random force, and then to compute the observed mean of the products of the random deformations, use is made of the correlation equations for random forces.[3)]

In the present work we show, first, that the sound fluctuations in the piezoelectric can be described in terms of equations of the type written down in [10-12], and that it is possible to calculate the correlators between the random forces in the nonequilibrium state which arises under the action of the stationary electric field. Second, from these equations we can derive directly an equation for the observed averages of the products of random quantities. This equation of the kinetics of the fluctuations is easily generalized for the spatially inhomogeneous case also, in which it takes the form (1.1). Its right hand side has the meaning of the power of the source of fluctuations and is expressed in terms of the coefficients that figure in the correlation relations between the random forces. The form of the left hand side is quite natural, although there it is necessary to note one particular circumstance.

The sound instability arises as the result of interaction of sound vibrations with vibrations of the electron density. The latter, in zeroth approximation in this interaction, satisfy an equation of first order in the time—the equation of continuity, in which the ohmic current figures as the density of electron current. There is no mechanical system which would satisfy an equation of such a type in the absence of dissipative processes. Therefore, it was not clear in advance that the mean square of the Fourier component of the fluctuating electron density would satisfy an equation of the type (1.1) with a left hand side characteristic for purely mechanical systems.

The difference between the system considered here and a purely mechanical system is clearly demonstrated in Sec. 4 in the example of a weakly nonstationary fluctuation. The nonstationarity is associated with the change in the external electric field **E**, and leads, for example, to the appearance of a contribution to the damping coefficient of the sound vibrations γ. However, this contribution is nowehere equal to $\dot{\omega}_Q/\omega_Q$, as it would have been according to the theory of adiabatic invariants for a purely mechanical system.

The introduction of equations of the type (1.1), which has been done in the present work in the concrete example of fluctuations in a piezoelectric, is easily generalized to include fluctuations of any other physical quantities which satisfy equations with random forces, the correlator between which is proportional to $\delta(t-t')$. The advantage of such an approach in the consideration of fluctuations in a piezoelectric in comparison with the procedure used in I is, in particular, that the limiting case of fluctuations whose frequency is larger than $1/\tau_M$ (where τ_M is the Maxwell relaxation time of the electron density fluctuations) cannot be considered by the methods of paper I, but can be studied by the methods developed in the present work.

2. INITIAL EQUATIONS IN THE THEORY OF SPATIALLY HOMOGENEOUS FLUCTUATIONS

In order not to write out the cumbersome tensor expressions, let us consider fluctuations of simple systems which are propagated in a direction close to some symmetry axis of the crystal (the x axis) and ultimately write down also the final result which is valid for the general case. As shown in I, the initial equations for the quantity

$$u_\mathbf{q} = V_0^{-1}\int d^3r\, e^{-i\mathbf{q}\mathbf{r}}\, u(\mathbf{r}), \qquad n_\mathbf{q} = V_0^{-1}\int d^3r\, e^{-i\mathbf{q}\mathbf{r}}\, n(\mathbf{r})$$

(where u is the x component of the fluctuation displacement vector, and $n(\mathbf{r})$ is the fluctuation electron density) have the form

$$\rho\frac{\partial^2 u_\mathbf{q}}{\partial t^2} = -\left(\lambda + \frac{4\pi\beta^2}{\varepsilon}\right) q_x^2 u_\mathbf{q} - \eta q_x^2 \frac{\partial u_\mathbf{q}}{\partial t} + \frac{4\pi e\beta}{\varepsilon} n_\mathbf{q} + i q_x s_\mathbf{q}, \tag{2.1}$$

$$\frac{\partial n_\mathbf{q}}{\partial t} + \left[i q_x V + \frac{1}{\tau_M}\left(1 + \frac{q_x^2}{\varkappa^2}\right)\right] n_\mathbf{q} - \frac{\beta}{e\tau_M} q_x^2 u_\mathbf{q} = \frac{i}{e} q_x g_\mathbf{q}, \tag{2.2}$$

where

$$\overline{g^*_\mathbf{q}(t)\, g_{\mathbf{q}'}(t')} = (2T_e\sigma/V_0)\,\delta_{\mathbf{q}'\mathbf{q}}\,\delta(t'-t), \tag{2.3}$$

$$\overline{s^*_\mathbf{q}(t)\, s_{\mathbf{q}'}(t')} = (2T\eta/V_0)\,\delta_{\mathbf{q}'\mathbf{q}}\,\delta(t'-t), \tag{2.4}$$

while $\overline{g^*_\mathbf{q}(t)\, s_{\mathbf{q}'}(t')} = 0$. Here $s_\mathbf{q}$ is the xx component of the tensor of random stresses, $g_\mathbf{q}$ is the density of the random current, ρ is the crystal density, e is the charge on the electron, λ_{iklm} is the tensor of the elastic moduli ($\lambda \equiv \lambda_{xxxx}$ and similarly for the other tensors), $\beta_{i,kl}$ is the tensor of piezoelectric moduli, ϵ_{ik} is the tensor of dielectric constant, η_{iklm} is the tensor of viscosity coefficients, σ_{ik} is the tensor of differential conductivity, D_{ik} is the tensor of diffusion coefficients, $\tau_M = \epsilon/4\pi\sigma$, $T_e = n_0 e^2 D/\sigma$, $\kappa^2 = 4\pi\sigma/\epsilon D$, n_0 is the stationary concentration of the electrons. The bar indicates averaging over the probabilities of all states which can participate in this product

3)The procedure applicable for electomagnetic fluctuations of such a type is described in detail in the books of Rytov[11] and Landau and Lifshitz [12].

