INTERACTION OF LONGITUDINAL AND TRANSVERSE WAVES IN A PLASMA

L. M. KOVRIZHNYKH and V. N. TSYTOVICH

P. N. Lebedev Physics Institute, Academy of Sciences U.S.S.R.

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The kinetic equations and the field equation are derived taking account of the interaction between longitudinal and transverse electromagnetic waves. It is shown that nonlinear interactions can lead to an instability for the longitudinal and transverse waves. An isotropic plasma and a beam-plasma system are considered.

1. INTRODUCTION

It is now evident that a characteristic feature of plasma, regarded as a state of matter, is the capability of supporting various kinds of oscillations (noise modes) under the effect of relatively small external perturbations (as a consequence of so-called plasma instabilities). On the other hand, nonlinear effects tend to couple longitudinal and transverse waves and it is possible that this coupling could have a stabilizing effect at high oscillation levels.[1)] In the present work we derive the particle-field equations for a plasma, taking account of the interaction of longitudinal and transverse waves and the interaction of these waves with the plasma particles. This interaction is essentially scattering of particles accompanied by induced conversion of the plasma wave into a transverse wave (or vice versa), and the absorption (emission) of two waves.[3] We limit ourselves to frequencies and wave numbers that correspond to the region of transmission and which do not satisfy the "decay" conditions. We assume, further, that the noise intensity is so large that the spontaneous processes can be neglected compared with the induced processes.

2. PARTICLE-WAVE EQUATIONS

The following equations describe the distribution functions for plasma particles of type α, $f^{\alpha}_{\mathbf{p}_\alpha}$, plasma waves, $N^{l}_{\mathbf{k}_1}$, and transverse waves, $N^{t}_{\mathbf{k}_2}$, taking account of the induced scattering (in which a longitudinal wave is converted into a transverse wave and vice versa), induced radiation and absorption of longitudinal and transverse photons:[2)]

$$\frac{df_{\mathbf{p}_\alpha}{}^{\alpha}}{dt} = -\int d\mathbf{k}_1 N_{\mathbf{k}_1}{}^{l}\{w_{\mathbf{p}_\alpha}{}^{\alpha}(\mathbf{k}_1)(f_{\mathbf{p}_\alpha}{}^{\alpha} - f_{\mathbf{p}_\alpha-\mathbf{k}_1}{}^{\alpha})$$
$$+ w_{\mathbf{p}_\alpha+\mathbf{k}_1}{}^{\alpha}(\mathbf{k}_1)(f_{\mathbf{p}_\alpha}{}^{\alpha} - f_{\mathbf{p}_\alpha+\mathbf{k}_1}{}^{\alpha})\}$$
$$-\int d\mathbf{k}_1\, d\mathbf{k}_2 N_{\mathbf{k}_1}{}^{l} N_{\mathbf{k}_2}{}^{t}\{W_{\mathbf{p}_\alpha}{}^{\alpha}(\mathbf{k}_1, \mathbf{k}_2)(f_{\mathbf{p}_\alpha}{}^{\alpha} - f_{\mathbf{p}_\alpha-\mathbf{k}_2+\mathbf{k}_1})^{\alpha}$$
$$+ W^{\alpha}_{\mathbf{p}_\alpha+\mathbf{k}_2-\mathbf{k}_1}(\mathbf{k}_1, \mathbf{k}_2)(f_{\mathbf{p}_\alpha}{}^{\alpha} - f^{\alpha}_{\mathbf{p}_\alpha-\mathbf{k}_1+\mathbf{k}_2})\widetilde{W}_{\mathbf{p}_\alpha}{}^{\alpha}(\mathbf{k}_1, \mathbf{k}_2)$$
$$\times (f^{\alpha}_{\mathbf{p}_\alpha} - f^{\alpha}_{\mathbf{p}_\alpha+\mathbf{k}_1+\mathbf{k}_2}) + \widetilde{W}^{\alpha}_{\mathbf{p}_\alpha+\mathbf{k}_1+\mathbf{k}_2}(\mathbf{k}_1, \mathbf{k}_2)(f^{\alpha}_{\mathbf{p}_\alpha} - f^{\alpha}_{\mathbf{p}_\alpha+\mathbf{k}_1+\mathbf{k}_2})\Big\};$$
$$\frac{dN_{\mathbf{k}_1}{}^{l}}{dt} = \gamma_{\mathbf{k}_1}{}^{l} N_{\mathbf{k}_1}{}^{l}; \tag{2.1}$$

$$\gamma_{\mathbf{k}_1}{}^{l} = (2\pi)^3 \sum_\alpha \int f^{\alpha}_{\mathbf{p}_\alpha} d\mathbf{p}_\alpha [w^{\alpha}_{\mathbf{p}_\alpha}(\mathbf{k}_1) - w^{\alpha}_{\mathbf{p}_\alpha+\mathbf{k}_1}(\mathbf{k}_1)]$$
$$+ \int N_{\mathbf{k}_2}{}^{t} \Phi^{l}(\mathbf{k}_1, \mathbf{k}_2)\, d\mathbf{k}_2,$$

