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ON THE FORMULATION OF A THEORY OF LOW ENERGY SCATTERING OF ELEMENTARY PARTICLES BASED ON THE SPECTRAL AND UNITARITY CONDITIONS

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Equations are proposed for the partial waves in the low energy region, which take into account exactly the requirements of analyticity of the scattering amplitude with respect to two variables, and the unitarity conditions in all three channels below the thresholds for inelastic processes.

1. INTRODUCTION

As is well known, it is possible to obtain an approximate system of equations for the scattering amplitude starting from the Mandelstam representation [1] and the unitarity condition in all three channels. Such a system of equations was formulated by Ter-Martirosyan.[2] Although recently doubts have been expressed as to the validity of the Mandelstam representation with a finite number of subtractions [3,4], equations of this type continue to be of interest in the analysis of low energy scattering. The Mandelstam representation is apparently violated only because of the increase without bound of the number of subtractions in the dispersion relations in momentum transfer as the energy is increased. The analytic properties of the amplitude for finite values of these variables are, probably, in agreement with the Mandelstam representation. As will be shown below it is possible in that case, confining oneself to the low energy region, to construct a system of equations for the amplitude similar in character to the equations of Ter-Martirosyan.

Independently of the complications connected with the infinite number of subtractions a direct solution of the Ter-Martirosyan equations presents great computational difficulties. Consequently a transition to equations for partial amplitudes is of great value. In such a transition one unavoidably loses a certain amount of information. In particular, in the methods of Chew and Mandelstam [5] and Shirkov et al.[6] the inelastic jumps[1)] in the amplitude are completely ignored. For this reason the analyticity properties of the amplitude originally postulated fail to be reflected in the equations for the partial waves.

In the present work we describe a method for obtaining equations for the partial waves in the low energy region, in which the analyticity properties of the amplitude are more fully taken into account. In the derivation of the equations the spectral functions of the Mandelstam representation are broken up into parts that are close to and far from the physical regions. With the help of the proposed method the contribution to the amplitude from the near part of the spectral function is taken into account fully. At the same time we are successful in avoiding the explicit inclusion of the spectral functions in the equations. This facilitates the writing of the equations in terms of partial waves. The most significant step in the derivation of the equations is the decomposition of the amplitude into separate terms for which dispersion relations can be written in a convenient form. The resultant equations take into account exactly the unitarity conditions in all three channels at energies not exceeding the thresholds for inelastic processes, and also the requirements of crossing symmetry.

The contribution of the distant part of the spectral function to the amplitude cannot be found within the framework of the present method. It is supposed that in the low-energy region this contribution may be approximated by a function of some small number of arbitrary constants which constitute the parameters of the theory. In addition, in the solutions of the equations there appear further arbitrary constants in agreement with the results of Castillejo, Dalitz, and Dyson.[7]

[1)]By inelastic we mean the jumps that are due only to intermediate states containing more than two particles.

2. THE SYSTEM OF EQUATIONS FOR THE AMPLITUDE

Let us consider the scattering of identical scalar particles of mass m. The generalization to more complex situations presents no difficulty in principle. Let us assume that transitions of an even number of particles into an odd number are forbidden (as in the case of $\pi\pi$ scattering). Let us introduce the Mandelstam variables s, u, t, equal to the energy squared in the first, second, and third channels respectively. Let z_1, z_2, z_3 be the cosines of the scattering angles in these channels so that

$$z_1 = 1 + 2t/(s - 4m^2), \qquad (1)$$

and two other equations obtained from the above by cyclic permutation of variables.

For simplicity let us suppose to begin with that the amplitude A(s, u, t) satisfies a double dispersion relation with no subtractions. We decompose the spectral functions of the Mandelstam representation $A_{ik}(x, y)$ into parts depending on their location with respect to the physical regions. Taking into account the identity of the particles we write these functions in the form

$$A_{ik}(x, y) = \rho(x, y) + \rho(y, x) + \sigma(x, y), \qquad (2)$$

where $\rho(x, y)$ is different from zero only for $4m^2 < x \le 16m^2$, $y > 16m^2$, and $\sigma_{ik}(x, y)$ only for $x \ge 16m^2$, $y \ge 16m^2$. By x and y we mean any two of the variables s, u, t, and the pair of indices i, k takes on the values 1, 2; 2, 3 and 3, 1. The function $\sigma(x, y)$ satisfies the condition

$$\sigma(x, y) = \sigma(y, x). \qquad (3)$$

The regions in which the functions $\rho(x, y)$ are nonvanishing are dashed in in Fig. 1.

