ON THE ANOMALOUS SINGULARITY AND DETERMINATION OF THE AMPLITUDES OF SOME PROCESSES

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It is shown that in the case of unstable particles the conditions for the anomalous singularity of a triangular diagram have a simple kinematical meaning. An expression for the amplitude of a triangular diagram is obtained, from which it is simple to single out the part containing the singularity. Under certain conditions the singularity occurs in the physical region of the variables. It is pointed out that an experimental study of this singularity in reactions involving the formation of resonances may permit the determination of the amplitudes of such processes as $\pi\pi \rightarrow \pi\pi$, $\pi\Lambda \rightarrow \pi\Lambda$, etc. Moreover, the anomalous singularity may result in the appearance of "humps," which imitate weak resonances, in the effective-mass distributions. (Examples are the hump in the $K_1^0K_1^0$ pair effective-mass distribution near 1 BeV and the so-called ABC-resonance, which indicates the possible existence of an excited nucleus with one nucleon replaced by the isobar $N^* - 1238$ MeV.)

1. INTRODUCTORY REMARKS

WE investigate here the anomalous [1] or, using Landau's terminology [2], the proper singularity of a triangular diagram, in the case when the particle masses may not satisfy the stability conditions. For the amplitudes of the production of several particles, the anomalous singularity was considered relatively recently in several papers [3-5], but attention was paid only to a root-type singularity. The purpose of the present work was to carry out the investigation in somewhat greater detail, to explain the meaning of the anomalous singularity in the case of unstable particles, especially to show that the anomalous singularity can be used to determine the amplitudes of such processes as $\pi\pi \rightarrow \pi\pi$, $\pi\Lambda \rightarrow \pi\Lambda$, etc., and also to interpret the irregularity in the effective-mass distributions observed in some reactions.

As will be shown below, it is essential that one of the particles corresponding to the internal diagram lines be unstable. To describe the unstable particle we propose that in first approximation it is sufficient to assume that the mass m in the corresponding propagation function has a constant negative imaginary increment $-i\Gamma/2$, where Γ — total width. Generally speaking, Γ is a function of q^2 (q —4-momentum of the particle) and of the particle masses. However, if we consider the amplitude near the proper singularity, for which $q^2 = m^2$, then Γ can be assumed constant if we deal with unstable particles for which the width is small compared with the energy released during the decay. This condition will henceforth be assumed satisfied.

We note that the amplitude has no singularities when $\Gamma \neq 0$ and for real values of the external variables. If the vertices are regarded as constant and independent of Γ, then as $\Gamma \rightarrow 0$ the amplitude corresponding to the triangular diagram has, generally speaking, a singularity of the form $\ln \Gamma$. However, the decay vertex is in fact proportional to $\sqrt{\Gamma}$. We can therefore speak of the singularity only in some arbitrary sense, taking it to mean the singularity of the amplitude when the vertices are assumed independent of Γ, and $\Gamma \rightarrow 0$. With such an approach, the role of the width does not differ in fact from the role played by the increment ϵ in the rules for going around the singularity, when m^2 is replaced by $m^2 - i\epsilon$. Therefore, when the stability conditions are satisfied, an amplitude with unstable particles can be regarded as an analytic continuation of the amplitude in the usual sense.

2. MEANING OF ANOMALOUS SINGULARITY IN THE CASE OF UNSTABLE PARTICLES

Let us consider the diagram shown in Fig. 1. For symmetry, the external 4-momenta p_i are assumed directed inward ($p_1 + p_2 + p_3 = 0$), and the 4-momenta of the virtual particles q_i are assumed to be directed clockwise. The vertex parts g_i are

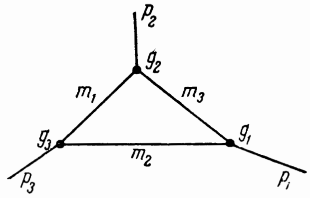

FIG. 1

assumed constant, and all the particles are assumed for simplicity to be scalar with nonzero masses. The necessary conditions for the proper singularity are, as is well known[2,6], of the form

$$\Sigma\alpha_i q_i = 0, \quad q_i^2 = m_i^2, \quad i = 1, 2, 3. \tag{1}$$

It follows from these conditions that $1 + 2y_1y_2y_3 - y_1^2 - y_2^2 - y_3^2 = 0$, i.e.,