$$\Phi^{l}(\mathbf{k}_1, \mathbf{k}_2) = -(2\pi)^3 \sum_\alpha \int d\mathbf{p}_\alpha [W^{\alpha}_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2) - W^{\alpha}_{\mathbf{p}_\alpha-\mathbf{k}_2+\mathbf{k}_1}(\mathbf{k}_1, \mathbf{k}_2)$$
$$- \widetilde{W}^{\alpha}_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2) + \widetilde{W}^{\alpha}_{\mathbf{p}_\alpha+\mathbf{k}_1+\mathbf{k}_2}(\mathbf{k}_1, \mathbf{k}_2)] f^{\alpha}_{\mathbf{p}_\alpha}; \tag{2.2}$$

$$\frac{dN_{\mathbf{k}_2}{}^{t}}{dt} = \gamma_{\mathbf{k}_2}{}^{t} N_{\mathbf{k}_2}{}^{t}, \qquad \gamma_{\mathbf{k}_2}{}^{t} = \int \Phi^{t}(\mathbf{k}_1, \mathbf{k}_2) N_{\mathbf{k}_1}{}^{l}\, d\mathbf{k}_1,$$

$$\Phi^{t}(\mathbf{k}_1, \mathbf{k}_2) = (2\pi)^3 \sum_\alpha \int d\mathbf{p}_\alpha [W^{\alpha}_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2) - W^{\alpha}_{\mathbf{p}_\alpha+\mathbf{k}_2-\mathbf{k}_1}(\mathbf{k}_1, \mathbf{k}_2)$$
$$+ \widetilde{W}^{\alpha}_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2) - \widetilde{W}^{\alpha}_{\mathbf{p}_\alpha+\mathbf{k}_1+\mathbf{k}_2}(\mathbf{k}_1, \mathbf{k}_2)] f^{\alpha}_{\mathbf{p}_\alpha}. \tag{2.3}$$

Here, $w^{\alpha}_{\mathbf{p}_\alpha}(\mathbf{k}_1)$ is the probability of Cerenkov radiation from a particle of type α with momentum $\mathbf{p}_\alpha$ yielding a plasma wave with momentum $\mathbf{k}_1$; $W^{\alpha}_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)$ is the probability of scattering of a particle α with momentum $\mathbf{p}_\alpha$ leading to the absorption of a plasma wave $\mathbf{k}_1$ and emission of a transverse wave $\mathbf{k}_2$; $\widetilde{W}^{\alpha}_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)$ is the probability of radiation from a particle α with momentum

1)The transformation of longitudinal waves into transverse waves in scattering on fluctuations has been treated in.[1,2]

2)We assume for simplicity that $\hbar = c = 1$.

$\mathbf{p}_\alpha$ yielding a plasma wave $\mathbf{k}_1$ and a transverse wave $\mathbf{k}_2$.

In the classical case, which is of greatest interest $k_1, k_2 \ll p$, expanding the right sides of Eq. (2.1)–(2.3) in powers of $\mathbf{k}_1$, and $\mathbf{k}_2$ and keeping the first nonvanishing terms we find

$$\frac{df^\alpha_{\mathbf{p}_\alpha}}{dt} = \frac{\partial}{\partial p_i^\alpha} D_{ij}^\alpha \frac{\partial}{\partial p_j^\alpha} f^\alpha_{\mathbf{p}_\alpha},$$

$$D_{ij}^\alpha = \int d\mathbf{k}_1 k_{1i} k_{1j} w^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1) N_{\mathbf{k}_1}{}^l + \int d\mathbf{k}_1\, d\mathbf{k}_2 N_{\mathbf{k}_1}{}^l N_{\mathbf{k}_2}{}^t$$
$$\times \{(k_{2i} - k_{1i})(k_{2j} - k_{1j}) W^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)$$
$$+ (k_{2i} + k_{1i})(k_{2j} + k_{1j}) \widetilde{W}^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)\}; \qquad (2.4)$$

$$\gamma_{\mathbf{k}_1}{}^l = (2\pi)^3 \sum_\alpha \int w^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1)\left(\mathbf{k}_1 \frac{\partial f^\alpha_{\mathbf{p}_\alpha}}{\partial \mathbf{p}_\alpha}\right) d\mathbf{p}_\alpha + \int N^t_{\mathbf{k}_2} \Phi^l(\mathbf{k}_1, \mathbf{k}_2)\, d\mathbf{k}_2,$$

$$\Phi^l(\mathbf{k}_1, \mathbf{k}_2) = (2\pi)^3 \sum_\alpha \int d\mathbf{p}_\alpha \left[W^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)\left((\mathbf{k}_1 - \mathbf{k}_2)\frac{\partial f^\alpha_{\mathbf{p}_\alpha}}{\partial \mathbf{p}_\alpha}\right)\right.$$
$$\left. + \widetilde{W}^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)\left((\mathbf{k}_1 + \mathbf{k}_2)\frac{\partial f^\alpha_{\mathbf{p}_\alpha}}{\partial \mathbf{p}_\alpha}\right)\right]; \qquad (2.5)$$

$$\gamma_{\mathbf{k}_2}{}^t = \int N_{\mathbf{k}_1}{}^l \Phi^t(\mathbf{k}_1, \mathbf{k}_2)\, d\mathbf{k}_1,$$

$$\Phi^t(\mathbf{k}_1, \mathbf{k}_2) = (2\pi)^3 \sum_\alpha \int d\mathbf{p}_\alpha \left[W^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)\left((\mathbf{k}_2 - \mathbf{k}_1)\frac{\partial f^\alpha_{\mathbf{p}_\alpha}}{\partial \mathbf{p}_\alpha}\right)\right.$$
$$\left. + \widetilde{W}^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)\left((\mathbf{k}_2 + \mathbf{k}_1)\frac{\partial f^\alpha_{\mathbf{p}_\alpha}}{\partial \mathbf{p}_\alpha}\right)\right]. \qquad (2.6)$$