(for more details in connection with this meaning of averaging, see I).

For compactness in writing the set (2.1)—(2.2), we put

$$u_{\mathbf{q}} \equiv \xi'^{(1)}_{\mathbf{q}}, \qquad \dot{u}_{\mathbf{q}} \equiv \xi'^{(2)}_{\mathbf{q}}, \qquad n_{\mathbf{q}} \equiv \xi'^{(3)}_{\mathbf{q}}.$$

Then, denoting by $\xi'_{\mathbf{q}}$ the set of three quantities $\xi'^{(n)}_{\mathbf{q}}$, we have

$$\partial \xi'_{\mathbf{q}} / \partial t - \alpha'(\mathbf{q}) \xi'_{\mathbf{q}} = y_{\mathbf{q}}'. \qquad (2.5)$$

Here

$$\alpha' = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\rho}\left(\lambda + \frac{4\pi\beta^2}{\varepsilon}\right) & -\frac{\eta}{\rho} q_x^2 & \frac{4\pi e\beta}{\varepsilon\rho} \\ \frac{\beta}{e\tau_M} q_x^2 & 0 & -iq_xV - \frac{1}{\tau_M}\left(1 + \frac{q_x^2}{\varkappa^2}\right) \end{pmatrix}, \qquad (2.6)$$

while the matrix Y' which enters into the relations

$$\overline{y'^{(n)*}_{\mathbf{q}}(t)\, y'^{(n')}_{\mathbf{q}'}(t')} = V_0^{-1} Y'_{n'n}(\mathbf{q})\, \delta_{\mathbf{q}'\mathbf{q}} \delta(t' - t), \qquad (2.7)$$

has two elements different from zero:

$$Y_{22}' = 2T\eta q_x^2 / \rho^2, \quad Y_{33}' = 2T_e\sigma / e^2.$$

Now let us introduce the linear transformation

$$\xi_{\mathbf{q}} = S\xi'_{\mathbf{q}} \quad \text{or} \quad \xi_{\mathbf{q}}^{(n)} = \sum_{n'} S_{nn'} \xi'^{(n')}_{\mathbf{q}}. \qquad (2.8)$$

Transforming the set (2.5), we have the form

$$\partial \xi_{\mathbf{q}} / \partial t - \alpha \xi_{\mathbf{q}} = y_{\mathbf{q}}; \qquad \alpha = S\alpha' S^{-1}, \quad y_{\mathbf{q}} = Sy'_{\mathbf{q}}. \qquad (2.9)$$

We shall see below that there are no multiple eigenvalues for the matrix α'. In such a case, as is well known, one can always select the matrix S in such a fashion that the matrix α becomes diagonal. Then the set (2.9) takes the form

$$\partial \xi_{\mathbf{q}}^{(n)} / \partial t - \alpha_n \xi_{\mathbf{q}}^{(n)} = y_{\mathbf{q}}^{(n)}. \qquad (2.10)$$

We separate the real and imaginary parts in the complex quantity $\alpha_n(\mathbf{q})$:

$$\alpha_n(\mathbf{q}) = -i\omega_{n\mathbf{q}} - \gamma_{n\mathbf{q}} / 2. \qquad (2.11)$$

As can easily be demonstrated, one can always choose the functions $\xi_{\mathbf{q}}^{(n)}$ in such a way that they satisfy the condition $\xi_{-\mathbf{q}}^{(n)} = \xi_{\mathbf{q}}^{(n)*}$. Below we shall assume this condition to be satisfied.

The random forces in (2.10) satisfy the correlation relations

$$\overline{y_{\mathbf{q}}^{(n)*}(t)\, y_{\mathbf{q}'}^{(n')}(t')} = V_0^{-1} Y_{n'n}(\mathbf{q})\, \delta_{\mathbf{q}'\mathbf{q}} \delta(t' - t), \qquad (2.12)$$

where $Y = SY'S^+$, where S^+ is the Hermitian conjugate of S.

Our goal is to calculate the average of the quantity

$$A_{\mathbf{q}'\mathbf{q}}^{n'n}(t) = \overline{\xi_{\mathbf{q}}^{(n)*}(t)\, \xi_{\mathbf{q}'}^{(n')}(t)},$$

which characterizes the fluctuations. In this section we consider spatially homogeneous fluctuations where $A_{\mathbf{q}'\mathbf{q}}^{n'n} = A_{\mathbf{q}\mathbf{q}}^{n'n}\delta_{\mathbf{q}'\mathbf{q}}$. Let us derive the equation satisfied by the functions $A_{\mathbf{q}\mathbf{q}}^{n'n}$. For this purpose, we consider, along with Eq. (2.10) for the functions $\xi_{\mathbf{q}}^{(n)}(t)$, the equation

$$\frac{\partial}{\partial t} \xi_{\mathbf{q}}^{(m)*}(t+\tau) + \left(-i\omega_{m\mathbf{q}} + \frac{\gamma_{m\mathbf{q}}}{2}\right) \xi_{\mathbf{q}}^{(m)*}(t+\tau) = y_{\mathbf{q}}^{(m)*}(t+\tau), \qquad (2.10a)$$

and the time $\tau > 0$ is chosen such that it is everywhere less than the period of the fluctuations.