$$y_i = y_j y_k \pm [(y_j^2 - 1)(y_k^2 - 1)]^{1/2}, \tag{2}$$

where

$$y_i \equiv y_{jk} = \frac{(q_j q_k)}{m_j m_k} = \frac{m_j^2 + m_k^2 - z_i}{2m_j m_k},$$

$$z_i = p_i^2, \quad p^2 = p_0^2 - \mathbf{p}^2.$$

The indices i, j, and k are cyclic permutations of 1, 2, and 3. We are interested only in the singularities for real values of the variables y. Then, as can be seen from (2), the singularity in question can appear either when all the $y_i^2 < 1$ or when all $y_i^2 > 1$. We recall that the condition $y_i^2 > 1$ signifies that the stability condition is not satisfied in the i-th vertex: when $y_i > 1$ one of the internal particles (j-th or k-th) is unstable, and $y_i < -1$ corresponds to instability of the external line.

In order for the singularity to appear on the "physical" sheet of the amplitude, it is necessary that the α_i obtained from (1) be positive[6]. This leads to the inequalities

$$y_i < y_j y_k \quad \text{for} \quad y^2 < 1,$$
$$y_i > y_j y_k \quad \text{for} \quad y^2 > 1. \tag{3}$$

From these inequalities it follows that it is necessary to take the minus sign in front of the root in (2) in the former case and the plus sign in the latter. In addition, it is easy to show that when $y^2 < 1$ the anomalous singularity can appear either when all the y_i are negative, or when one of the y_i is positive and the two other are negative. For $y^2 > 1$ the singularity can appear only when one of the y_i is positive and the two others are negative[1)].

[1)]When $y^2 < 1$ the conditions for the anomalous singularity can be formulated with the aid of a dual diagram (see[2]). In the the case of $y^2 > 1$ it is impossible to construct a corresponding diagram in Euclidean space, and it is therefore convenient to formulate these conditions algebraically.

Let us consider the case $y^2 > 1$, and let for concreteness $y_2 > 1$ and $y_1, y_3 < -1$. We change over to new variables putting $q_2 = -q_2'$, $q_1 = q_1'$, and $q_3 = q_3'$ (or $q_2 = q_2'$, $q_1 = -q_1'$, and $q_3 = -q_3'$). Then all the $y_i' = (q_j' q_k')/m_j m_k$ will be larger than +1, and $q_i'^2/m^2 = 1$, i.e., we obtain the relations characterizing the four-velocities of the free particles are obtained. Consequently, the anomalous singularity can occur when the particles are real in the intermediate state. We then have two possibilities, which differ only in the signs of q_i' and correspond to mutually reversible reactions.

Let us consider the case when particles 1 and 2 emerge from vertex 3 and, making the diagram more concrete (see Fig. 2), let us assume that one line corresponding to a particle with mass m_1' emerges from vertex 2, and two lines with masses m_2' and m_3' emerge from vertex 1. Particle 1′ should emerge from vertex 2 if $m_1 > m_3$ and go into this vertex if $m_1 < m_3$. Inasmuch as the latter possibility corresponds to a reaction in which three particles are converted into two, we shall assume that $m_1 > m_3$, i.e., we shall consider a reaction in which two particles are converted into three.

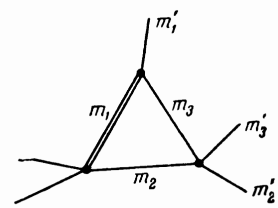

FIG. 2

Let us discuss the meaning of the conditions (2) which connect the relative particle velocities and are therefore purely kinematic. We introduce variables ϑ_{ik}, defined by the formula $\cosh \vartheta_{ik} = y_{ik}'$, and compare relations (2) with the expression for the law of velocity addition in relativistic kinematics (see, for example[7,8]):*

$$\operatorname{ch} \vartheta_{ik} = \operatorname{ch} \vartheta_{ij} \operatorname{ch} \vartheta_{jk} - \operatorname{sh} \vartheta_{ij} \operatorname{sh} \vartheta_{jk} \cos \alpha_{ik},$$

where α_{ik} —angle between velocities of particles i and k in a system where particle j is at rest. Taking into account the choice of the signs preceding the root in (2), we obtain $\cos \alpha_{23} = \cos \alpha_{12} = 1$ and $\cos \alpha_{13} = -1$. Thus, if in the general case the three free particles correspond to a triangle on a sphere with imaginary radius, then the conditions for anomalous singularity stipulate that this triangle degenerate into a "straight line." The equality $\vartheta_{13} = \vartheta_{12} + \vartheta_{23}$ must also be satisfied. The intuitive physical meaning of these conditions is readily understood: particle 3 should be emitted in the

