REFLECTION OF "ZERO SOUND" FROM A RIGID WALL

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The reflection of "zero sound" from an infinite rigid wall is studied on the basis of the Landau theory of a Fermi liquid. The reflection of the quasiparticles from the wall is considered to be diffuse and the wall immovable. It is shown that there is a dissipation of the energy at the wall upon reflection of zero sound; this dissipation reaches a considerable magnitude at certain angles of incidence; the dependence of the reflection coefficient on the angle of incidence is obtained.

ACCORDING to the Landau theory,[1] the so-called "zero sound" can also be propagated in a Fermi liquid, in addition to ordinary hydrodynamic sound. While ordinary hydrodynamic sound is associated in its propagation with real collisions between the excitations, and is thus realizable under the condition of the smallness of the free path length in comparison with the wavelength, i.e., under the condition $\omega\tau \ll 1$ (ω is the frequency of the sound, τ is the time between collisions), the propagation of the zero sound is due to the self-consistent interaction of the excitations and is realizable in the other limiting case $\omega\tau \gg 1$. Inasmuch as $\tau \sim T^{-2}$ in the Fermi liquid, the zero-sound oscillations guarantee the propagation of the sound through the liquid at $T = 0$. The problem of the propagation of zero sound in an unbounded medium, in particular, the problem of the damping with distance, has been treated theoretically in connection with the dissipation of acoustic energy. The calculation of the absorption coefficient of zero sound was carried out both by means of the kinetic equation [2] (through an imaginary contribution to the sound velocity) and also from the viewpoint of the theory of a Fermi liquid [3] (through an imaginary part of the pole of the scattering amplitude).

In the propagation of sound close to the rigid wall, the presence of viscosity and thermal conductivity can lead to an additional dissipation of the energy in connection with the generation of large temperature gradients and velocity gradients in the layer close to the wall.

In the present research, the problem of zero-sound propagation in a semi-infinite medium, i.e., of the reflection of zero sound from an infinite, immovable wall, is considered for the elementary excitations in a Fermi liquid by means of the kinetic equation. The dissipative processes, which are connected in the kinetic consideration with the redistribution of the momenta of the quasiparticles at the walls, also determine the reflection coefficient of the zero sound from the rigid wall. The dependence of the reflection coefficient on the angle of incidence is investigated; this is shown to be far from trivial.

1. The distribution function $n(\mathbf{p}, \mathbf{r}, t)$ of the elementary excitations in a Fermi liquid satisfies the kinetic equation

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\frac{\partial \varepsilon}{\partial \mathbf{r}} = I(n), \tag{1}$$

where ϵ is the energy of the excitation and $I(n)$ is the collision integral.

We seek the distribution function in the form

$$n = n_0 + \delta n, \tag{2}$$

where n is the equilibrium Fermi function for $T = 0$. The energy of the excitation, which is a functional of n, is expressed by the formula

$$\varepsilon = \varepsilon_0(\mathbf{p}) + \int f(\mathbf{p}, \mathbf{p}')\,\delta n(\mathbf{p}')\,d\mathbf{p}', \tag{3}$$

where $\epsilon_0(\mathbf{p})$ is the equilibrium excitation energy for the temperature of absolute zero, while $f(\mathbf{p}, \mathbf{p}')$ characterizes the interaction between the excitations and is determined as the second variational derivative of the energy density with respect to δn.

Inasmuch as the case $\omega\tau \gg 1$ is considered, then it would have been possible to neglect the collision integral. However, this would have led to a singular integral equation or to the Riemann boundary problem. Therefore, we keep the collision integral in its simplest form, $I = -\delta n/\tau$, just as did Bekarevich and Khalatnikov, [4] and eliminate τ in the final expressions in the limiting transition $\omega\tau \to \infty$.