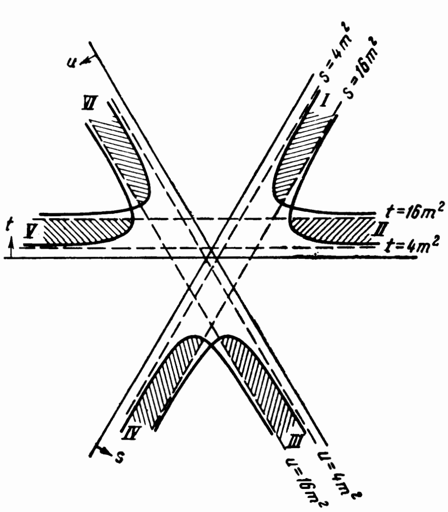

FIG. 1

We represent the scattering amplitude A(s, u, t) in the form

$$A(s,u,t) = \beta(s,u) + \beta(s,t) + \beta(u,t) + \beta(u,s) + \beta(t,s) + \beta(t,u) + f(s,u,t); \qquad (4)$$

$$\beta(x, y) = \frac{1}{\pi^2}\int_{4m^2}^{16m^2} dx' \int_{\gamma(x')}^{\infty} dy' \frac{\rho(x', y')}{(x'-x)(y'-y)}, \qquad (5)$$

$$f(s, u, t) = \frac{1}{\pi^2}\int_{16m^2}^{\infty} dx' \int_{16m^2}^{\infty} dy' \sigma(x', y') \times \left\{\frac{1}{(x'-s)(y'-u)} + \frac{1}{(x'-u)(y'-t)} + \frac{1}{(x'-t)(y'-s)}\right\}. \qquad (6)$$

Here $\gamma(x') > 16m^2$. We want the amplitude A(s, u, t) in all three physical regions for $s \le 16m^2$, $u \le 16m^2$, $t \le 16m^2$ (see Fig. 1). Our aim then is to construct a system of equations which takes into account the entire contribution to the amplitude from the near parts of the spectral functions $\rho(x, y)$ and is at the same time in a convenient form for passage to partial waves.

We write the quantity β in the form

$$\beta(x, y) = \frac{1}{2\pi}\int_{4m^2}^{16m^2} \frac{\alpha(x', y)}{x'-y} dx'; \qquad (7)$$

$$\alpha(x, y) = \frac{2}{\pi}\int_{\gamma(x)}^{\infty} \frac{\rho(x, y')}{y'-y} dy'. \qquad (8)$$

On substitution of Eq. (7) into Eq. (4) we obtain the following representation for the amplitude:

$$A(s, u, t) = \frac{1}{2\pi}\int_{4m^2}^{16m^2} \frac{ds'}{s'-s}[\alpha(s', u) + \alpha(s', t)] + \frac{1}{2\pi}\int_{4m^2}^{16m^2} \frac{du'}{u'-u}[\alpha(u', t) + \alpha(u', s)] + \frac{1}{2\pi}\int_{4m^2}^{16m^2} \frac{dt'}{t'-t}[\alpha(t', s) + \alpha(t', u)] + f(s, u, t). \qquad (9)$$

The functions α that appear in this formula are simply related to the elastic jumps of the amplitude. In particular the elastic jump in the amplitude in the s channel, A_{1el}, is equal to

$$A_{1el}(s, z_1) = \tfrac{1}{2}(\alpha(s, t(s, z_1)) + \alpha(s, u(s, z_1))). \qquad (10)$$

It is not hard to see that for $4m^2 \le s \le 16m^2$ the quantity $\alpha(s, t(s, z_1))$ has a Legendre polynomial expansion in the same region as the function $A_{1el}(s, z_1)$. At the same time Eq. (9) makes it possible to express in terms of α the contribution to the amplitude of the spectral functions $\rho(x, y)$. It is therefore convenient to choose the quantities

α as the unknown functions when constructing the equations. In order to obtain a closed system of equations it is necessary to write for α a relation analogous to the conventional unitarity condition. To this end we decompose the amplitude $A(s, u, t)$ into two parts:

$$A(s, u, t) = {}^1/_2[A^{II}(s, u, t) + A^{III}(s, u, t)]; \qquad (11)$$

$$A^{II}(s, u, t) = \frac{2}{\pi}\int_{4m^2}^{\infty}\frac{A_2(u', z_2(u', s))}{u' - u}du', \qquad (12a)$$

$$A^{III}(s, u, t) = \frac{2}{\pi}\int_{4m^2}^{\infty}\frac{A_3(t', z_3(t', s))}{t' - t}dt'. \qquad (12b)$$

Here $A_2(u, z_2)$ and $A_3(t, z_3)$ are the full jumps in the amplitude in the u and t channels respectively. It is obvious that for $s = \text{const} \geq 4m^2$ the function $A^{II}(s, u, t)$ has a cut only for $u \geq 4m^2$, and the function $A^{III}(s, u, t)$ only for $t \geq 4m^2$. In our case $A_2(x, z) = A_3(x, z)$ and therefore

$$A^{II}(s, t, u) = A^{III}(s, u, t). \qquad (13)$$

The singularities of the function $A^{III}(s, u, t)$ are shown in Fig. 2.