*ch = cosh, sh = sinh.

center-of-mass system of the reaction, in a direction opposite to that of the velocity of particle 1, and its velocity must be sufficiently high to "overtake" particle 2.

Thus, the anomalous singularity arises when the momenta of the intermediate particles are such that the triangle diagram describes a three-step reaction with free particles: first particles 1 and 2 are produced, and then particle 1 breaks up into particles 3 and 1′, where particle 3 moves backward relative to the direction of the flight of 1 (in the c.m.s.), "catches up" with particle 2, and we obtain either scattering or a reaction. Such multistage reactions were already considered before, but the role of the anomalous singularity was not noticed. For example, in connection with an analysis of the possibility of measuring the lifetime of the Σ^0 particle, the author discussed the reaction K^- + nucleus $Z \rightarrow \Sigma^0$ + nucleus $(Z-1)$, when the produced Σ^0 particle breaks up into a Λ particle and a γ quantum, latter producing with the nucleus $(Z-1)$ an electron-positron pair[9]. This process can be schematically described by a triangular diagram, and its probability depends on the velocity of the Σ^0 particle and on its lifetime. The possibility of using three-step reactions for the determination of the lifetime and the scattering cross sections of unstable particles was investigated in the non-relativistic case by Fox[10].

3. EXPRESSION FOR THE AMPLITUDE

We separate now that part of the amplitude which corresponds to the diagram of Fig. 1, which contains an anomalous singularity. The invariant amplitude U is of the form[2)] $U = g_1 g_2 g_3 \pi^2 A$, where

$$A = \int \frac{d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\Sigma\alpha_i)}{a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 - z_1\alpha_2\alpha_3 - z_2\alpha_1\alpha_3 - z_3\alpha_1\alpha_2} = \int \frac{d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\Sigma\alpha_i)}{D}. \quad (4)$$

Here

$$a_i = m_i^2, \quad D = \sum_{i,j} \alpha_i\alpha_j\zeta_{ij},$$

$$\zeta_{ij} \equiv \zeta_k = m_i m_j y_{ij} \quad \text{for } i \neq j, \ \zeta_{ii} = a_i.$$

We can obtain a more convenient integral representation of A if we use the fact that $\Sigma_i \partial A/\partial a_i$ can be expressed in terms of elementary functions ([11], see also [12]):

$$\sum_{i=1}^{3} \frac{\partial A}{\partial a_i} = -\frac{1}{4\Phi} \sum_i^3 \frac{P_i}{\sqrt{R_i}} \ln\left(\frac{\zeta_i + \sqrt{R_i}}{\zeta_i - \sqrt{R_i}}\right);$$

$$\Phi = \det|\zeta_{ij}| = a_1 a_2 a_3 (1 + 2y_1y_2y_3 - y_1^2 - y_2^2 - y_3^2),$$

$$P_i = \partial\Phi / \partial a_i = a_j a_k - \zeta_i^2 + \zeta_i(\zeta_j + \zeta_k) - a_j\zeta_j - a_k\zeta_k,$$

$$R_i = \zeta_i^2 - a_j a_k = a_j a_k (y_i^2 - 1). \quad (5)$$

We shall also make use of the quantity

$$\lambda = \sum_i P_i = -\frac{1}{4}\left(\sum_i z_i^2 - 2\sum_{i<j} z_i z_j\right).$$

These quantities are related by

$$P_i^2 + \lambda R_i + z_i \Phi = 0. \quad (6)$$

The anomalous singularity corresponds to the vanishing of Φ. It is easy to see that when $y_i^2 > 1$ and when conditions (2) and (3) are satisfied, all $P_i < 0$ $(\lambda < 0)$, i.e., $P_i = -\sqrt{-\lambda}\sqrt{R_i}$.