The system (2.4)–(2.6) contains the classical probabilities where $W^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)$ and $\widetilde{W}^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)$ are related by[3)]

$$\widetilde{W}^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2) = W^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1', \mathbf{k}_2)\Big|_{\substack{\mathbf{k}_1' = -\mathbf{k}_1 \\ \omega_1' = -\omega_1}}. \qquad (2.7)$$

If the imaginary part of the frequency is much smaller than the real part, the phase velocity of the longitudinal waves is much greater than the mean thermal velocity (but much smaller than the velocity of light) and $\omega_1 \approx \omega_0$ while the frequency of the transverse waves $\omega_2 \gg \omega_0$ so that $\omega_2 = |\mathbf{k}_2|$; hence, the expressions for $w^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1)$ and $W^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)$ assume the form[3]

$$w^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1) = (e_\alpha^2 \omega_0 / 2\pi k_1^2)\, \delta(\omega_0 - \mathbf{k}_1 \mathbf{v}_\alpha),$$
$$\mathbf{v}_\alpha = \mathbf{p}_\alpha / \sqrt{\mathbf{p}_\alpha^2 + m_\alpha^2}, \qquad (2.8)$$
$$\omega_0^2 = 4\pi N_e e^2 / m_e;$$

$$W^\alpha_{\mathbf{p}_\alpha}(\mathbf{k}_1, \mathbf{k}_2)$$
$$= \frac{e_\alpha^4 \omega_0 \mathbf{k}_1^2}{m_e^2 k_2}\left[\beta^2 - \frac{(\mathbf{k}_2\beta)^2}{k_2^2}\right] \delta(|\mathbf{k}_2| - \omega_0 - \mathbf{k}_2\mathbf{v}_\alpha + \mathbf{k}_1\mathbf{v}_\alpha),$$
$$\beta_e = \beta_e(\mathbf{v}) = \frac{\mathbf{k}_1}{|\mathbf{k}_2|}\left[\frac{\sqrt{1 - v^2}}{(1 - \mathbf{k}_2\mathbf{v}/|\mathbf{k}_2|)\mathbf{k}_1^2} - \frac{1}{(\mathbf{k}_1 - \mathbf{k}_2)^2 - \mathbf{k}_2^2}\right]$$
$$+ \mathbf{v}\left[\frac{\sqrt{1 - v^2}}{(\mathbf{k}_1\mathbf{v})^2\, \mathbf{k}_1^2}(\mathbf{k}_1\mathbf{k}_2 - |\mathbf{k}_2|\mathbf{k}_1\mathbf{v}) - \frac{1}{(\mathbf{k}_2 - \mathbf{k}_1)^2 - \mathbf{k}_2^2}\right]; \qquad (2.9)$$

$$\beta_i = \beta_i(\mathbf{v}) = \frac{\mathbf{k}_1/|\mathbf{k}_2| + \mathbf{v}}{(\mathbf{k}_1 - \mathbf{k}_2)^2 - k_2^2}. \qquad (2.10)$$

The obtained system (2.4)–(2.6) obviously satisfies energy conservation

$$\frac{d}{dt}\left\{\sum_\alpha \int \sqrt{\mathbf{p}_\alpha^2 + m_\alpha^2} f^\alpha_{\mathbf{p}_\alpha} d\mathbf{p}_\alpha + W^l + W^t\right\} = 0,$$
$$W^l \equiv \langle (E^l)^2 \rangle / 8\pi = (2\pi)^{-3} \int \omega_0 N_{\mathbf{k}_1}{}^l\, d\mathbf{k}_1,$$
$$W^t \equiv \langle (E^t)^2 \rangle / 8\pi = (2\pi)^{-3} \int |\mathbf{k}_2| N_{\mathbf{k}_2}{}^t\, d\mathbf{k}_2. \qquad (2.11)$$

Thus, the interaction between the longitudinal and transverse waves results in a modification of both the diffusion tensor and the growth rates (damping rates) $\gamma^l_{\mathbf{k}_1}$, $\gamma^l_{\mathbf{k}_2}$ (cf. [5-7]). It follows from (2.8) and (2.9) that the essential role is played by particles whose velocities exceed the phase velocity of the plasma waves. In other words, the effect being considered (the interaction between longitudinal and transverse waves) can be important if there is a relatively large number of superthermal particles. Such cases are not unusual and are of definite interest. Indeed, superthermal particles are necessary for turbulent heating; high noise intensity of this kind also is produced in a plasma having superthermal particles.[8,9] Hence we shall be concerned primarily with cases in which the plasma contains a relatively large number of superthermal particles.

Without going through a detailed investigation of the equations that have been derived we proceed directly to a number of results that follow in certain particular cases.