By linearizing the kinetic equation (1), we note that $\delta n \sim \partial n_0/\partial\epsilon \sim \delta(p - p_0)$, where p_0 is the limiting momentum. Therefore, we can write δn in the form

$$\delta n = \frac{\partial n_0}{\partial \varepsilon} \nu(\mathbf{r}, \theta, \chi) e^{-i\omega t}, \tag{4}$$

where θ and χ are the spherical coordinates of the vector **p**: θ is the angle between the direction of **p** and the Z axis, which is perpendicular to the wall and is directed into the liquid; χ is the azimuthal angle in the plane of the wall (i.e., in the XY plane), measured from the X axis.

For simplicity, we limit ourselves to the first term in the expansion of the function $f(\mathbf{p}, \mathbf{p}')_{p_0}$ in terms of spherical harmonics, i.e., we consider it to be constant $[f(\mathbf{p}, \mathbf{p}')_{p_0} p_0 m_{\mathrm{eff}}/\pi^2\hbar^3 \equiv F_0]$. Then, with account of (3) and (4), we get

$$\varepsilon = \varepsilon_0 - F_0 \int \nu(\theta', \chi', \mathbf{r}) \frac{d\Omega'}{4\pi} e^{-i\omega t}. \tag{5}$$

In the reflection problem, the small, nonequilibrium contribution δn can be represented in the form of the sum

$$\delta n = \delta n^{\mathrm{inc}} + \delta n^{\mathrm{ref}}, \tag{6}$$

where δn^{inc} is the contribution which describes the zero sound incident on the wall, and which has the form [2]

$$\delta n^{\mathrm{inc}} = \frac{\partial n_0}{\partial \varepsilon} \frac{\cos(\mathbf{p}\hat{\mathbf{k}}^{\mathrm{inc}})}{s - \cos(\mathbf{p}\hat{\mathbf{k}}^{\mathrm{inc}})} \exp\{i(\mathbf{k}^{\mathrm{inc}}\mathbf{r} - \omega t)\}, \tag{7}$$

s is the ratio of the propagation velocity of the wave u to the velocity of the excitations on the Fermi surface, v_0; $\mathbf{k}^{\mathrm{inc}}$ is the wave vector of the incident wave:

$$k^{\mathrm{inc}} = is_0 / v_0\tau,$$

where $s_0 = (1 - i\omega\tau)/s \equiv x_0(1 - i\omega\tau)$. The imaginary part of k gives the damping brought about by the collisions.

Substituting the relations (5)—(7) and δn^{ref} in the form

$$\delta n^{\mathrm{ref}} = \frac{\partial n_0}{\partial \varepsilon} \nu^{\mathrm{ref}}(\mathbf{r}, \theta, \chi) e^{-i\omega t}$$

in the linearized Eq. (1), we get the equation relative to ν^{ref}:

$$-i\omega\nu^{\mathrm{ref}} + \mathbf{v}_0 \frac{\partial \nu^{\mathrm{ref}}}{\partial \mathbf{r}} + F_0 \mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}} \int \nu^{\mathrm{ref}} \frac{d\Omega}{4\pi} = -\frac{\nu^{\mathrm{ref}}}{\tau}. \tag{8}$$

Furthermore, starting out from the axial symmetry of the system liquid-wall, we can assume that $\mathbf{k}^{\mathrm{inc}}$ lies in the XZ plane, without limitation of the generality of the system. Then ν^{ref} is also independent of y. Introducing the new function

$$\Psi^{\mathrm{ref}} = \nu^{\mathrm{ref}}(\mathbf{r}, \theta, \chi) + F_0 \int \nu^{\mathrm{ref}}(\mathbf{r}, \theta', \chi') \frac{d\Omega'}{4\pi}, \tag{9}$$

we get the following equation from (8)

$$(1 - i\omega\tau)\left[\Psi^{\mathrm{ref}}(x, z, \theta, \chi) - \frac{F_0}{1 + F_0}\Psi_0^{\mathrm{ref}}(x, z)\right] + \sin\theta\cos\chi\frac{\partial\Psi^{\mathrm{ref}}}{\partial x} + \cos\theta\frac{\partial\Psi^{\mathrm{ref}}}{\partial z} = 0. \tag{10}$$

Here

$$\Psi_0^{\mathrm{ref}}(x, z) \equiv \int \Psi^{\mathrm{ref}}(x, z, \theta, \chi)\frac{d\Omega}{4\pi}, \tag{11}$$

z is measured in lengths of the mean free path $l = \nu_0\tau$.