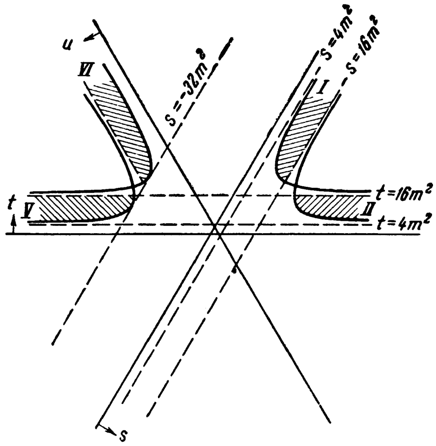

FIG. 2

Let us show now that the quantity $\alpha(s, t)$ is related to the function $A^{III}(s, u, t)$ by a relation analogous to the conventional unitarity condition:

$$\alpha(s, t(s, z_1)) = \frac{1}{4\pi}\left[\frac{s - 4m^2}{s}\right]^{1/2} \times \int d\Omega' A^{III}(s, u', t')\, A^{III\,*}(s, u'', t''), \qquad (14)$$

where $d\Omega'$ is a surface element of the unit sphere in the space of the momenta of the intermediate particles in the c.m.s. The validity of Eq. (14) is simplest to establish by making use of the concept of complex angular momentum.[8] In order to make use of this method we first expand the functions α and A^{III} in Legendre polynomials in the s channel by making use of the formulas

$$\alpha(s, t(s, z_1)) = \sum_{l=0}^{\infty}(2l + 1)\,\alpha^{(l)}(s)\,P_l(z_1), \qquad (15)$$

$$A^{III}(s, u(s, z_1), t(s, z_1)) = \sum_{l=0}^{\infty}(2l + 1)\,A^{III\,(l)}(s)\,P_l(z_1), \qquad (16)$$

where

$$\alpha^{(l)}(s) = \frac{1}{2}\int_{-1}^{1}\alpha(s, t(s, z_1))\,P_l(z_1)\,dz_1, \qquad (17)$$

$$A^{III\,(l)}(s) = \frac{1}{2}\int_{-1}^{1}A^{III}(s, u(s, z_1), t(s, z_1))\,P_l(z_1)\,dz_1. \qquad (18)$$

By making use of Eqs. (8) and (12b) the last two expressions may be written as

$$\alpha^{(l)}(s) = \frac{2}{\pi}\int_{z_0(s)}^{\infty}Q_l(z_1)\,\rho(s, t(s, z_1))\,dz_1, \qquad (19)$$

$$A^{III(l)}(s) = \frac{2}{\pi}\int_{z_0'(s)}^{\infty}Q_l(z_1)\,A_3(t(s, z_1), z_2(s, z_1))\,dz_1, \qquad (20)$$

where $Q_l(z_1)$ are the Legendre functions of the second kind, $z_0(s) > 1$, $z_0'(s) > 1$.

Equations (19) and (20) make it possible to continue the functions $\alpha^{(l)}(s)$ and $A^{III(l)}(s)$ to complex l values. Under the assumptions that were made about the properties of the amplitude these functions are analytic in the l plane to the right of the line $\text{Re}\, l = 0$. As is well known, the quantities $A^{III(l)}(s)$ satisfy the unitarity condition which, in view of the identity of all particles, may be written for arbitrary real positive l in the form

$$\text{Im}\, A^{III(l)}(s) = \omega(s)\,|A^{III(l)}(s)|^2, \qquad (21)$$

$$\omega(s) = [(s - 4m^2)/s]^{1/2}. \qquad (22)$$

For $4m^2 \leq s \leq 16m^2$

$$\text{Im}\, A_3(t, z_2(t, s)) = \rho(s, t) \qquad (23)$$

and in accordance with (8), (12b), (17), and (18)

$$\text{Im}\, A^{III(l)}(s) = \alpha^{(l)}(s). \qquad (24)$$

Therefore the unitarity condition (21) takes on for $4m^2 \leq s \leq 16m^2$ the form

$$\alpha^{(l)}(s) = \omega(s)\,|A^{III(l)}(s)|^2. \qquad (25)$$

This relation is valid, in particular, for all integer (even and odd) values of l. Formula (14), therefore, follows from it as in the case of the conventional unitarity condition.