3. EXCITATION AND DAMPING IN AN ISOTROPIC PLASMA

If the particle velocity distribution is isotropic, i.e., $f^\alpha_{\mathbf{p}_\alpha} = f^\alpha_0(v)$, in the nonrelativistic case $(k_2/k_1 \ll 1)$

[3)]The system of equations that has been obtained cannot be written in the form used in[4]. This follows because it is necessary to take account of processes that correspond, in the classical limit, to the transition radiation of a charge interacting with the inhomogeneities produced by the plasma wave. These processes make a contribution to the probability that is of the same order as that corresponding to radiation of a particle oscillating in the field of a longitudinal wave. The effect of such processes in the interaction of longitudinal waves is discussed, for example, in.[4]

$$\Phi^{l,t}(\mathbf{k}_1, \mathbf{k}_2) = -\sum_\alpha \frac{64\pi^4 e^4 \omega_0}{|\mathbf{k}_1|^3}\Big\{\sin^2 2\theta \Big[f_0^{(e)}\Big(\Big|\frac{\omega_2+\omega_0}{\mathbf{k}_2+\mathbf{k}_1}\Big|\Big) \mp f_0^{(e)}\Big(\Big|\frac{\omega_2-\omega_0}{\mathbf{k}_1-\mathbf{k}_2}\Big|\Big)\Big] + \frac{k_1^2}{k_2^2}\frac{1}{4\pi m_e^2} \times \Big[\cos^4\theta + \frac{\sin^2\theta}{2} - \sin^2\theta\cos^2\theta\Big] \times \Big[\varphi^{(e)}\Big(\Big|\frac{\omega_2+\omega_0}{\mathbf{k}_1+\mathbf{k}_2}\Big|\Big) \mp \varphi^{(e)}\Big(\Big|\frac{\omega_2-\omega_0}{\mathbf{k}_1-\mathbf{k}_2}\Big|\Big)\Big] + \frac{m_i^2}{m_e^2}\frac{k_1^2}{k_2^2}\frac{\sin^2\theta}{4} \times \Big[f_0^{(i)}\Big(\Big|\frac{\omega_2+\omega_0}{\mathbf{k}_1+\mathbf{k}_2}\Big|\Big) \mp f_0^{(i)}\Big(\Big|\frac{\omega_2-\omega_0}{\mathbf{k}_1-\mathbf{k}_2}\Big|\Big)\Big]\Big\}, \tag{3.1}$$

where θ is the angle between the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, m_e and m_i are the masses of the electron and ion respectively, and $\varphi^{(e)}(v) = \int f_0^{(e)} dp_\perp$ is the one-dimensional distribution function; $p_\perp$ is the component of p perpendicular to $\mathbf{k}_1 - \mathbf{k}_2$.

The ion contribution is unimportant in most cases because the corresponding term contains the small factor m_e/m_i.[4] For Eq. (3.1) to hold it is necessary that the distribution function be isotropic only in the vicinity of the point $v = |(\omega_2 \pm \omega_0)/(\mathbf{k}_1 \pm \mathbf{k}_2)|$. In other regions, specifically, at lower velocities, it need not be isotropic (for example, a plasma in the presence of a beam with velocity much smaller than $|(\omega_2 \pm \omega_0)/(\mathbf{k}_1 \pm \mathbf{k}_2)| \approx \omega_2/|\mathbf{k}_1|$.

1. We now consider damping of transverse waves in an isotropic plasma. Since the expression appearing in the curly brackets in Eq. (3.1) for Φ^t is essentially positive, it follows from Eq. (2.6) that transverse waves are always damped ($\gamma^t_{\mathbf{k}_2} < 0$) if the particle momentum distribution is isotropic. In general this damping is small; however, if there is a relatively large number of superthermal particles and a high noise level $N^l_{\mathbf{k}_1}$ this damping can become appreciable, even exceeding the damping caused by collisions (γ_c).

To estimate the mean-square noise amplitude $\langle(E^l)^2\rangle$ at which the damping $\gamma^t_{\mathbf{k}_2}$ becomes comparable with γ_c we assume that the noise is distributed uniformly over the interval between some k_{min} and $k_{max} \gg k_{min}$ and that the electron distribution function in the region $v_1 < v < v_2$ (where $v_2 \gg \Delta v = v_2 - v_1$) is constant and equal to $n_1/4\pi v_2^2 \Delta v m_e^3$, where $v_1 \gg v_T = \sqrt{2kT/m}$; outside this region it is assumed that the electron distribution is Maxwellian with temperature T. Assuming further that $k_{max} = \omega_0/3v_T$ and that the Coulomb logarithm in the expression for the collision frequency is of order 10, we find that the damping is nonvanishing for frequencies $\omega_2 < 3 \times 10^4 \omega_0 v_2/\sqrt{T}$ where

$$\frac{\gamma^t_{k_2}}{\gamma_c} = 5\cdot 10^{-2}\frac{n_1}{n_0}\frac{\omega_2^2}{\omega_0^2}\frac{1}{v_2^2\Delta v}\Big(\frac{T^2}{n_0}\Big)^{3/2}\langle(E^l)^2\rangle. \tag{3.2}$$

It then follows that the nonlinear effects being treated here are strongest in a high-temperature low-density plasma. For example, assuming that $v_2 = 10^{-1}$, $T = 10^6\,^\circ K$, $n_1/n_0 = 10^{-2}$, $n_0 = 10^{11}\ cm^{-3}$, $\Delta v/v_2 = 10^{-1}$, $E^l = 1$ (CGS) we find that $\gamma^t = 5 \times 10^2\ \gamma_c$, i.e., nonlinear effects lead to a very strong damping even at a relatively low noise level.