Transforming to the analogous notation in (5), and omitting the periodic time dependence, we get

$$\varepsilon = \varepsilon_0 - \frac{F_0}{1 + F_0}\Psi_0 \tag{12}$$

(here Ψ, Ψ_0, and in what follows Ψ^{inc}, are determined by Eq. (9) with the corresponding substitution $\nu^{\mathrm{ref}} \to \nu$, ν^{inc}.

2. We now proceed to the derivation of the boundary conditions for Eq. (10).

The greatest interest lies in the case of diffuse reflection of the quasiparticles from the wall, which is very close to the real conditions in liquid He^3. We note that there is no dissipation of the energy at the wall in specular reflection of quasiparticles from a fixed wall, and the coefficient of reflection of zero sound is equal to unity, independent of the angle of incidence.

In diffuse reflection, particles reflected from the wall are distributed over the equilibrium Fermi function n_0. Thus, the following condition is satisfied on the wall:

$$n(z = 0, p_z > 0) = n_0(\varepsilon_0 + \delta\varepsilon - \delta\mu) = n_0(\varepsilon_0) + \frac{\partial n_0}{\partial \varepsilon_0}(\delta\varepsilon - \delta\mu);$$

the change in the chemical potential $\delta\mu$ is also included in the argument of the distribution function, guaranteeing the vanishing of mass flow across the plane z = 0 (the wall is considered immovable).

On the other hand,

$$n = n_0 + \frac{\partial n_0}{\partial \varepsilon_0}\nu(\mathbf{r}, \theta, \chi).$$

Comparing the last two expressions, with account of (9) and (12), we get

$$\Psi = -\delta\mu, \qquad z = 0, \qquad p_z > 0; \tag{13}$$

$\delta\mu$ is found from the condition

$$m\int n(\partial\varepsilon/\partial\mathbf{p})_z d\mathbf{p} = 0, \qquad z = 0,$$

which is equivalent to the condition

$$\int \cos\theta \Psi' \, d\Omega = 0, \qquad z = 0. \tag{14}$$

The relations (13) and (14) would obviously be satisfied for the total function $\Psi = \Psi^{inc} + \Psi^{ref}$. Inasmuch as, in accord with (7) and (9),

$$\Psi^{inc} = (s + a\cos\theta - \sqrt{1-a^2}\sin\theta\cos\chi)^{-1} \times \exp\{-xs_0\sqrt{1-a^2} + zas_0\}, \tag{15}$$

the corresponding conditions for Ψ^{ref} have the form

$$\Psi^{ref}(x,\chi,z=0,\cos\theta>0) = -\delta\mu(x) - \frac{\exp\{-xs_0\sqrt{1-a^2}\}}{s + a\cos\theta - \sqrt{1-a^2}\sin\theta\cos\chi};$$

$$\int \cos\theta\Psi^{ref}\, d\Omega = -\exp\{-s_0x\sqrt{1-a^2}\} \times \int \frac{\cos\theta\, d\Omega}{s + a\cos\theta - \sqrt{1-a^2}\sin\theta\cos\chi}, \quad z = 0.$$

Here $a = \cos\vartheta > 0$, ϑ is the acute angle between $\mathbf{k}^{inc}$ and the normal to the surface of the wall, i.e., the angle of incidence (Fig. 1).