We remark that as a consequence of the relations (11) and (13) the functions $A^{III(l)}(s)$ coincide for integer even l with the partial amplitudes $A^{(l)}(s)$ obtained by expanding the full amplitude $A(s, u(s, z_1), t(s, z_1))$ in Legendre polynomials. Correspondingly the quantities $\alpha^{(l)}(s)$ are equal for even l to the elastic jumps $A^{(l)}{}_{1el}(s)$ of these partial amplitudes. For integer odd l the functions $\alpha^{(l)}(s)$ have no direct physical meaning, they are needed however to reconstitute the quantity $\alpha(s, t)$ by means of Eq. (15). We have made use of the complex angular momentum technique only for the purpose of extending the unitarity condition (25) to odd values of l.

To obtain the desired system of equations it is still necessary to express the function A^{III} by means of dispersion relations in terms of the quantity α, i.e. to write for the function A^{III} a formula analogous to the relation (9). It is easy to show that this formula is of the form

$$A^{III}(s, u, t) = \frac{1}{\pi}\int_{4m^2}^{16m^2} \frac{\alpha(s', t)}{s' - s} ds'$$

$$+ \frac{1}{\pi}\int_{4m^2}^{16m^2} \frac{dt'}{t' - t}[\alpha(t', 4m^2 - s - t') + \alpha(t', s)]$$

$$+ \frac{1}{\pi}\int_{4m^2}^{16m^2} \frac{du'}{u' - u}[\alpha(u', t) - \alpha(u', 4m^2 - s - u')]$$

$$+ f^{III}(s, u, t); \tag{26}$$

$$f^{III}(s, u, t) = \frac{2}{\pi^2}\int_{16m^2}^{\infty} \frac{dt'}{t' - t}\int_{16m^2}^{\infty} \frac{ds'}{s' - s}\sigma(s', t')$$

$$+ \frac{2}{\pi^2}\int_{16m^2}^{\infty} \frac{dt'}{t' - t}\int_{16m^2}^{\infty} \frac{du'}{u' - 4m^2 + t' + s}\sigma(u', t'). \tag{27}$$

Indeed, upon substitution into Eq. (26) of the values of the function α obtained from relation (8), and making use of Eq. (2), we arrive at expression (12b) for the quantity A^{III}, in which

$$A_3(t, z_3) = \frac{1}{\pi}\int \frac{ds'}{s' - s(t, z_3)} A_{13}(s', t)$$

$$+ \frac{1}{\pi}\int \frac{du'}{u' - u(t, z_3)} A_{23}(u', t). \tag{28}$$

The integration here is over regions in which the corresponding spectral functions are nonvanishing.

Under the assumption that the function $f^{III}(s, u, t)$ is known in all three physical regions for $s \le 16m^2$, $u \le 16m^2$, $t \le 16m^2$, the relations (14) and (26) constitute an exact system of equations which make possible the determination of the amplitude $A(s, u, t)$ for these values of the variables s, u, t. The function $f^{III}(s, u, t)$ can not be obtained within the framework of the present method. It is easy to see that it has no singularities inside of the triangle defined by the lines $s = 16m^2$, $u = 16m^2$, $t = 16m^2$ (see Fig. 1). We suppose that the function $f^{III}(s, u, t)$ can be approximated with sufficient accuracy inside the triangle by a polynomial of low degree in the variables s, u, t. The coefficients in this polynomial serve as the parameters of the theory. For simplicity we use this approximation in the present work all the way up to the boundaries of the triangle $s \le 16m^2$, $u \le 16m^2$, $t \le 16m^2$, although it is justified only inside the triangle at a certain distance from its boundaries.

The solutions of the system of equations (14) and (26), obtained under these assumptions, can make physical sense only for values of the variables s, u, t somewhat smaller than $16m^2$. Indeed, under the stated conditions the quantity $\alpha(x, y)$ should for $x = 16m^2$ vanish identically in y. Otherwise the function $A^{III}(s, u, t)$ would, according to Eq. (26), have a logarithmic singularity at $s = 16m^2$ and $t = 16m^2$ in the physical regions. It is obvious that in this case the elastic jump in the amplitude $A_{1el}(s, z_1)$, also must vanish for $s = 16m^2$ identically in z_1. Had we started from the exact values of the spectral functions of the Mandelstam representation and determined the quantities $\alpha(x, y)$ and $f^{III}(s, u, t)$ by Eqs. (8) and (27), we would find that the function $f^{III}(s, u, t)$ as well as the remaining three terms on the right side of (26) have logarithmic singularities for $s = 16m^2$ and $t = 16m^2$. These singularities compensate each other and the quantity $A^{III}(s, u, t)$ would have no singularities. The appearance of these singularities is due to the decomposition of the spectral functions into parts [Eq. (2)]. It should be noted that within the framework of this method the function $f^{III}(s, u, t)$ may be approximated by an expression which has logarithmic singularities at $s = 16m^2$ and $t = 16m^2$. This would require a different choice for the arbitrary parameters that enter the theory. In this way one could obtain a solution of (14) and (26) which would be physically meaningful up to the lines $s = 16m^2$, $u = 16m^2$, $t = 16m^2$.