We multiply (2.10) by $\xi_{\mathbf{q}}^{(m)*}(t+\tau)$ and (2.10a) by $\xi_{\mathbf{q}}^{(n)}(t)$, carry out the averaging, and add the two equations. On the left hand side of the resulting equation we can obviously set $\tau = 0$ everywhere, since this quantity is much less than the period of fluctuations. In the calculation of the right hand side, we shall take into account that

$$\overline{\xi_{\mathbf{q}}^{(n)}(t)\, y_{\mathbf{q}}^{(m)*}(t+\tau)} = 0,$$

inasmuch as the value of the function ξ at a much earlier time cannot depend on the values of the random force at the later time. Finally, for $\omega_n, \tau \ll 1$ and $|\gamma_{n\mathbf{q}}/\tau| \ll 1$ we get

$$\overline{\xi_{\mathbf{q}'}^{(m)*}(t+\tau)\, y_{\mathbf{q}}^{(n)}(t)} = V_0^{-1} Y_{nm}(\mathbf{q})\, \delta_{\mathbf{q}'\mathbf{q}}. \qquad (2.12a)$$

As a result we obtain the following relation for $A_{\mathbf{q}'\mathbf{q}}^{n'n}$:

$$\frac{\partial}{\partial t} A_{\mathbf{q}'\mathbf{q}}^{n'n} + i(\omega_{n'\mathbf{q}'} - \omega_{n\mathbf{q}}) A_{\mathbf{q}'\mathbf{q}}^{n'n} + {}^1\!/_2 (\gamma_{n'\mathbf{q}'} + \gamma_{n\mathbf{q}}) A_{\mathbf{q}'\mathbf{q}}^{n'n} = V_0^{-1} \delta_{\mathbf{q}'\mathbf{q}} Y_{n'n}(\mathbf{q}). \qquad (2.13)$$

This equation is suitable also for description of nonstationary fluctuations if the nonstationarity is brought about by the explicit dependence of $Y_{n'n}$ on t. If now this nonstationarity is connected with the time dependence of $\omega_{n\mathbf{q}}$ or $\gamma_{n\mathbf{q}}$, then (2.13) no longer holds and the corresponding equation for the description of the functions will be obtained in Sec. 4.

The averages of the initial values of $\overline{\xi_{\mathbf{q}}'^{(n)*} \xi_{\mathbf{q}'}'^{(n')}}$ form a matrix A' which is obtained from A by means of the inverse transformation: $A' = S^{-1} A (S^{-1})^+$.

We shall now determine the matrix α, i.e., we shall find the eigenvalues $\alpha_n(\mathbf{q})$ of the matrix α'. For this purpose, it is necessary to solve the equation $\operatorname{Det}(\alpha' - \alpha I) = 0$ (where I is the unit matrix), or in explicit form [7]

$$(\alpha - i\omega_q^-)(\alpha + i\omega_q^+)\left[\alpha + iq_xV + \frac{1}{\tau_M}\left(1 + \frac{q_x^2}{\varkappa^2}\right)\right] = \frac{4\pi\beta^2 q_x^2}{\varepsilon\rho\tau_M}, \qquad (2.14)$$

where

$$\omega_q^{\mp} = \omega_q' \pm \frac{i\gamma_l}{2}, \quad \gamma_l = \frac{\eta q_x^2}{\rho}, \quad \omega_q'^2 = \omega_q^2 - \frac{\gamma_l^2}{4},$$

$$\omega_q^2 = \left(\lambda + \frac{4\pi\beta^2}{\varepsilon}\right)\frac{q_x^2}{\rho}.$$

We shall solve (2.14) by the method of successive approximations, assuming the right hand side to be small. In the zeroth approximation,

$$\alpha_1^{(0)} = -i\omega_{q0}^{+}, \quad \alpha_2^{(0)} = i\omega_{q0}^{-},$$

$$\alpha_3^{(0)} = -iq_xV - \frac{1}{\tau_M}\left(1 + \frac{q_x^2}{\varkappa^2}\right), \qquad (2.15)$$

where

$$\omega_{q0}^{\pm} = \omega_{q0}' \mp \frac{i\gamma_l}{2}, \quad \omega_{q0}'^2 = \omega_{q0}^2 - \frac{\gamma_l^2}{4}, \quad \omega_{q0}^2 = \frac{\lambda q_x^2}{\rho}.$$

We find the contribution of a first approximation to $\alpha_1^{(0)}$, $\delta\alpha_1^{(1)}$, by substituting on the left hand side of (2.14) in place of the first factor $\delta\alpha_1^{(1)}$, and by setting $\alpha = \alpha_1^{(0)}$ in the second and third factors. In similar fashion, computing the contributions to $\alpha_2^{(0)}$ and $\alpha_3^{(0)}$, we have

$$\delta\alpha_1^{(1)} = \frac{2\pi\beta^2 q_x^2 A_+}{\varepsilon\rho\tau_M\omega_{q0}'},$$

$$\delta\alpha_2^{(1)} = -\frac{2\pi\beta^2 q_x^2 A_-}{\varepsilon\rho\tau_M\omega_{q0}'}, \quad \delta\alpha_3^{(1)} = -\frac{4\pi\beta^2 q_x^2 A_+ A_-}{\varepsilon\rho\tau_M}; \quad (2.16)$$

where

$$A_{\pm} = [q_xV \mp \omega_{q0}' + i\gamma_l/2 - i\tau_M^{-1}(1 + q_x^2\varkappa^{-2}]^{-1}.$$

We shall determine when one can use the method of successive approximations to calculate the corrections (2.16). For calculation of $\delta\alpha_2^{(1)}$, it is necessary that the following inequality be satisfied

$$|\delta\alpha_2^{(1)}| \ll |\alpha_2^{(0)}|. \qquad (2.17)$$

But in piezoelectrics, as a rule,

$$4\pi\beta^2/\varepsilon\lambda \ll 1, \qquad (2.18)$$

and then (2.17) is a direct consequence of (2.18).