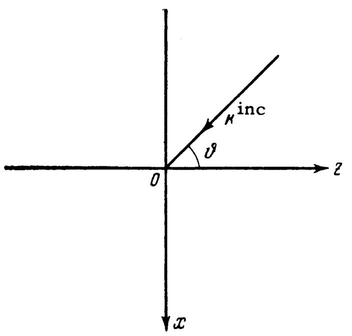

FIG. 1

Starting out from the boundary conditions and the content of the problem, we can represent Ψ^{ref} in the form

$$\Psi^{ref}(z, x, \theta, \chi) = \exp\{-s_0x\sqrt{1-a^2}\}\Phi(z, \theta, \chi).$$

Thus the problem reduces to the solution of the equation

$$\mu\, \partial\Phi(z,\mu,\chi)/\partial z + \Phi(z,\mu,\chi)\{1 - i\omega\tau - \sqrt{1-\mu^2}\cos\chi s_0\sqrt{1-a^2}\} - \frac{F_0}{1+F_0}(1-i\omega\tau)\Phi_0(z) = 0 \tag{16}$$

with the boundary conditions

$$\Phi(z=0,\mu>0,\chi) = \xi - (s + a\mu - \sqrt{1-a^2}\sqrt{1-\mu^2}\cos\chi)^{-1}, \tag{17}$$

$$\int \mu\Phi(z=0,\mu,\chi)\, d\Omega = -\int \frac{\mu\, d\Omega}{s + a\mu - \sqrt{1-a^2}\sqrt{1-\mu^2}\cos\chi} \tag{18}$$

and $\Phi(z\to\infty) \to 0$. Here the following notation is used:

$$\mu = \cos\theta; \qquad \Phi_0(z) = \int \Phi(z,\mu,\chi)\frac{d\Omega}{4\pi},$$

$$\delta\mu(x) = -\xi\exp\{-xs_0\sqrt{1-a^2}\}, \qquad \xi = \text{const}.$$

3. Multiplying (16) by e^{-uz} and integrating over z, we get an equation for the Laplace transformations

$$\varphi(u,\mu,\chi)\{1 - i\omega\tau - \sqrt{1-\mu^2}\cos\chi s_0\sqrt{1-a^2} + \mu u\} - \mu\Phi(z=0,\mu,\chi) = \frac{F_0}{1+F_0}(1-i\omega\tau)\varphi_0(u)$$

$$\left(\varphi(u,\mu,\chi) = \int_0^\infty \Phi(z,\mu,\chi)e^{-uz}\,dz\right), \tag{16'}$$

whence, after division by the coefficient for $\varphi(u, \mu, \chi)$ and integration over the angles, we find the equation for $\varphi_0(u)$:

$$\varphi_0(u)\Delta(u) = \frac{1}{4\pi}\int \frac{\mu\Phi(z=0,\mu,\chi)\,d\Omega}{1 - i\omega\tau - \sqrt{1-\mu^2}\cos\chi s_0\sqrt{1-a^2} + \mu u}; \tag{19}$$

$$\Delta(u) = 1 - \frac{F_0}{1+F_0}\frac{i(1-i\omega\tau)}{2\sqrt{c}}\ln\frac{1-i\omega\tau - i\sqrt{c}}{1-i\omega\tau+i\sqrt{c}} = \frac{F_0}{1+F_0}\left\{\frac{1}{F_0} - W\left[\frac{i(1-i\omega\tau)}{\sqrt{c}}\right]\right\},$$

$$W(s) = \frac{s}{2}\ln\frac{s+1}{s-1} - 1, \qquad c = s_0^2(a^2-1) - u^2. \tag{20}$$

Solution of the integral equation (19) is given in the Appendix I because of the cumbersome calculations. In the half-plane Re u > 0, the desired representation of $\varphi_0(u)$ is shown to be equal to

$$\varphi_0(u) = \frac{1+F_0}{F_0}\left\{\frac{\xi u}{u^2 + (1-a^2)s_0^2} - \frac{x_0}{u - as_0} + \frac{g_-(u)[u + (1-i\omega\tau)\sqrt{1-x_0^2(1-a^2)}]}{u + as_0}\left[\frac{\gamma}{u - as_0} - \frac{\xi}{2}\frac{u(\beta+\alpha) + i\sqrt{1-a^2}s_0(\alpha-\beta)}{2[u^2 + s_0^2(1-a^2)]}\right]\right\}, \tag{21}$$