2. We now consider longitudinal waves and examine the conditions under which the presence of "transverse" noise $N^t_{\mathbf{k}_2}$ can cause an instability of the longitudinal waves in an isotropic plasma. Neglecting terms proportional to $f_0^{(i)}$ and $\varphi^{(e)}$ in Eq. (3.1) and assuming that the noise is concentrated in a narrow band around $\omega_2 = \omega_2^0$, we find[5]

$$\gamma^l_{\mathbf{k}_1} = \frac{(2\pi)^6 e^4\omega_0 \sin^2 2\theta}{\omega_2^0 k_1^3} \times \Big[f_0^{(e)}\Big(\Big|\frac{\omega_2^0-\omega_0}{\mathbf{k}_1-\mathbf{k}_2^0}\Big|\Big) - f_0^{(e)}\Big(\Big|\frac{\omega_2^0+\omega_0}{\mathbf{k}_1+\mathbf{k}_2^0}\Big|\Big)\Big] \times \langle(E^t)^2\rangle - \frac{2\pi^2\omega_0^4 m_e^3}{|\mathbf{k}_1|^3 n_0} f_0^{(e)}\Big(\frac{\omega_0}{|\mathbf{k}_1|}\Big). \tag{3.3}$$

The last term in Eq. (3.3) is obviously always negative but the first can be smaller or greater than zero. Thus, if there is a group of superthermal particles for some $v = v_0 \gg v_T$, i.e., if $f_0^{(e)}$ exhibits a maximum at the point v_0, then at large value of $\langle(E^t)^2\rangle$ and appropriate values of ω_2^0 and v_0 it turns out that $\gamma^l_{|\mathbf{k}_1|}$ can be positive, that is to say, the longitudinal waves can become unstable.

If the energy spread of the superthermal particles Δv_0 is narrow[6] the instability condition can be written in the form

$$2\pi^3\frac{\omega_0}{\omega_2^0}\sin^2 2\theta\frac{f_0^{(e)}(\omega_2^0/|\mathbf{k}_1|)}{f_0^{(e)}(\omega_0/|\mathbf{k}_1|)}\frac{\langle(E^t)^2\rangle}{n_0 m_e} > 1. \tag{3.4}$$

To make an estimate we assume that $f_0^{(e)}$ is constant and equal to $n_1/4\pi m_e^3 v_0^2 \Delta v_0$ in the range $v_0 \pm \Delta v_0$; outside this range we assume a Max-

[4]In certain cases, all of the ions can make a contribution of the same order as the electrons because in addition to the small factor m_e/m_i the expression in question contains the large quantity $\sim 1/v_0^2 \gg 1$, where v_0 is the mean characteristic velocity of the superthermal particles.

[5]It is valid to neglect these terms if the superthermal particles are primarily electrons and if the velocity spread of these electrons is small enough, i.e., $\Delta v/v \ll 1$. This case is the one that is treated in this work.

[6]$\Delta v_0 \ll \max\{\omega_0/k_1, \omega_2^0 k_2^0/k_1^2\}$.

wellian distribution with temperature T, $\omega_2^0 = 4\omega_0$, $n_1/n_0 = 10^{-2}$, $\sin^2 2\theta = 1/\pi$, $v_0 = 3 \times 10^9$ cm/sec, $\Delta v_0/v_0 = 10^{-1}$; for waves characterized by $k_1 = 10^{-9}\omega_0$ cm^{-1} the relation in (3.4) assumes the form

$$10^{-5} n_0^{-1} T^{3/2} e^{4\cdot 10^9/T} \langle (E^t)^2 \rangle > 1. \qquad (3.4')$$

The growth rate then becomes

$$\gamma_{\mathbf{k}_1}{}^l = \pi^3 \sin^2 2\theta \frac{n_1}{n_0} \left(\frac{\omega_0}{\omega_2^0}\right)^4 \frac{v_0}{\Delta v_0} \omega_0 \frac{\langle (E^t)^2 \rangle}{n_0 m_e}. \qquad (3.5)$$

In Eq. (3.4′) we take $n_0 = 10^{11}$ cm^{-3}, $T = 10^5$ and find that waves characterized by $k_1 = 17$ cm^{-1} are unstable even when $\sqrt{\langle (E^t)^2 \rangle} = 3 \times 10^{-5}$ cgs esu. The growth rate in this case is $\sim 10^3 \langle (E^t)^2 \rangle$, i.e., for a mean field intensity $E^t = 10$ cgs esu the growth time for the instability is very small, amounting to 10^{-5} sec.

4. BEAM INSTABILITY FOR TRANSVERSE AND LONGITUDINAL WAVES

We now consider further effects of the nonlinear interaction between longitudinal and transverse waves in the presence of beams in a plasma.