To obtain $\delta\alpha_1^{(1)}$ and $\delta\alpha_3^{(1)}$, the procedure is more complicated. For $q_xV \approx \omega_{q0}$, the real parts of $\alpha_1^{(0)}$ and $\alpha_3^{(0)}$ are close to one another; for example, in calculation of the correction $\delta\alpha_1^{(1)}$, one can set $\alpha = \alpha_1^{(0)}$ in the third factor in the left hand side of (2.14), only if

$$|\alpha_1^{(0)} - \alpha_3^{(0)}| \gg |\delta\alpha_1^{(1)}|. \qquad (2.19)$$

The inequality (2.19) is satisfied for all values of V only if

$$\left|\frac{1}{\tau_M}\left(1 + \frac{q_x^2}{\varkappa^2}\right) - \frac{\gamma_l}{2}\right|^2 \gg \frac{2\pi\beta^2 q_x^2}{\varepsilon\rho\tau_M\omega_{q0}'}. \qquad (2.20)$$

Furthermore, it will be assumed that (2.19) does hold. Then the amplified oscillations are close in their character to mechanical (the case of "weak interaction"). For the reverse inequality, there exists a range of values of V in which the first and third formulas of (2.16) are unsuitable, and the corrections $\delta\alpha_1^{(1)}$ and $\delta\alpha_3^{(1)}$ must be determined simultaneously from the quadratic equation. In this case (which we shall not consider here) the vibrations of the amplified branch arise as the result of the strong interaction of the mechanical and electronic systems, brought about by the closeness of the corresponding frequencies. These equations are not similar to any oscillations existing for $\beta = 0$.

It is convenient to find the matrix S in two steps, setting $S = S_2S_1$. The matrix

$$S_1 = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2}\omega_q^- & 0 \\ 1/\sqrt{2} & -i/\sqrt{2}\omega_q^+ & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.21)$$

brings about a partial diagonalization of the matrix α by means of the transformation from the quantities u_q and $\dot{u}_q$ to the quantity $b_q^{(1,2)} = 2^{-1/2}(u_q \pm i\dot{u}_q/\omega_q^{\mp})$. The latter represent (with accuracy up to the factor $\sqrt{2}$) the amplitudes of two traveling sound waves propagating in opposite directions. In the calculation of the matrix S_2 and in what follows, we shall assume that the equality $\gamma_l \ll \tau_M^{-1}(1 + q_x^2\kappa^{-2})$ holds, which is satisfied in practice in the majority of cases of interest, and limit ourselves everywhere to the lowest approximation in $\gamma_l/\omega_{q0} \ll 1$. Then, with the accepted accuracy,

$$S_2 = \begin{pmatrix} 1 & \dfrac{i\pi\beta^2 q_x^2 A_+}{\varepsilon\rho\tau_M\omega_{q0}^2} & \dfrac{4\pi e\beta A_+}{\sqrt{2}\varepsilon\rho\omega_{q0}} \\ \dfrac{i\pi\beta^2 q_x^2 A_-}{\varepsilon\rho\tau_M\omega_{q0}^2} & 1 & -\dfrac{4\pi e\beta A_-}{\sqrt{2}\varepsilon\rho\omega_{q0}} \\ \dfrac{i\beta q_x^2 A_+}{\sqrt{2}e\tau_M} & \dfrac{i\beta q_x^2 A_-}{\sqrt{2}e\tau_M} & 1 \end{pmatrix}. \qquad (2.22)$$

S_2^{-1} is obtained in our approximation from S_2 by means of a change of sign in all the nondiagonal matrix elements. We note that in the case of "weak interaction," [(2.19)] the quantities $\xi_q^{(1)}$, $\xi_q^{(2)}$, and $\xi_q^{(3)}$ retain approximately the meaning of the amplitudes of sound waves traveling in the direction **q** and in the opposite direction, and the amplitudes of the electron density oscillations, respectively.