where β, γ, α depend on α known coefficients [see (I.2)—(I.4)] and hence the following notation is used:

$$g_\pm(u) = \exp\left\{\frac{1}{2\pi i}\int_{\pm b - i\infty}^{\pm b + i\infty}\frac{\ln G(s)}{s-u}\,ds\right\},$$

$$0 < b < \sqrt{1 - x_0^2(1-a^2)};$$

$$G(u) = \Delta(u)\frac{u^2-(1-i\omega\tau)^2[1-x_0^2(1-a^2)]}{u^2-a^2s_0^2}.$$

The value of ξ entering into Eq. (21) for $\varphi_0(u)$ is determined by the relation (18), which, as the result of complicated mathematical transformations (see Appendix II) gives

$$\xi = \frac{4ax_0[a(\alpha+\beta)-i\sqrt{1-a^2}(\alpha-\beta)](1+F_0)}{g_-(as_0)[ax_0+\sqrt{1-x_0^2(1-a^2)}]}$$
$$\times\left\{\frac{(\alpha^2-\beta^2)(1+F_0)}{2ix_0\sqrt{1-a^2}}\right.$$
$$-\frac{2}{3}(3+F_0)\sqrt{1-x_0^2(1-a^2)}+\frac{2a}{x_0}+F_0$$
$$-\frac{2}{3}F_0\frac{1-\sqrt{1-x_0^2(1-a^2)}}{(1-a^2)x_0^2}$$
$$\left.+\frac{2}{\pi}\int_1^\infty\frac{y^2-2x_0^2(1-a^2)}{y^3\sqrt{y^2-x_0^2(1-a^2)}}\operatorname{arc\,tg} S\,dy\right\}^{-1}, \qquad (22)^*$$

where

$$S=\pi F_0/2y\left[1+F_0\left(1-\frac{1}{2y}\ln\frac{y+1}{y-1}\right)\right].$$

4. We now transform from the Laplace transform to the original function

$$\Phi_0(z)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\varphi_0(u)e^{uz}\,du.$$

For calculation of the integral, we take $\epsilon = 0$ [this is permitted by the analytical properties of $\varphi_0(u)$] and the closed contour of integration of the arc of radius $R \to \infty$ in the left half-plane (Fig. 2). In accord with (21), the integrand has a pole at the point $u = -as_0$; moreover, if we represent $g_-(u)$ in the form (II.2), then the integrand will have an ambiguity in the left half-plane, for the removal of which it is necessary to bypass the cut along the curve

$$u=-x(1-i\omega\tau)\sqrt{1-x_0^2(1-a^2)},\quad x\geqslant 1 \qquad (M'N'P').$$

The pole at the point $u = -as_0$ corresponds to reflection of the zero sound; the reflection amplitude is proportional to the residue of the integrand at this point. The integral over the cut corresponds to the energy dissipation near the boundary separating the liquid and the solid.

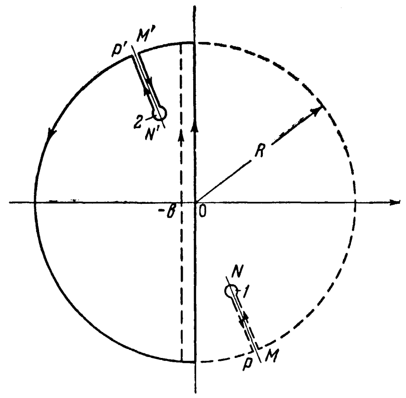

FIG. 2. The integration contour for computation of the integrals $\Phi_0(z)$ and I'' is shown by the solid curve. The contour integration along which leads to the expression for $g_-(u)$ for Re u = 0 to the form (II.2) is indicated by the dashed curve; 1, 2 are the branch points of the function G(u): $u = \pm(1-i\omega\tau)\sqrt{1-x_0^2(1-a^2)}$.