If $\zeta_i > 0$ $(i = 1, 2, 3)$, then A has no singularities. We introduce the function

$$A|_{a\rightarrow a+t} = \int \frac{d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\Sigma\alpha_i)}{D+t}.$$

For $\zeta_i > 0$ and $t \geq 0$, it has likewise no singularities. Since

$$\sum_i \frac{\partial A(a+t)}{\partial a_i} = \frac{dA(a+t)}{dt},$$

we have

$$A = -\int_0^\infty dt \sum_i \frac{\partial A(a+t)}{\partial a_i}.$$

If all $a_j \rightarrow a_j + t$, then $\zeta_i \rightarrow \zeta_i + t$, $\Phi \rightarrow \Phi + \lambda t$, $R_i \rightarrow R_i - z_i t$, and $P_i \rightarrow P_i$. We also take account of the fact that in order to go around the singularities in the analytic continuation in ζ it is necessary to replace a_j by $a_j - i\epsilon$ $(\epsilon > 0)$. As a result we obtain

$$A = \frac{1}{4}\int_{0-i\varepsilon}^{+\infty-i\varepsilon} \frac{dt}{\Phi + \lambda t} \sum_i \frac{P_i}{(R_i - z_i t)^{1/2}} \times \ln\left(\frac{\zeta_i + t + (R_i - z_i t)^{1/2}}{\zeta_i + t - (R_i - z_i t)^{1/2}}\right). \quad (7)$$

The "physical" branches of the multiply-valued functions in the right sides of (5) and (7) are determined by the requirement that when $\zeta_i > 0$ the function A must have no singularities, as can be seen from (4). The functions A and $\Sigma_i \partial A/\partial a_i$ have singularities at identical points. Therefore A has only singularities that correspond to singularities of the integral in (7) at $t = 0$. [No singularities arise in connection with the pinching of the integration contour in the representation (7)]. It can be shown (see [11]) that

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[1)]The normalization is such that the matrix element

$$T_{ba} = \Pi(2E_j)^{-1/2} U_{ba} \delta^{(4)}(p_b - p_a),$$

where $iT = S - 1$, S – scattering matrix, E_j – energies of initial and final particles, and the constants g_i – invariant amplitudes located at the vertices of the diagrams.

$$\sum_i \frac{P_i}{\sqrt{R_i}} \ln \left(\frac{\zeta_i + \sqrt{R_i}}{\zeta_i - \sqrt{R_i}} \right) \Bigg|_{\Phi=0} \sim \ln 1.$$

Therefore for $\zeta_i > 0$ it is necessary to take the branch of the logarithm for which $\ln(1 + i\epsilon) = 0$ as $\epsilon \to 0$. In the analytic continuation in ζ_i, from $\zeta_i > m_j m_k$ to $\zeta_i < -m_j m_k$, the argument of the logarithm in (7) receives an increment $2\pi i$ when $0 < t < R_i/z_i$. Therefore the "normal" imaginary part of A for $y_i < -1$ and $y_j, y_k > 0$ is equal to

$$\operatorname{Im} A\,|_{y_i<-1} = \frac{1}{4} \int\limits_0^{R_i/z_i} \frac{dt}{\Phi + \lambda t} \frac{2\pi P_i}{(R_i - z_i t)^{1/2}}.$$

Assume now that y_1 and $y_3 < -1$, while $y_2 > 1$. Then

$$A = \frac{\pi i P_1}{2} \int\limits_0^{R_1/z_1} \frac{dt}{(\Phi + \lambda t)(R_1 - z_1 t)^{1/2}} + \frac{\pi i P_3}{2} \int\limits_0^{R_3/z_3} \frac{dt}{(\Phi + \lambda t)(R_3 - z_3 t)^{1/2}} + A_0,$$

$$A_0 = \frac{1}{4} \int\limits_0^{\infty} \frac{dt}{\Phi + \lambda t} \sum_i \frac{P_i}{(R_i - z_i t)^{1/2}} \times \ln \left(\frac{\zeta_i + t + (R_i - z_i t)^{1/2}}{\zeta_i + t - (R_i - z_i t)^{1/2}} \right)_{(0)}. \quad (8)$$

The subscript (0) denotes that it is necessary to take the branch for which $\ln(1 + i\epsilon) = 0$. The term A_0 is real and has no singularity at $\Phi = 0$ and if conditions (3) are satisfied.