In the same approximation

$$\omega_{1q} = \omega_{q0}, \quad \omega_{2q} = -\omega_{q0}, \quad \omega_{3q} = q_xV,$$

$$\gamma_{1q}(V) = \gamma_{2q}(-V)$$
$$= \gamma_l + \frac{4\pi\beta^2}{\varepsilon\lambda}\frac{\omega_{q0}(\omega_{q0} - q_xV)\tau_M}{(1 + q_x^2\varkappa^{-2})^2 + (\omega_{q0} - q_xV)^2\tau_M^2},$$

$$\gamma_{3q} = (1 + q_x^2\varkappa^{-2})\tau_M^{-1}. \quad (2.23)$$

Finally,

$$Y_{11}(V) = Y_{22}(-V) = \frac{T\gamma_l}{\rho\omega_{q0}^2} + \frac{4\pi\beta^2}{\varepsilon\lambda}\frac{T_e}{\rho}\frac{\tau_M}{(1+q_x^2\varkappa^{-2})^2 + (\omega_{q0} - q_xV)^2\tau_M^2},$$

$$Y_{12} = Y_{21}^* = -\frac{T\gamma_l}{\rho\omega_{q0}^2} + \frac{4\pi\beta^2}{\varepsilon\lambda}\frac{T_e}{\rho}\frac{\tau_M}{[\omega_{q0}\tau_M + i(1+q_x^2\varkappa^{-2})]^2 - (q_xV\tau_M)^2}$$

$$Y_{33} = Y_{33}', \qquad Y_{13}(V) = Y_{31}^*(V) = -Y_{23}(-V) = -Y_{32}^*(-V) = \frac{4\pi\sqrt{2}\,\beta\sigma q_x^2T_e}{e\varepsilon\rho\omega_{q0}}\frac{1}{q_xV - \omega_{q0} - i(1+q_x^2\varkappa^{-2})\tau_M^{-1}}. \quad (2.24)$$

We shall give without derivation the expressions for γ_{nq} and Y_{nq} in the case in which the wave vector $\mathbf{q}$ is directed not along the axis of symmetry of the piezoelectric but in arbitrary fashion (summation is carried out over repeated indices):

$$\gamma_1(\mathbf{V}) = \gamma_2(-\mathbf{V}) = \frac{\eta_{iklm}q_iq_le_ke_m}{\rho} + \frac{(\beta_{l,ab}q_lq_ae_b)^2(\omega_{q0} - \mathbf{qV})}{\rho\sigma_q[(1+q^2\varkappa^{-2})^2 + (\omega_{q0}-\mathbf{qV})^2\tau_M^2]\,q^2\omega_{q0}}, \quad (2.25)$$

$$\gamma_3 = (1+q^2\varkappa^{-2})\,\tau_M^{-1}; \qquad Y_{11}(\mathbf{V}) = Y_{22}(-\mathbf{V}) = \gamma_0U_0,$$

$$U_0 = \frac{T}{\rho V_0\omega_{q0}^2}, \qquad Y_{33} = \frac{2T_e\sigma_q}{e^2},$$

$$\gamma_0 = \frac{\eta_{iklm}q_iq_le_ke_m}{\rho} + \frac{T_e}{T}\frac{(\beta_{l,ab}q_lq_ae_b)^2}{\rho\sigma_qq^2[(1+q^2\varkappa^{-2})^2 + (\omega_{q0}-\mathbf{qV})^2\tau_M^2]}. \quad (2.26)$$

Here ω_{q0} is the frequency of elastic vibrations for $\beta_{i,kl} = 0$, $\mathbf{e}$ is its polarization vector, $\sigma_q = \sigma_{mn}q_mq_n/q^2$ (and similarly for the other tensors),

$$\varkappa^2 = 4\pi\sigma_q/\varepsilon_qD_q, \qquad \tau_M = \varepsilon_q/4\pi\sigma_q, \qquad T_e = n_0e^2D_q/\sigma_q.$$

For $\mathbf{q}\cdot\mathbf{V} > \omega_{q0}$, we have $\gamma_i < 0$ (which also indicates instability), while $\gamma_2 > 0$ and $\gamma_3 > 0$.

We want to calculate the mean square of the amplitude of the traveling sound wave $\overline{b_q^{(1)*}b_q^{(1)}}$, which will be denoted by the symbol U_q. We have

$$U_{\mathbf{q}} = (S_2^{-1})_{1k}A^{kl}(S^{-1})_{1l}^*.$$

In the case of "weak interaction," when the condition (2.19) holds, the matrix S_2 is "nearly diagonal," and, with the accuracy that we have used, $U_q = A^{11}$. When γ_1 becomes negative, the solution of Eq. (2.13) for $A^{11} = U_q$ increases as a function of t. In other words, for $\gamma_1 < 0$, a real instability arises in the spatially homogeneous case: the sound fluctuations, in accord with linear theory, increase in time without limit.

In practice, each real system is spatially bounded and the boundary effects play a special role in it. If all $\gamma_n > 0$, then for sufficiently large dimensions of the system, the role of these effects is negligible. If some $\gamma_n < 0$, then these effects can play a principal role, independent of the dimensions of the system. Thanks to it, the fluctuations become spatially inhomogeneous. It is essential that here they become stationary in time at any fixed point and for $\gamma_n < 0$ but on the other hand they exhibit spatial growth "along the current." This is the case of convective instability of the system.

In order to consider convective instability, it is tempting simply to add a convective term on the left hand side of Eq. (2.13) for the diagonal elements of the matrix A [see (1.1)]. However, to do so would have heen careless, inasmuch, as was pointed out above, there does not exist a mechanical system which (in the absence of damping) would be described by the set of equations (2.1)–(2.2). Therefore, an equation of the type (1.1) calls in this case for a somewhat more careful derivation, the basic ideas of which will be set forth in the next section.