3. SYSTEM OF EQUATIONS FOR THE PARTIAL AMPLITUDES

In order to derive a system of equations for the partial amplitudes one must substitute the expansions (15) and (16) into the relation (26) and make use of the unitarity condition in the form (25). It is first necessary to verify that the sought for functions will enter the equations only for those values of the independent variables for which the corresponding Legendre series converges. It follows from (8) that the region of convergence of the series (15) for the function $\alpha(s, t)$ for $4m^2 < s \le 16m^2$ lies between the boundaries of the spectral functions $\rho(s, t)$ and $\rho(s, u)$ (lines I and IV in Fig. 1). If the function $\alpha(s, t)$ is known in that region then the quantity $\beta(s, t)$ can be found from Eq. (7) in the entire triangle $s \leq 16m^2$, $u \leq 16m^2$, $t \leq 16m^2$. This is true of all six functions β that appear in Eq. (4) and, consequently, of the complete amplitude $A(s, u, t)$. In the same way one easily verifies that if the function α is known within the region of convergence of the series (15), then the function $A^{III}(s, u, t)$ can be obtained inside the triangle $s \leq 16m^2$, $u \leq 16m^2$, $t \leq 16m^2$ from Eq. (26). On the other hand the function $\alpha(s, t)$ is determined for $4m^2 \leq s \leq 16m^2$ with the help of the unitarity condition (25) in the entire region of convergence of its Legendre expansion, if the quantity $A^{III}(s, u, t)$ is known inside the triangle $s \leq 16m^2$, $u \leq 16m^2$, $t \leq 16m^2$. This justifies the transition to partial amplitudes in the Eqs. (14) and (26).

The system of equations for the partial amplitudes assumes a convenient form if the following abbreviations are introduced:

$$\tilde{\alpha}^{(l)}(s) = (s - 4m^2)^{-l}\alpha^{(l)}(s), \tag{29}$$

$$\tilde{A}^{III(l)}(s) = (s - 4m^2)^{-l}A^{III\,(l)}(s), \tag{30}$$

$$\tilde{\omega}^{(l)}(s) = s^{-1/2}(s - 4m^2)^{l+1/2}. \tag{31}$$

Upon substitution of Eq. (15) into Eq. (26), followed by expansion of the resultant value for the function $A^{III}(s, u, t)$ in Legendre polynomials, we arrive at the following relation, valid for $4m^2 \leq s \leq 16m^2$:

$$\tilde{A}^{III(l)}(s) = \frac{1}{\pi}\int_{4m^2}^{16m^2} \frac{\tilde{\alpha}^{(l)}(s')}{s'-s} ds' + \eta^{(l)}(s), \tag{32}$$

$$\eta^{(l)}(s) = \frac{2}{\pi(s-4m^2)^l}\sum_{k=0}^{\infty}\frac{4k+1}{2^{2k}}\int_{z_0'(s)}^{z_0''(s)} dz_1\, Q_l(z_1)$$
$$\times [(s-4m^2)(z_1-1) - 8m^2]^{2k} P_{2k}$$
$$\times\left(\frac{(s-4m^2)(3+z_1)+8m^2}{(s-4m^2)(z_1-1)-8m^2}\right)\tilde{\alpha}^{(2k)}\left(\frac{(s-4m^2)(z_1-1)}{2}\right)$$
$$+\frac{1}{2\pi(s-4m^2)^l}\int_{-1}^{1} dz_1\, P_l(z_1)\sum_{l'=0}^{\infty}(2l'+1)\int_{4m^2}^{16m^2} ds'\,\frac{\tilde{\alpha}^{(l')}(s')}{s'-s}$$
$$\times(s'-4m^2)^{l'}\left[P_{l'}\left(1+\frac{(s-4m^2)}{(s'-4m^2)}(z_1-1)\right)\right.$$
$$\left.-\frac{(s-4m^2)^{l'}}{(s'-4m^2)^{l'}}P_{l'}(z_1)\right] + \frac{1}{\pi(s-4m^2)^l}\int_{-1}^{1} dz_1\, P_l(z_1)\sum_{l'=0}^{\infty}$$
$$\times(2l'+1)\int_{4m^2}^{16m^2} du'\,\frac{\tilde{\alpha}^{(l')}(u')}{2u'+(s-4m^2)(z_1+1)}$$
$$\times(u'-4m^2)^{l'}\left[P_{l'}\left(\frac{4m^2-2s+u'+(s-4m^2)(z_1+1)}{u'-4m^2}\right)\right.$$
$$\left.-P_{l'}\left(\frac{4m^2-2s-u'}{u'-4m^2}\right)\right]$$
$$+\frac{1}{2(s-4m^2)^l}\int_{-1}^{1} dz_1\, P_l(z_1) f^{III}(s, t(s, z_1), u(s, z_1)). \tag{33}$$

Here $Q_l(z)$ is the Legendre function of the second kind,

$$z_0'(s) = \frac{s+4m^2}{s-4m^2}, \qquad z_0''(s) = \frac{s+28m^2}{s-4m^2}.$$

In the new notation the unitarity condition (25) for $4m^2 \leq s \leq 16m^2$ becomes

$$\tilde{\alpha}^{(l)}(s) = \tilde{\omega}^{(l)}(s)|\tilde{A}^{III\,(l)}(s)|^2. \tag{34}$$

We note that the unitarity condition is not used for $s \geq 16m^2$.