1. We first treat the excitation of transverse waves by a nonrelativistic beam in the simplest case, in which the longitudinal noise is essentially along the beam. Assuming that the spread in transverse beam velocity is small compared with the characteristic range in longitudinal beam velocities and that for the transverse noise $\mathbf{k}_1 \cdot \mathbf{v} \approx k_1 v > 0$, we find

$$\gamma_{\mathbf{k}_2}^t = -\frac{32 e^4 \omega_0^2 \pi^3}{m_e^2 k_2^3} \sin^2 2\theta \int \varphi(v)\, dv \left.\frac{dN^l(v_{\rm ph})}{dv_{\rm ph}}\right|_{v_{\rm ph} = \omega_0 v / |\mathbf{k}_2|};$$

$$\varphi(v) = \int f^{(e)}(\mathbf{p})\, d\mathbf{p}_\perp, \quad N^l(v_{\rm ph}) = \int N_{\mathbf{k}_1}{}^l\, d\mathbf{k}_{1\perp}|_{k_\parallel = \omega_0/v_{\rm ph}}, \qquad (4.1)$$

where $\varphi(v)$ is the one-dimensional particle distribution function in the beam, $N^l(v_{ph})$ is the one-dimensional distribution function for the longitudinal noise, θ is the angle between $\mathbf{k}_2$ and $\mathbf{v}$ while $\mathbf{p}_\perp$ and $\mathbf{k}_\perp$ are the components of the corresponding vectors perpendicular to the mean beam velocity.

In the most interesting case, in which $N^l(v_{ph})$ decreases with increasing v_{ph}, it is found that the transverse waves are unstable. Suppose that N^l is a weak function of v_{ph} and falls off rapidly for an amount Δv_{ph} when[7] $v \sim v_{ph_1}$. If the velocity spread in the beam is relatively small ($\Delta v \ll v_{ph}\omega_2/\omega_0$), the following frequencies are excited:

$$\omega_2 \approx \omega_0 v / v_{\rm ph_1} \qquad (\Delta\omega_2 \approx \omega_2 \Delta v_{\rm ph} / v_{\rm ph_1}) \qquad (4.2)$$

with a growth rate

$$\gamma_{\mathbf{k}_2} = 16\pi^4 \omega_0 \left(\frac{\omega_0}{\omega_2}\right)^3 \frac{n_1}{n_0} \frac{v_{\rm ph_1}}{\Delta v_{\rm ph_1}} \sin^2 2\theta \frac{W^l}{n_0 m_e}, \qquad (4.3)$$

where W^l is the energy of the longitudinal waves per cm^3. Assuming for example $n_0 = 10^{10}$ cm^{-3}, $\omega_0 = 5 \times 10^9$ sec^{-1}, $n_1/n_0 = 3 \times 10^{-2}$; $\Delta v_{ph_1}/v_{ph_1} = 1/10$, $v_{ph_1} = 10^{-1}$, $W^l = \frac{1}{10} n_1 m_e v_{ph_1}^2$; $k_2 \approx 1$ cm^{-1}, we find $\gamma_{k_2}^t \approx 10^6$ sec^{-1}.

We note that Eq. (4.1) applies even for monoenergetic beams if Δv_f is sufficiently large. This follows from

$$\gamma_{\mathbf{k}_2}{}^t \ll \max\{k_1 v \Delta v_{\rm ph} / v_{\rm ph};\ k_1, \Delta v\}. \qquad (4.4)$$

2. Now, using the example of excitation of transverse waves by a nonrelativistic beam we analyze the change in beam parameters in the development of the instability. The mean value of the quantity L is denoted by the symbol $\langle L \rangle = \int f_p L\, dp / \int f_p\, dp$; from Eq. (2.4) we find the mean change in the energy of the superthermal particles

$$\frac{d}{dt}\langle \varepsilon \rangle = \langle \Lambda \rangle; \qquad \Lambda = \Lambda|_{N^t = 0} + \frac{\partial \mathbf{j}}{\partial \mathbf{p}}, \qquad (4.5)$$

where

$$\mathbf{j} = \int \{(\omega_2 - \omega_0)(\mathbf{k}_2 - \mathbf{k}_1) W_{\mathbf{p}}(\mathbf{k}_1, \mathbf{k}_2) + (\omega_2 + \omega_0)(\mathbf{k}_2 + \mathbf{k}_1) \widetilde{W}_{\mathbf{p}}(\mathbf{k}_1, \mathbf{k}_2)\} N_{\mathbf{k}_1}^l N_{\mathbf{k}_2}{}^t\, d\mathbf{k}_1\, d\mathbf{k}_2. \qquad (4.6)$$

The sign of $d\langle \epsilon \rangle/dt$ is determined by the sign of $\partial j/\partial p$. For a beam of nonrelativistic particles and the one-dimensional longitudinal noise considered above we find that $\mathbf{j}$ has a single component along $\mathbf{v}$ and that

$$\mathbf{j} = \frac{4 e^4 \omega_0}{v m^2} \int \frac{d\mathbf{k}_2}{k_2} N_{k_2}{}^t \sin^2 2\theta_2 N^l\left(\frac{\omega_0}{k_2} v\right). \qquad (4.7)$$

The function $v^{-1} N^l(\omega_0 v/k_2)$ is a diminishing function in the region of interest and consequently $dj/dv < 0$. This means that the beam is retarded as a result of excitation of transverse waves.

3. We now consider the case in which a beam of relativistic particles with mean momentum $p = p_0 = m\epsilon_0$ ($\epsilon_0 \gg 1$) and dispersion Δp_0 propagates in a plasma. Assuming that the relativistic beam interacts most strongly with the transverse waves propagating along the beam we limit ourselves to waves characterized by $k_2 \parallel p_0$ and a well focused beam.[8] After some simple calculations we find

[7] This case can be realized if the longitudinal noise is formed in the beam itself while it is accelerated in the electric field if the longitudinal instability occurs when $v < v_{ph\,1}$ where the limit on v_{ph} is due either to the bounded system (cf.[10]) or to nonlinear effects.