3. SPATIALLY INHOMOGENEOUS FLUCTUATIONS

Here we derive an equation which makes it possible to determine the diagonal ($n = n'$) elements of the matrix A for the case of a spatial inhomogeneity, in which the characteristic length l over which the growth or decay of the fluctuations takes place is sufficiently large. We shall write down the initial equations and the correlation relations in the coordinate representation:[4)]

$$\frac{\partial\xi^{(n)}(\mathbf{r},t)}{\partial t} - \int d^3r_1\,\alpha_n(\mathbf{r}-\mathbf{r}_1)\,\xi^{(n)}(\mathbf{r}_1) = y^{(n)}(\mathbf{r},t), \quad (3.1)$$

4)Fundamental interest is attached to the behavior of the long-wave fluctuations with wave vectors $q \lesssim \kappa$ (maximum amplification corresponds to $q = \kappa$; for larger q, the amplification coefficient $-\gamma_1$ falls off as q^{-2}); therefore, if for example the integral $\int d^3q\,\alpha(q)$ diverges, it suffices to cut off the integrand when $q \gg \kappa$. The cut-off method has no effect on the form of an equation of the type (1.1) in the significant interval of q.

$$\overline{y^{(n)}(\mathbf{r}; t)\, y^{(n)}(\mathbf{r}', t')} = Y_{nn}(\mathbf{r}' - \mathbf{r})\,\delta(t - t'), \tag{3.2}$$

$$\alpha_n(\mathbf{r}) = (2\pi)^{-3}\int d^3q\,\alpha_n(\mathbf{q})\, e^{i\mathbf{qr}},$$

$$Y_{nn}(\mathbf{r}) = (2\pi)^{-3}\int d^3q\, Y_{nn}(\mathbf{q})\, e^{i\mathbf{qr}}. \tag{3.3}$$

In the case of spatially homogeneous fluctuations,

$$C^{nn}(\mathbf{r}, \mathbf{r}') = \overline{\xi^{(n)}(\mathbf{r})\,\xi^{(n)}(\mathbf{r}')}$$

depends only on $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}'$, while the matrix $A^{nn}_{q'q}$ is proportional to $\delta_{q'q}$. In the presence of spatial damping or growth of the fluctuations, $C^{nn}(\mathbf{r}, \mathbf{r}')$ begins to depend on $\mathbf{r}_0 = (\mathbf{r} + \mathbf{r}')/2$ also, while the matrix $A^{nn}_{q'q}$ differs from zero even for $q \neq q'$. Here, $C^{nn}(\mathbf{r}, \mathbf{r}')$ is a sufficiently steep function of $\Delta\mathbf{r}$ and a smooth function of $\mathbf{r}_0$; $A^{nn}_{q'q}$, on the other hand, depends smoothly on $\mathbf{q}_0 = (\mathbf{q} + \mathbf{q}')/2$ and quite sharply on $\Delta\mathbf{q}$. To be precise, we can state that the matrix elements $A^{nn}_{q'q}$ are essentially different from zero in the interval $\Delta q \approx 1/l$.

We introduce a representation with the aid of wave packets, in which one can write down the equation for the increasing fluctuations. Here we construct the system of functions [5)]

$$\psi_{\mathbf{QR}}(\mathbf{r}) = \psi_{\mathbf{Q}}(\mathbf{r} - \mathbf{R}) = \left(\frac{a^3}{V_0}\right)^{1/2}\sum_{\mathbf{k}} e^{i\mathbf{Qr}} e^{i\mathbf{k}(\mathbf{r}-\mathbf{R})}, \tag{3.4}$$

where the summation over k is carried out in the limits $-\pi/a < k_x, k_y, k_z < \pi/a$. These functions satisfy the orthogonality and normalization relation

$$\int d^3r\, \psi^*_{\mathbf{QR}}(\mathbf{r})\, \psi_{\mathbf{Q'R'}}(\mathbf{r}) = V_0 \delta_{\mathbf{Q'Q}}\delta_{\mathbf{R'R}}. \tag{3.5}$$

Here $\mathbf{R}$ is a discrete set of points, the distance between which $\Delta X = \Delta Y = \Delta Z = a$, and $\Delta Q_x = \Delta Q_y = \Delta Q_z = 2\pi a^{-1}$, so that $\Delta X \Delta Q_x = 2\pi$, etc. We choose the length a such that

$$q \gg a^{-1} \gg l^{-1}, \tag{3.6}$$

and in all other respects we assume it to be arbitrary.

We proceed to the derivation of the equation for the diagonal $(n = n')$ elements of the matrix

$$B^{n'n}_{\mathbf{Q'R}',\mathbf{QR}} = V_0^{-2}\int d^3r \int d^3r'\, \overline{\xi^{(n)}(\mathbf{r})\,\xi^{(n')}(\mathbf{r}')}\,\psi_{\mathbf{QR}}(\mathbf{r})\,\psi^*_{\mathbf{Q'R'}}(\mathbf{r}')$$

$$= \overline{\xi^{(n)*}_{\mathbf{QR}}(t)\,\xi^{(n')}_{\mathbf{Q'R'}}(t)} = \frac{a^3}{V_0}\sum_{\mathbf{kk'}} A^{n'n}_{\mathbf{Q'+k'},\,\mathbf{Q+k}}\, e^{-i\mathbf{kR}+i\mathbf{k'R'}}, \tag{3.7}$$

where $\xi^{(n)}_{\mathbf{QR}}$ is a coefficient in the expansion of the function $\xi^{(n)}(\mathbf{r}, t)$ in a series in the functions $\psi_{\mathbf{QR}}(\mathbf{r})$. Taking into account the properties of the matrix $A^{nn}_{q'q}$ indicated above, it is not difficult to establish the fact that, with the accuracy assumed here,

$$B^{nn}_{\mathbf{Q'R'},\,\mathbf{QR}} = B^{(n)}_{\mathbf{Q}}(\mathbf{R})\,\delta_{\mathbf{Q'Q}}\delta_{\mathbf{R'R}}.$$