Formulas (32), (33), and (34) constitute the desired system of equations. They contain "nonphysical" partial waves with odd l, which is their characteristic peculiarity.

Let us also note the following circumstance. In the second and third terms on the right side of Eq. (33) the numerators in the integrands vanish simultaneously with the denominators. These integrands are, consequently, polynomials in the variables s and z_1. If in the solution of these equations only the first N partial waves are taken into account then the contribution from these two terms to the quantity $\eta^{(l)}(s)$ is equal to a polynomial of degree $N - 1 - l$ for $l \leq N - 1$, and is equal to zero for $l > N - 1$. Let at the same time the function $f^{III}(s, u, t)$ be approximated by a polynomial of degree K. Then the last term on the right side of Eq. (33) equals a polynomial in s of degree $K - l$. It is clear that the second and third terms of the formula contribute substantially to $\eta^{(l)}(s)$ only under the condition $N - 1 > K$. In the opposite case, when $N - 1 \leq K$, the contribution of these terms reduces to a modification of the arbitrary parameters that enter the polynomial used to approximate the function $f^{III}(s, u, t)$. On the other hand the "nonphysical" functions $\alpha^{(l')}$ with odd l' occur only in the second and third terms of Eq. (33). Consequently the properties of these equations due to the presence of these functions manifest themselves only when the number of partial waves N taken into account is sufficiently large.

Let us discuss one of the possible methods of solving Eqs. (32) and (34).[2)] We define the function $F^{(l)}(s)$ by

$$F^{(l)}(s) = \frac{1}{\pi}\int_{4m^2}^{16m^2} ds'\,\frac{\tilde{\alpha}^{(l)}(s')}{s'-s}. \tag{35}$$

[2)]An analogous method was used by Mandelstam[3] and Chew[9] in the complex angular momentum theory.

Equations (34) can be reduced with the help of (32) and (35) to the form

$$\operatorname{Im} F^{(l)}(s) = \tilde{\omega}^{(l)}(s)\,|F^{(l)}(s) + \eta^{(l)}(s)|^2. \qquad (36)$$

For the solution of these equations one may utilize the method of successive approximations to determine $F^{(l)}(s)$, and therefore also $\tilde{\alpha}^{(l)}(s) = \operatorname{Im} F^{(l)}(s)$, from some given functions $\eta^{(l)}(s)$, and then determine the next approximation to $\eta^{(l)}(s)$ from Eq. (33). In this way the problem reduces to the solution of Eq. (36) with respect to $F^{(l)}(s)$ for known functions $\eta^{(l)}(s)$.

Omitting the superscript l, we set

$$F(s) + \eta(s) = N(s)\,/\,D(s). \qquad (37)$$

Here $N(s)$ is a function that is continuous and real for $4m^2 \le s \le 16m^2$; $D(s)$ is a function analytic in the entire complex s plane except for the cut $4m^2 \le s \le 16m^2$, which tends to unity as $s \to \infty$. On the cut itself the function $D(s)$ may have poles. As is well known [5], the function $F(s) + \eta(s)$ may always be represented in the form (37). From Eq. (36) it follows that for $4m^2 \le s \le 16m^2$

$$\operatorname{Im}\frac{1}{F(s)+\eta(s)} = -\tilde{\omega}(s) - \sum_{i=1}^{r} a_i \delta(s - s_i), \qquad (38)$$

where the a_i are the arbitrary parameters of Castillejo, Dalitz, and Dyson.[7] It follows from (37) and (38) and from the assumed behavior of $D(s)$ as $s \to \infty$ that

$$D(s) = 1 - \frac{1}{\pi}\int_{4m^2}^{16m^2}\left[\tilde{\omega}(s) + \sum_{i=1}^{r} a_i\delta(s'-s_i)\right]\frac{N(s')}{s'-s}\,ds'. \qquad (39)$$