We assume that particle 1 is unstable, and take into account the width Γ_1 only in the terms with the singularity, replacing a_1 by $a_1 - i\gamma_1$ ($\gamma_1 = m_1\Gamma_1$, $\Phi \to \Phi - i\gamma_1 P_1$). Then

$$A = -\frac{\pi i}{\sqrt{-\lambda}} (\ln M + i\varphi) + A_0,$$

$$M = \left[\frac{(\Phi^2 + \gamma_1^2 P_1^2) z_1 z_3}{(-P_1 + \sqrt{-\lambda}\sqrt{R_1})^2 (-P_3 + \sqrt{-\lambda}\sqrt{R_3})^2} \right]^{1/2}, \quad (9)$$

$\varphi = \arg(-\Phi + i\gamma_1 P_1)$, $-\pi < \varphi < 0$, since $P_1 < 0$ in the region of interest to us. We see that the singularity at $\Phi \to 0$, $\Gamma_1 \to 0$ is logarithmic and is the more sharply pronounced the smaller Γ_1. However, the amplitude U itself is proportional to $\sqrt{\Gamma_1}$.

For the reaction with three particles in the final state (Fig. 2) it is convenient to fix the energy of the incident particle (i.e., y_3) and to consider the distribution with respect to $y_1(z_1)$. The value of y_2 for this diagram is fixed in the case of two-particle decay of particle 1. In this case the anomalous singularity should be observed for

$$y_1^{(+)} = y_2 y_3 + [(y_2^2 - 1)(y_3^2 - 1)]^{1/2}.$$

Using this formula, we can easily find the positions of the singularities for concrete reactions (see the examples below). If $y_1^{(+)} = -1$ (the anomalous and threshold singularities coincide), then the singularity remains logarithmic. When $y_2 \neq 1$ coincidence is possible only with one of the threshold singularities. On the other hand, if $y_2 = 1$, then the singularity for $y_1 = -1$ and $y_3 = -1$ is of the form $1/\sqrt{-\lambda}$, $\lambda \to 0$. This interesting case was already considered in several papers [3-5]. To observe a root singularity it is necessary, generally speaking, to investigate reactions with four particles in the final state, since it is necessary to satisfy the requirements $y_1 = -1$, $y_2 = +1$, and $y_3 = -1$ simultaneously, which is difficult for reactions with three particles in the final state, for usually y_2 differs from $+1$. An exception is the φ meson ($y_2 = +1.03$).

4. DETERMINATION OF THE AMPLITUDES FROM THE ANOMALOUS SINGULARITY

The amplitudes g_i depend in the general case on q_i. However, if they have no singular points near values of q_i satisfying the conditions (1) with $\alpha_i > 0$, then they can be regarded as constant near these values, and the contribution of the singularity will be given as before by formula (9). Inasmuch as the virtual particles become real near the singularities, the expression for the singular part of the amplitude will contain g_i for physical values of the variables. It is thus necessary to replace g_1 by the corresponding scattering amplitude. Separation of the contribution of the anomalous singularity actually corresponds to the realization, for example, of such a process as scattering of a free pion by a free pion. We note that nothing is changed in the preceding arguments if the g_i are regarded as functions of the form $g_i(z_i)$. Then formula (9) is valid also away from the singularity, and we can use for the amplitude g_1 the formula from the theory with effective radius.

In order to determine g_1 from experiment, it is necessary to have definite information concerning g_2 and g_3. Let us consider the simplest case, when the g_i can be regarded as constants and the reaction is described only by the diagrams of Figs. 2 and 3. This assumption may not be very far from reality, for there are cases when the reaction is well described by the diagram of Fig. 3 (see [13,14]). If the mass of the resonance is set equal to $m_1 - i\Gamma_1/2$, then this diagram gives the Breit-Wigner formula, with which the effective-mass distributions of particles 1′ and 3 were indeed compared. The diagram of Fig. 2 then plays the role of a cor-