In order to derive the equation for $B^{(n)}_{\mathbf{Q}}(\mathbf{R})$, we derive in the coordinate representation of (3.1) an equation similar to (2.13):

$$\frac{\partial}{\partial t}\overline{\xi^{(n)}(\mathbf{r})\,\xi^{(n)}(\mathbf{r}')} - \int d^3r_1\,\alpha_n(\mathbf{r} - \mathbf{r}_1)\,\overline{\xi^{(n)}(\mathbf{r}_1)\,\xi^{(n)}(\mathbf{r}')}$$

$$- \int d^3r_1\,\alpha_n(\mathbf{r}' - \mathbf{r}_1)\,\overline{\xi^{(n)}(\mathbf{r})\,\xi^{(n)}(\mathbf{r}_1)}$$

$$= \overline{y^{(n)}(\mathbf{r}, t)\,\xi^{(n)}(\mathbf{r}', t + \tau)}, \tag{3.8}$$

multiply it by $V_0^{-2}\psi_{\mathbf{QR}}(\mathbf{r})\,\psi^*_{\mathbf{QR}}(\mathbf{r}')$, and integrate over $\mathbf{r}$ and $\mathbf{r}'$. Then the first component on the left hand side gives $\partial B^{(n)}_{\mathbf{Q}}/\partial t$. Using Eq. (3.4) we represent the sum of the second and third components in the form

$$a^3 V_0^{-1}\sum_{\mathbf{k}}\sum_{\mathbf{k}'}[i(\omega_{n,\,\mathbf{Q+k'}} - \omega_{n,\,\mathbf{Q+k}})$$

$$+ {}^1\!/_2(\gamma_{n,\,\mathbf{Q+k'}} + \gamma_{n,\,\mathbf{Q+k}})]\, A^{nn}_{\mathbf{Q+k'},\,\mathbf{Q+k}}\, e^{-i(\mathbf{k}-\mathbf{k'})\mathbf{R}}. \tag{3.9}$$

We expand $\gamma_{n,Q+k}$ and $\omega_{n,Q+k}$ in powers of $\mathbf{k}$, limiting ourselves in the first case to the zeroth and in the second to zeroth and first terms of the expansion. Then (3.9) can be rewritten in the form

$$\mathbf{w}_{n\mathbf{Q}}\;\partial B^{(n)}_{\mathbf{Q}}/\partial\mathbf{R} + \gamma_{n\mathbf{Q}}\; B^{(n)}_{\mathbf{Q}} \quad (\mathbf{w}_{n\mathbf{Q}} \equiv \partial\omega_{n\mathbf{Q}}/\partial\mathbf{Q}).$$

Finally, in analogy with (2.12a), the right hand side of (3.9) can be represented as

$$a^3 V_0^{-2}\sum_k Y_{nn}(\mathbf{Q} + \mathbf{k}) \approx V_0^{-1} Y_{nn}(\mathbf{Q}). \tag{3.10}$$

We then obtain the following equation:

$$\frac{\partial B_{\mathbf{Q}}^{(n)}}{\partial t} + \frac{\partial\omega_{n\mathbf{Q}}}{\partial\mathbf{Q}}\frac{\partial B_{\mathbf{Q}}^{(n)}}{\partial\mathbf{R}} + \gamma_{n\mathbf{Q}} B_{\mathbf{Q}}^{(n)} = \frac{1}{V_0} Y_{nn}(\mathbf{Q}). \tag{3.11}$$

The reverse transition to the q representation is carried out in accord with the formula

$$A^{nn}_{\mathbf{Q'+k'},\,\mathbf{Q+k}} = a^3 V_0^{-1}\sum_{\mathbf{RR'}} B^{nn}_{\mathbf{Q'R'},\,\mathbf{QR}}\, e^{i(\mathbf{kR}-\mathbf{k'R'})}. \tag{3.12}$$

Boundary and initial conditions for $B^{n}_{\mathbf{Q}}(\mathbf{R})$ are specified separately in each case, starting from the concrete physical situation.

By analogy with the spatially homogeneous case, it can be assumed that $B^{(1)}_{\mathbf{Q}}(\mathbf{R})$ is identical with $U_{\mathbf{Q}}(\mathbf{R}) = \overline{b^{(1)*}_{\mathbf{QR}}\, b^{(1)}_{\mathbf{QR}}}$ —the mean square of the amplitude of the sound wave packet. For $\omega_{q_0}\tau_M \ll 1$ and $qV\tau_M \ll 1$, the expression (2.23) and (2.24) for the quantities γ_1 and $Y_{11}(q)$, which enter in (3.11), go over into the formula (5.20) of I, as they should.

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5)Such a representation was first introduced by McIrvine and Overhauser. [13] I am thankful to A. N. Ansel'm who pointed out this work to me.