In accordance with the definitions of the functions $F(s)$ and $D(s)$ the function $F(s)D(s) = N(s) - \eta(s)D(s)$ is analytic in the entire complex s plane except for the cut $4m^2 \le s \le 16m^2$. Writing a dispersion relation for this function and making use of Eq. (39) we arrive at a Fredholm type integral equation for the function $N(s)$:

$$N(s) = \frac{1}{\pi}\int_{4m^2}^{16m^2} ds'\,\frac{\eta(s') - \eta(s)}{s'-s}\,\tilde{\omega}(s')\,N(s')$$
$$+\,\eta(s) + \sum_{i=1}^{r} b_i\frac{\eta(s_i) - \eta(s)}{s_i - s}, \qquad (40)$$

where in place of the a_i we have introduced new arbitrary constants

$$b_i = \pi^{-1} a_i N(s_i). \qquad (41)$$

Having solved Eq. (40), we can determine $D(s)$ from (39) with (41) taken into account and then determine $F(s)$ from Eq. (37). The solution has physical meaning if the function $D(s)$ so obtained has no zeros in the complex s plane. This condition imposes certain restrictions on the arbitrary parameters b_i, and also on the function $\eta(s)$, which is to say on the polynomial used to approximate the quantity $f^{III}(s, u, t)$ in Eq. (33).

4. SUBTRACTION TERMS

Up to now it has been assumed that the amplitude $A(s, u, t)$ satisfies a double dispersion relation with no subtractions. We shall now show that the system of equations obtained above remains in essence unchanged when one considers the more general case. Suppose that for finite values of the variables s, u, and t the analyticity properties of the amplitude $A(s, u, t)$ are as before in agreement with the Mandelstam representation. Let us assume, however, that the number of necessary subtractions in the dispersion relations in momentum transfer in each channel can increase without bound with increasing energy. Under these conditions it follows that for $4m^2 \le s \le 16m^2$ the amplitude A is bounded by

$$|A(s, t(s, z_1), u(s, z_1))| \leqslant c\,|z_1|^{n-\varepsilon}, \qquad (42)$$

where n is a positive integer and ϵ is an arbitrarily small positive number. Analogous inequalities hold in the other two channels.

Let us define the functions $\alpha(s, t)$, $A^{II}(s, u, t)$ and $A^{III}(s, u, t)$ for $4m^2 \le s \le 16m^2$ by means of (8), (12a), and (12b), having performed in these formulas n subtractions and added to the right sides arbitrary polynomials in t (in u) of degree $n - 1$ with s-dependent coefficients. Further let us require that Eqs. (10), (11), and (13) be satisfied, i.e., that the even with respect to z_1 part of the function $A^{III}(s, u(s, z_1), t(s, z_1))$ coincide with the amplitude $A(s, u(s, z_1), t(s, z_1))$ and the even part of $\alpha(s, t(s, z_1))$ coincide with the elastic jump of the amplitude $A_{1el}(s, z_1)$. In addition we subject the functions A^{III} and α for $4m^2 \le s \le 16m^2$, $-1 \le z_1 \le 1$ to the relation

$$\operatorname{Im} A^{III}(s, u(s, z_1),\, t(s, z_1)) = \alpha(s, t(s, z_1)). \qquad (43)$$

All these requirements can be satisfied for any given amplitude $A(s, u, t)$. Their imposition, therefore, involves no loss of generality.

Formula (26) and the resultant equalities (32) and (33) remain unchanged in the passage to these new conditions as a consequence of the definitions used for the quantities A^{III} and α. Indeed, by comparing the jump of A^{III} and of the first three terms on the right side of Eq. (26) one can show, as before, that these jumps coincide in the strips

$4m^2 \leq s \leq 16m^2$ and $4m^2 \leq t \leq 16m^2$.[3)] The function $f^{III}(s, u, t)$, as before, has no singularities inside the triangle $s \leq 16m^2$, $u \leq 16m^2$, $t \leq 16m^2$ and can be approximated there by a polynomial. Its analytic properties for finite values of the variables s, u, t are in agreement with the integral representation (27). However now such a representation for the function $f^{III}(s, u, t)$ can not, generally speaking, be written down because of the infinite number of necessary subtractions. This circumstance does not, apparently, invalidate formula (26).

The unitarity condition (34) for all even l as a consequence of the definition of the functions A^{III} and α coincides, as before, with the analogous condition for physical partial amplitudes and consequently should be fulfilled. This condition is also valid for odd l larger than $n - \epsilon$, since, as is well known, it can be continued in the complex l plane up to the line $\operatorname{Re} l = n - \epsilon$. On the other hand the unitarity condition can not be extended to values of l satisfying the relation $l = 2r + 1 < n - \epsilon$ (r—an integer), since we make no assumptions about the analytic properties of the functions $\widetilde{A}^{III(l)}(s)$ and $\widetilde{\alpha}^{(l)}(s)$ in the l plane to the left of the line $\operatorname{Re} l = n - \epsilon$.