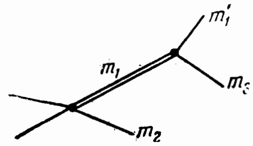

FIG. 3

rection that takes into account the interaction of particles 2 and 3 and can be quite appreciable near the threshold of production of particles 1 and 2. Let us assume that the vertex 1 corresponds to elastic scattering. Then, taking into account diagrams of Figs. 2 and 3, the differential cross section for the formation of particles 1′, 2, and 3 can be represented in the form

$$d\sigma \sim |\mathfrak{M}|^2 \delta\left(\sum s_i - s_0 - \sum b_i\right) \frac{ds_1\, ds_2\, ds_3}{s_0}, \qquad (10)$$

where

$$s_1 = (k_2 + k_3)^2, \quad s_2 = (k_1 + k_3)^2, \quad s_3 = (k_1 + k_2)^2,$$

k_i —4-momenta of the particles in the final state, $b_i = k_i^2$ —squares of the masses of these particles, and s_0 —square of total energy in the c.m.s.,

$$\mathfrak{M} = g_2 g_3 \left[-\frac{1}{s_2 - a_1 + i\gamma_1} + \frac{g_1}{16\pi^2}\left(A_0 - \frac{\pi i \ln M}{\sqrt{-\lambda}} + \frac{\pi\varphi}{\sqrt{-\lambda}} \right) \right]. \qquad (11)$$

The values of M, φ, λ, and A_0 for a given total energy depend only on the variable s_1. The variables s_i vary inside the Dalitz-diagram region bounded by the curve

$$s_1 s_2 s_3 - \sum s_i (b_i s_0 + b_j b_k) + 2 s_0 (b_1 b_2 + b_2 b_3 + b_1 b_3) + 2 b_1 b_2 b_3 = 0,$$
$$s_1 + s_2 + s_3 = s_0 + \sum b_i. \qquad (12)$$

In this case it is convenient to consider a Dalitz diagram in the variables s_1 and s_2 (Fig. 4). An anomalous singularity appears on the line $s_1 = s_{10}$, and passes through the left point of intersection of the line $s_2 = a_1$ with the boundary (12). To demonstrate this, it is convenient to write the equation of the boundary in terms of the variables

$$\eta_1 = \frac{b_2 + b_3 - s_1}{2\sqrt{b_2 b_3}}, \quad \eta_2 = \frac{s_2 + b_3 - b_1}{2\sqrt{s_2 b_3}}, \quad \eta_3 = \frac{s_2 + b_2 - s_0}{2\sqrt{s_2 b_2}}.$$

Then Eq. (12) assumes the form

$$1 + 2\eta_1\eta_2\eta_3 - \eta_1^2 - \eta_2^2 - \eta_3^2 = 0,$$

i.e., it coincides with (2) when $b_2 = a_2$, $b_3 = a_3$, $b_1 = z_2$, $s_2 = a_1$, and $s_0 = z_3$.

A general expression for $|\mathfrak{M}|^2$ can be obtained from (11). A readily-analyzed formula is obtained if the amplitude g_1 is small and real. Then, neglecting the terms with g_1^2, we have

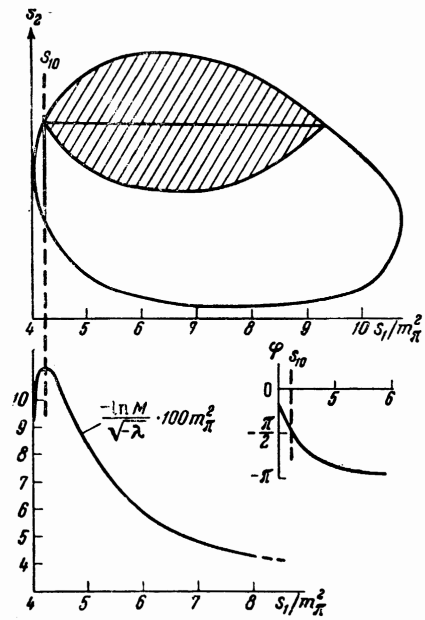

FIG. 4. Dalitz diagram and position of singularities for the reaction $K^- p \to \Lambda\pi^+\pi^-$ at $p_{K.lab} = 520$ MeV/c (see the text).

$$|\mathfrak{M}|^2 = \frac{|g_2 g_3|^2}{(s_2 - a_1)^2 + \gamma_1^2} \left\{ 1 - \frac{g_1 \gamma_1 \ln M}{8\pi\sqrt{-\lambda}} + \frac{g_1}{8\pi^2}(a_1 - s_2)\left(A_0 + \frac{\pi\varphi}{\sqrt{-\lambda}} \right) \right\}. \qquad (13)$$