[8] We assume that the angle θ is not too close to $\pi/2$ and also that $\theta_1 \ll 1/\epsilon_0$ for $k_\parallel \gtrsim \omega_0$ and $\theta_1 \ll \epsilon_0^{-3}$ for $k_\parallel \ll \omega_0$, where θ_1 is the angle between $\mathbf{k}_2$ and $\mathbf{p}$;

$$k_\perp v_\perp < \max\left\{\frac{\Delta\omega_2}{2\varepsilon^2}, \frac{\omega_2}{\varepsilon_0^2}\frac{\Delta\varepsilon_0}{\varepsilon_0}\right\}.$$

$$\Delta\gamma_{\mathbf{k}_1}^l = \gamma_{\mathbf{k}_1}^l - \gamma_{\mathbf{k}_1}^l\big|_{N^t=0} = \frac{16\pi^3 e^4\omega_0 \sin^2\theta}{m_e^2 (k_\parallel - \omega_0)^2}\int \frac{\partial\varphi}{\partial\varepsilon} d\varepsilon$$
$$\times \{N_\parallel^t (2\varepsilon^2 (k_\parallel - \omega_0)) - N_\parallel^t (-2\varepsilon^2 (k_\parallel - \omega_0))\}, \quad (4.8)$$

where

$$N_\parallel^t (k_\parallel) = \int N^t (\mathbf{k})\, d\mathbf{k}_\perp, \quad \varphi = \int f^{(e)} d\mathbf{p}_\perp, \quad \mathbf{k}_\perp = \mathbf{k} - \frac{\mathbf{p}_0 (\mathbf{k}\mathbf{p}_0)}{\mathbf{p}_0},$$
$$\mathbf{p}_\perp = \mathbf{p} - \frac{\mathbf{p}_0 (\mathbf{p}\mathbf{p}_0)}{\mathbf{p}_0^2}, \qquad k_\parallel = \frac{(\mathbf{k}\mathbf{p}_0)}{p_0},$$
$$p_\parallel = \frac{(\mathbf{p}\mathbf{p}_0)}{p_0}, \qquad \varepsilon = \frac{p_\parallel}{m}.$$

It is then evident that the quantity $\Delta\gamma^l$ (depending on the form of the spectrum N^t) can be either larger or smaller than zero. In other words for a sufficiently large "transverse" noise the nonlinear action can lead to additional excitation or damping of longitudinal waves with the appropriate values of $\mathbf{k}_1$.

As an example let us consider the effect of a "monochromatic" beam of waves on the longitudinal waves. Using in (4.8) the following approximation

$$N_{k_2}^t = \pi^2 (\omega_2^0)^{-1} \langle (E^t)^2 \rangle\, \delta (k_2 - k_2^0),$$

we find

$$\Delta\gamma^l = \frac{k_\parallel - \omega_0}{|k_\parallel - \omega_0|} 2\pi^3\omega_0 \left(\frac{\omega_0}{\omega_2^0}\right)^4 \sin^2\theta$$
$$\times \frac{\langle (E^t)^2 \rangle}{n_0^2} \varepsilon^5 \frac{\partial\varphi}{\partial\varepsilon}\bigg|_{\varepsilon=\sqrt{\omega_2^0/2(k_\parallel - \omega_0)}}, \quad (4.9)$$

i.e., $\Delta\gamma^l > 0$ for waves characterized by $k_\parallel > \omega_0 + \omega_2^0/2\epsilon_0^2$ and waves[9] characterized by $\omega_0 - \omega_2^0/2\epsilon_0^2 < k_\parallel < \omega_0$.

It is interesting to note that waves propagating against the beam are also unstable when $\omega_2^0 > 2\epsilon_0^2\omega_0$. To make an estimate we assume that $\partial\varphi/\partial\epsilon \sim m_e n_1/(\Delta p_0)^2$, a beam density $n_1 = 10^7$ cm^{-3}, a plasma $n_0 = 10^{15}$ cm^{-3}, $\Delta p_0/p_0 = 10^{-3}$, $\epsilon_0 = 10^2$, $E^t = 10^5$ cgs esu, $\omega_2^0 = 2 \times 10^{15}$ sec^{-1}, and find that the growth rate for plasma waves characterized by $k_\parallel = (\omega_0 \pm 10^{11}$ sec$^{-1})/3 \times 10^{10}$ cm^{-1} is $10^7 - 5 \times 10^6$ sec^{-1}, which is very large.

We note that the expression obtained for $\Delta\gamma^l$, Eq. (4.8), is valid when $\Delta\gamma^l$ is not too large, specifically, when

$$\Delta\gamma^l \ll \max\left\{\frac{\Delta\omega_2}{2\varepsilon_0^2};\ \frac{\omega_2}{\varepsilon_0^2}\frac{\Delta p_0}{p_0}\right\},$$

where $\Delta\omega_2$ is the characteristic spread and frequencies of the transverse waves.

[9]We recall that the formulas obtained here apply only when $|k_\parallel| \gg \omega_0/2\varepsilon_0^2$, $\omega_0 \ll k_1 \ll k_2\sqrt{2}\varepsilon_0$.

An analysis similar to the one given above shows that the development of the instability is accompanied by spreading of the beam. In this case the energy going into excitation of waves characterized by $k_\parallel > \omega_0$ is taken from particles characterized by $p < p_0$. These same particles simultaneously excite transverse waves at frequencies close to ω_2^0. However, particles characterized by $p > p_0$ are accelerated by virtue of the transverse waves and frequently lose energy in the excitation of longitudinal waves characterized by $k_\parallel < \omega_0$.