Let us determine now what is the contribution to the right side of Eq. (33) from those functions $\widetilde{\alpha}^{(l')}(s)$ for which $l' = 2r + 1 < n - \epsilon$, i.e., $l' = 2r + 1 \leq n - 1$. It follows from the remark made at the end of the derivation of Eqs. (32) and (34) that that contribution is equal to a polynomial in s of degree $n - 2 - l$ for even n and of degree $n - 3 - l$ for odd n. It can be discarded without loss of generality if the degree K of the arbitrary polynomial used to approximate the function $f^{III}(s, u, t)$ in Eq. (33) satisfies the conditions

$$K \geqslant n-2 \quad \text{for even n},$$
$$K \geqslant n-3 \quad \text{for odd n}. \tag{44}$$

In that case one may assume that

$$\widetilde{\alpha}^{(2r+1)}(s) = 0 \quad \text{for} \quad 2r+1 \leqslant n-1 \tag{45}$$

and exclude from consideration the equalities (32)—(34) with indices l satisfying the condition $l = 2r + 1 \leq n - 1$. The remaining equations (32), (33), and (34) form a closed system.

We note that the number n, which characterizes the rate of growth of the amplitude in the variable t for $4m^2 \leq s \leq 16m^2$, can not be found by theoretical considerations without making some additional hypothesis about the analytic properties of the function $A^{III(l)}(s)$ in the complex l plane for $\operatorname{Re} l > 0$. If no such hypothesis is used then the quantity K, and consequently the number of arbitrary parameters entering the equations, may be determined only by comparing the theory with experiment. This applies also to the arbitrary Castillejo-Dalitz-Dyson parameters which occur in the solution of the equations.

5. CONCLUDING REMARKS

Equations (32) and (34) are similar in form to the Chew-Mandelstam equations.[5] There are, however, significant differences. First of all, in Eq. (32) the right-hand cut has been cut off at $s = 16m^2$. For this reason the well known proof that the Chew-Mandelstam equations can not be solved without a cut-off on the left cut [10] loses its force. This proof is based on the fact that the left and right hand cuts meet at infinity in the s plane, combined with the fact that the unitarity condition holds on the right-hand cut for arbitrarily large s. Such considerations are not applicable to the system of equations (32) and (34). Secondly, the functions $\eta^{(l)}$ in (32) are determined indirectly and not by means of dispersion relations for partial amplitudes. The jumps in these amplitudes across the left-hand cut are not involved in the equations. This makes it possible to avoid going outside the region of convergence of the corresponding Legendre series. Finally, Eqs. (32) and (34) reflect more fully the analytic properties of the amplitude because the "unphysical" partial waves with odd l are taken into account.

The number of arbitrary parameters entering these equations is in no way related to the number of partial waves taken into account. As is well known such a relation does hold in the case of the Chew-Mandelstam equations cut off on the left-hand cut. Equations with the number of arbitrary parameters independent of the number of partial waves considered, were also obtained by Malakhov.[11] In his work, however, the "unphysical" partial waves are not used. Therefore the contribution of the near part of the spectral function to the amplitude is not fully taken into account.

As already noted, one can obtain with the help of (32) and (34) a more exact value for the scattering amplitude in the low energy region if the function $f^{III}(s, u, t)$ in Eq. (33) is approximated not by a polynomial but by an expression having logarithmic singularities for $s = 16m^2$ and

[3)]These functions, just like $f^{III}(s, u, t)$, have no jump in the u variable (see Fig. 2).

$t = 16m^2$. Such an approximation, of course, requires a different choice for the arbitrary parameters entering the theory. In that case the quantities $\eta^{(l)}(s)$, obtained from Eq. (33), will also have logarithmic singularities for $s = 16m^2$ and, consequently, the integral equation (40) will become singular. The method for solving equations of this type is outlined in the work of Chew[9] where it is shown that the diffulties connected with the singularity can be overcome.

The system (32) and (34) permits one to find the amplitude only below the threshold for inelastic processes. On the other hand the energy region of greatest importance from the point of view of explaining experimental data lies above these thresholds. In particular, resonances in the $\pi\pi$ scattering amplitude are observed only at energies in excess of four pion masses. It would therefore be of great value to generalize the method described here to include higher energies. The main difficulty that one encounters here has to do with the fact that the region of convergence of the Legendre series is limited. This difficulty may be circumvented with the help of the method described in [12], which, however, substantially complicates the system of equations.

It should be remarked that recently a system of equations has been proposed for the analysis of low-energy scattering, based on the assumption that the scattering amplitude is a meromorphic function in the complex l plane for $\mathrm{Re}\, l > 0$.[9] These equations are analogous in form to Eq. (40) except that the index l is allowed to assume a continuous set of values. It is obvious that such equations are substantially more complicated than the partial wave equations for integer l. Therefore the equations containing only functions with integer values of l continue to be of substantial significance in application to the problem of low energy scattering, even if one starts with the hypothesis that the amplitude is a meromorphic function in the l plane.

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