If we select only the events that fall in a region that is symmetrical about the line $s_2 = a_1$ (shaded in Fig. 4), then the last term in (13) makes no contribution to the distribution of the number of events as a function of s_1, while the second term contains only one unknown quantity g_1. Thus, if the reaction can be described by the diagrams of Figs. 2 and 3, then the amplitude g_1 can in principle be readily determined. We note that in this case the sign of the amplitude is also determined. Thus, for $g_1 > 0$ (attraction forces) the contribution of the second term in (13) is positive ($\ln M < 0$ near the singularity if the width Γ_1 is not very large.)

The Dalitz diagram on Fig. 4 is drawn for the reaction $K^- p \to \Lambda\pi^+\pi^-$ for a total energy 570 MeV ($p_{K.lab} = 520$ MeV/c). It is assumed that this reaction proceeds principally via formation of the resonance $Y_1^* - 1385$ MeV. (For simplicity the diagram of Fig. 4 shows only the contribution of resonance with a charge of one polarity.) The diagram of Fig. 2, corresponding to $\pi\pi$ scattering, has in this case a singularity at $s_1 = 4.14\, m_\pi^2$.

If the $\pi\pi$ scattering length is $1/m_\pi$, then

$$\left(-\frac{g_1 \gamma_1 \ln M}{8\pi\sqrt{-\lambda}} \right)_{max} = \frac{1}{2} \quad \text{for } \Gamma_1 = 35 \text{ MeV},$$

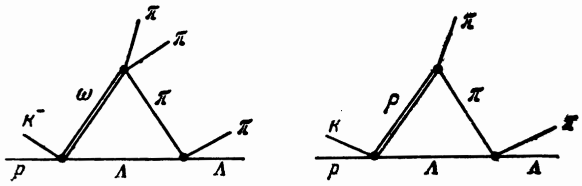

FIG. 5

i.e., the contribution of the triangular diagram is appreciable ($g_1 = 8(m_2 + m_3)c$, c —scattering length). The dependence of the terms $-(-\lambda)^{-1/2} \ln M$ on φ and s_1 is shown in Fig. 4.

Examples of processes from which information on $\pi\Lambda$ scattering can be obtained are given in Fig. 5.

5. "FALSE" RESONANCES

As can be seen from Fig. 4, an anomalous singularity can lead to the appearance of a "hump" in the distribution relative to the effective-mass of the two particles. The position and the very existence of this hump depend on the total energy (or y_3). Inasmuch as $y_2 \lesssim 2$ for most resonances, the singularity appears as a rule, by virtue of inequalities (3), near $y_1 = -1$, i.e., at an effective mass $\sim (m_2 + m_3)$. If the information on the reaction is insufficient, such singularities can be interpreted as weak resonances. We present examples of reactions in which the role of the anomalous singularity is appreciable.

1. In the reaction $\pi^- p \to K\bar{K}N$ at $p_{\pi.\text{lab}} \sim 1.95$ BeV/c, a relatively large number was observed of cases of $K_1^0 K_1^0$ pairs with effective mass near 1 BeV[15]. Inasmuch as a noticeable role in the production of $K\bar{K}$ pairs at this energy is played by the resonance Y — 1520 MeV, the diagram of Fig. 6a, which has a singularity at an $K\bar{K}$ effective mass equal to 1 BeV (for $p_{\pi\,\text{lab}} = 1.95$ BeV/c), should be significant. The maximum value of $-2(\gamma_1 m_K c/\sqrt{-\lambda}) \ln M$ at $\Gamma_1 = 16$ MeV is in this case $cm_K/4$, i.e., when the scattering length is $c \sim 1/m_\pi$ the contribution of the triangular diagram is comparable with the contribution of the pole diagram of Fig. 3 (m_K —mass of K meson). At higher energy, the contribution of the diagram with emission of one or several pions may be appreciable (Fig. 6b).

The diagram with φ-meson production (Fig. 6c) is interesting because a root singularity of the form $1/\sqrt{-\lambda}$ is important in it. The contribution of this diagram is maximal when K, $\bar{K}$, and n have a low c.m.s. kinetic energy, i.e., the effective mass of the K mesons is close to the mass of the φ meson. However, unlike the φ meson, the appearance of $K_1^0 K_1^0$ is possible here as a result of the $\bar{K}n$ scattering.