4. LIMITS OF APPLICABILITY OF THE THEORY AND ITS POSSIBLE GENERALIZATIONS

In order to be able to select the length a in correspondence with the inequalities (2.2), it is necessary that $q \gg l^{-1}$. Inasmuch as, on the other hand, $l^{-1} \approx |\gamma_n w_n^{-1}|$, this inequality can also be rewritten in the form

$$w_{nq} q \gg |\gamma_{nq}|. \qquad (4.1)$$

The second inequality is the condition for the possibility of expanding $\omega_{n,Q+k}$, $\gamma_{n,Q+k}$ and $Y_{n'n}(\mathbf{Q}+\mathbf{k})$ in a series in powers of $\mathbf{k}$. Inasmuch as $k \lesssim a^{-1}$, then in order for this to be the case, $|\omega_n^{-1}\, \partial\omega_n/\partial\mathbf{Q}| \ll a$. But $a \ll l$; therefore, in any case,

$$|\omega_n^{-1}\, \partial\omega_{n\mathbf{Q}}/\partial\mathbf{Q}| \ll l. \qquad (4.2)$$

Similar inequalities must be satisfied for γ_{nQ} and $Y_{nn(Q)}$. However, we emphasize that if we can discard the term with the space derivative in the solution of (3.11), then these conditions may also not be satisfied. An essential condition is also the absence of temporal dispersion in the kinetic coefficients which enter into the problem, for example in the conductivity tensor σ_{ik}.

Finally, we note that the given linear theory is applicable so long as the nonlinear effects of scattering of the growing vibrations by one another do not play a role, and as long as the growing oscillations do not change the correlation relations (2.3) and (2.4).

We now consider the generalization of the theory to the case of nonstationary external conditions. If the function $\alpha'_{n'n}$ in (2.5) depends weakly on the time, Eq. (2.13) must be modified somewhat. A dependence will be called weak if

$$|(\alpha'_{nn'})^{-2}\, \partial\alpha'_{nn'}/\partial t| \ll 1.$$

Then the matrix S which reduces $\alpha'_{n'n}(q)$ to diagonal form is also a weak function of time. As before we shall denote the diagonal matrix which enters into Eqs. (2.10) by α. However, it will now no longer be equal to the matrix $\alpha^{(0)} = S\alpha' S^{-1}$. The point is that the substitution $\xi'_q = S^{-1}\xi$ in Eq. (2.5) does not allow us to reduce α' to diagonal form, since in the calculation of the time derivative, one must also differentiate S. Therefore we set $\xi_q = (1+\Delta)S\xi'$, where Δ is a small matrix which is also obtained from the requirement of diagonality of α. Limiting ourselves to the first approximation, we shall neglect both higher orders of Δ and their time derivatives. Substituting the expression $\xi'_q = S^{-1}(1-\Delta)\xi_q$ in (2.5), we get

$$S\frac{\partial S^{-1}}{\partial t}\xi_{\mathbf{q}} + \frac{\partial \xi_{\mathbf{q}}}{\partial t} - \alpha^{(0)}\xi_{\mathbf{q}} - [\Delta, \alpha^{(0)}]\,\xi_{\mathbf{q}} = (1+\Delta)\, S y'_{\mathbf{q}}. \qquad (4.3)$$

We choose the nondiagonal elements of the matrix Δ in correspondence with the condition

$$(S\partial S^{-1}/\partial t)_{nn'} = [\Delta, \alpha^{(0)}] \quad (n \neq n'), \qquad (4.4)$$

and set the diagonal elements equal to zero. Then Eq. (4.3) takes the form

$$\partial\xi_{\mathbf{q}}^{(n)}/\partial t + [i\omega_{n\mathbf{q}}(t) + {}^{1}\!/_{2}\gamma_{n\mathbf{q}}(t)]\,\xi_{\mathbf{q}}^{(n)} = y_{\mathbf{q}}^{n}(t), \qquad (4.5)$$

where

$$i\omega_{n\mathbf{q}} - {}^{1}\!/_{2}\gamma_{n\mathbf{q}} = \alpha_{nn}(\mathbf{q}) = \alpha_{nn}^{(0)}(\mathbf{q}) + \delta\alpha_{nn}(\mathbf{q}),$$

$$\delta\alpha_{nn} = -i\delta\omega_{n\mathbf{q}} - {}^{1}\!/_{2}\delta\gamma_{n\mathbf{q}} = -(S\partial S^{-1}/\partial t)_{nn}$$

$$= [S^{-1}\partial S/\partial t)_{nn}. \qquad (4.6)$$

The functions $\omega_{nq}(t)$, $\gamma_{nq}(t)$, and $Y_{n'n}(q,t)$ computed in this fashion, which also depend on time, must be substituted in (2.13).

Temporal nonstationarity in a piezoelectric is generally brought about by a change in the external electric field and is associated with a change in the drift velocity V. For this case it is easy to get, for example, the following expression for the correction to the damping coefficient of the growing oscillations:

$$\delta\gamma_{1\mathbf{q}} = \mathrm{Im}\frac{2\pi\beta^2 q_x^2}{\varepsilon\rho\omega_{\mathbf{q}0}\tau_M}\frac{\partial A_+^2}{\partial t}. \qquad (4.7)$$

As was emphasized above, it never has the form $\delta\gamma = -\omega_q^{-1}\,\partial\omega_q/\partial t$, which is obtained for purely mechanical systems from the theory of adiabatic invariants.

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Translated by R. T. Beyer
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