4. We consider finally transverse waves in a plasma in the presence of a relativistic beam. Using the same assumptions as in the preceding section we find

$$\gamma_{\mathbf{k}_2}^t = \frac{32\pi^3 e^4\omega_0}{m_e^2 k_2^2}\int \varepsilon^2 \frac{\partial\varphi}{\partial\varepsilon} d\varepsilon$$
$$\times \left[N_\parallel^l \left(\frac{k_2}{2\varepsilon^2} + \omega_0\right) + N_\parallel^l \left(\omega_0 - \frac{k_2}{2\varepsilon^2}\right)\right], \quad (4.10)$$

where, as before, $\varphi(\epsilon)$ is the one-dimensional beam distribution function while

$$N_\parallel^l (k_\parallel) = \int N_k^l \frac{k_\perp^2}{k_\parallel^2 + k_\perp^2} dk_\perp.$$

In the general case the value of γ^t depends on the form of the spectrum $N_\parallel^l$. However, if the function $N_\parallel^l (\omega_0 \pm k_2/2\epsilon^2)$ varies slowly in the range $\epsilon_0 \pm \Delta\epsilon_0$ the expression for $\gamma_{\mathbf{k}_2}^t$ assumes the simple form

$$\gamma_{\mathbf{k}_2}^t = -\frac{32\pi^3 e^4\omega_0}{m^3 k_2^2} n_1 \frac{\partial}{\partial\varepsilon}\varepsilon^2$$
$$\times \left[N_\parallel^l \left(\omega_0 + \frac{k_2}{2\varepsilon^2}\right) + N_\parallel^l \left(\omega_0 - \frac{k_2}{2\varepsilon^2}\right)\right]\bigg|_{\varepsilon=\varepsilon_0}, \quad (4.11)$$

and the instability condition obviously leads to the requirement

$$\frac{\partial}{\partial\varepsilon}\varepsilon^2\left[N_\parallel^l \left(\omega_0 + \frac{k_2}{2\varepsilon^2}\right) + N_\parallel^l \left(\omega_0 - \frac{k_2}{2\varepsilon^2}\right)\right]\bigg|_{\varepsilon=\varepsilon_0} < 0. \quad (4.12)$$

If $N_\parallel^l$ is large in the region $k_\parallel \lesssim \omega_0/v_T$ and if (4.12) is satisfied then waves are excited with frequencies $\omega_2 \sim \omega_0\epsilon_0^2/v_T$ ($v_T \ll 1$), which can lie in the optical region even for plasmas of relatively low density.

In principle, the transverse instability arising as a result of the nonlinear interaction of waves with a beam opens the possibility of using such systems for amplification and generation of millimeter waves.

To make an estimate we assume that the "longitudinal" noise exists only in a range from $k_\parallel = \omega_0/v_0$ to $k_\parallel + \Delta k_\parallel$ where $\Delta k_\parallel = k_\parallel \Delta v_0/v_0$ in which $\Delta\epsilon_0/\epsilon_0 \ll \Delta v_0/v_0 \ll 1$; from Eq. (4.10) we find that waves are excited at frequencies $\omega_2 = 2\epsilon_0^2\omega_0/v_0$ with characteristic growth rate

$$\gamma_{k_2}{}^t = \omega_0 \frac{\pi^3 v_0{}^3}{\varepsilon_0{}^3}\left(\frac{v_0}{\Delta v_0}\right)^2 \frac{n_1}{n_0} \frac{\langle (E^l)^2\rangle}{mn_0}, \qquad (4.13)$$

which for $n_0 = 10^{10}\ \text{cm}^{-3}$, $n_1 = 10^8\ \text{cm}^{-3}$, $\Delta v/v = 10^{-1}$, $v_0 = 0.5$, $E^l = 10^2$ cgs esu, $\epsilon_0 = 10$ yields

$$\omega_2 = 2\cdot 10^{12}\ \text{sec}^{-1}, \quad \gamma_{k_2}^t = 2\cdot 10^7\ \text{sec}^{-1}.$$

This result applies when

$$\gamma_{k_2}{}^t \ll \max\left\{\frac{\Delta\varepsilon_0}{\varepsilon_0}\omega_2,\ k_\parallel \frac{\Delta v_0}{v_0}\right\}.$$

The interaction of transverse and longitudinal plasma waves has been investigated above. As we have noted, the relative weakness of these effects is due to the fact that the longitudinal and transverse waves interact only via superthermal particles, which are generally present in small numbers. This is due to the fact that the phase velocity of the longitudinal wave exceeds the thermal velocity. However, other wave modes can propagate in a magnetized plasma; in particular, waves whose phase velocity is smaller than the thermal velocity (for example, waves propagating across the magnetic field with $\omega_1 = n\omega_H$). Such waves must interact with the transverse waves for which $\omega_2 \gg \omega_H$ through particles with velocities less than or of the order of the thermal velocity, and the number of such particles is large. In turn this effect can lead to an appreciable intensification of the interaction of the transverse waves with the plasma.

In conclusion we note that nonlinear effects analogous to those considered here can also occur in solid-state plasmas; these are of interest since the values of ω_0 (and consequently the values of ω_2) for which growing waves are possible are very large.

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