Thus, the experimentally observed increase in the number of K_1^0-meson pairs with effective mass near 1 BeV can be partially due to the diagrams considered here.

2. In the reaction $p + d \to He^3 + 2\pi$ a hump is observed in the He^3 momentum distribution corresponding to an effective pion mass ~ 300—310 MeV[16]. If we assume that the production of nuclei of the type NNN* (N —nucleon, N* —1238-MeV isobar), which decay into $He^3(H^3) + \pi$ (Fig. 7a), is important in this reaction, then the experimentally observed anomaly (the so-called ABC resonance) can be due to the singularity of the triangular diagram which takes into account the $\pi\pi$ scattering (Fig. 7b). At an NNN* nucleus mass of $m_{He^3} + 300$ MeV and at an initial proton kinetic energy of 743 MeV, the singularity appears at $s_1 = 4.8\,m^2$, i.e., at $m_{\pi\pi} \sim 310$ MeV. At lower proton energies the singularity already corresponds to complex values of s_1, amounting physically to a "smearing" of the hump. This agrees with the data of Abashian et al.[16] A similar interpretation of the ABC anomaly was proposed recently by Anisovich and Dakhno[17], and reactions with nuclei containing resonances, which are of independent interest, were considered by Grishin and Podgoretskiĭ[18].

Figure 8 shows the results of calculations by formula (13) without account of the term A_0, which contains no singularities, for a proton kinetic energy 743 MeV. The width Γ_1 is set equal to 100 MeV, and the $\pi\pi$-scattering length is $1/m_\pi$. The same figure shows the contribution of the terms quadratic in g_1 with a singularity. The terms with A_0 plus possible constants, which are not taken into

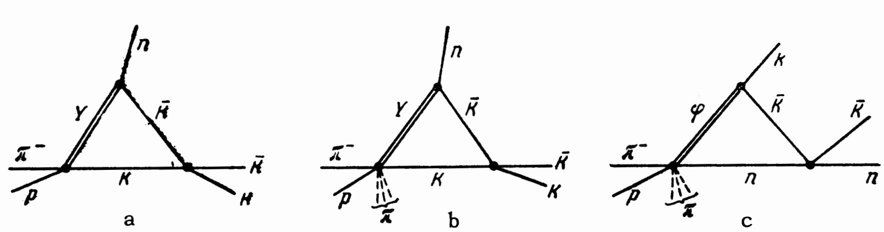

FIG. 6

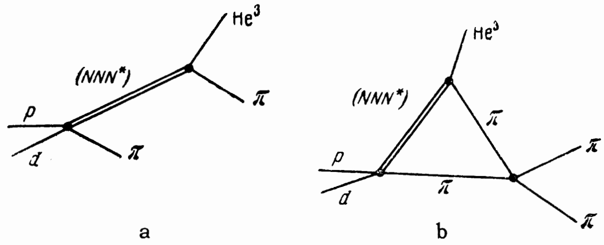

FIG. 7

account by the diagrams of Figs. 7a and b, add to the distribution over s_1 a contribution that behaves in analogy with the phase-volume curve.

The foregoing examples show how important it is to take into account triangular diagrams in reactions with resonance production. It is of interest to consider also reactions in which two resonances can be produced. In this case the square diagram describing the scattering of the decay products has a singularity of the form $1/\sqrt{s - s_0}$.

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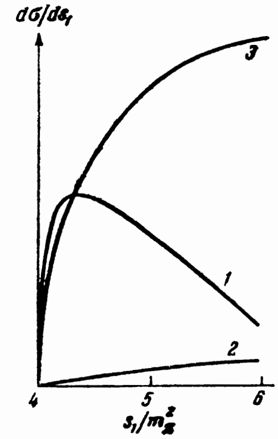

FIG. 8. Dependence of $d\sigma/ds_1$ on s_1/m_π^2 for the reaction $p + d \to He^3 + 2\pi$ in accordance with formula (13) with $A_0 = 0$ (curve 1), contribution of the terms quadratic in g_1 with singularity (curve 2), and dependence of $d\sigma/ds_1$ for $|\mathfrak{M}|^2 = \text{const}$ (curve 3).

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