To find the reflection coefficient, we need to know the residue of the function $\varphi_0(u)$ at the point $u = -as_0$ (since, on the wall, $z = 0$):

$$\operatorname{res}\varphi_0(u=-as_0)=\frac{1+F_0}{2F_0x_0}g_-(-as_0)[-ax_0$$
$$+\sqrt{1-x_0^2(1-a^2)}]\{\xi[a(\beta+\alpha)$$
$$+i\sqrt{1-a^2}(\beta-\alpha)]-\gamma/a\}.$$

The function φ_0 considered by us corresponds to δn^{ref}, i.e., this is φ_0^{ref}. If we introduce φ_0^{inc} in similar fashion, which is in turn proportional to the amplitude of the incident wave, then, in accord with (15), we get

$$\operatorname{res}\varphi_0^{\text{inc}}(u=as_0)=x_0(1+F_0)/F_0$$

for all angles of incidence.

Thus the reflection coefficient R of zero sound from a rigid, immovable wall is equal to the ratio of the time average of the energy flux in the reflected wave to the mean energy flux in the incident wave, and is expressed by the formula

$$R=\frac{[\operatorname{res}\varphi_0^{\text{ref}}(u=-as_0)]^2}{[\operatorname{res}\varphi_0^{\text{inc}}(u=as_0)]^2}$$
$$=\frac{g_-^2(-as_0)}{4x_0^4}[-ax_0+\sqrt{1-x_0^2(1-a^2)}]^2$$
$$\times\{\xi[a(\beta+\alpha)+i\sqrt{1-a^2}(\beta-\alpha)]-\gamma/a\}^2.$$

For analysis and calculation, it is convenient to use a formula which is obtained from the previous after transformation of the integrals in the complex plane [$g_-(-as_0)$, ξ, α, β, γ] to integrals over the real variable and transition to the limit $\omega\tau \to \infty$:

$$R=P^2, P=\frac{\operatorname{res}\varphi_0^{\text{ref}}(u=-as_0)}{\operatorname{res}\varphi_0^{\text{inc}}(u=as_0)}=\exp\{2ax_0I_1\}\frac{r-ax_0}{r+ax_0}$$
$$\times\left\{8a\cos^2(\delta-\vartheta)\Big/\left[\frac{\sin 2\delta}{\sqrt{1-a^2}}\right.\right.$$
$$-\frac{2}{3}(3+F_0)x_0r+2a+F_0x_0$$

*tg = tan.

$$-\frac{2}{3}F_0\frac{1-r}{(1-a^2)x_0}+2x_0I_2-4x_0^3(1-a^2)I_3\Big]-1\Big\};\qquad(23)$$

here, we use the notation

$$I_1=\frac{1}{\pi}\int_1^\infty\frac{y\operatorname{arc\,tg}S\,dy}{(y^2-x_0^2)\sqrt{y^2-x_0^2(1-a^2)}};$$

$$I_2=\frac{1}{\pi}\int_1^\infty\frac{\operatorname{arc\,tg}S\,dy}{y\sqrt{y^2-x_0^2(1-a^2)}}$$

$$I_3=\frac{1}{\pi}\int_1^\infty\frac{\operatorname{arc\,tg}S\,dy}{y^3\sqrt{y^2-x_0^2(1-a^2)}};$$

$$r=\sqrt{1-x_0^2(1-a^2)},$$

$$\delta=\sqrt{1-a^2}\,x_0I_2-\arcsin(\sqrt{1-a^2}\,x_0)+\vartheta.$$

We now analyze the dependence of P on the angle of incidence. In accord with Eq. (23), for a = 1, i.e., for normal incidence,

$$P=\exp\{2x_0I_1(a=1)\}\frac{1-x_0}{1+x_0}$$
$$\times\{2/[1-x_0+x_0I_2(a=1)]-1\}.\qquad(24)$$

For the existence of zero sound, it is necessary that the wave propagation velocity u be larger than the velocity of the excitations on the Fermi surface v_0;[2] therefore, x_0, which is equal to the ratio of v_0 to u, is a positive quantity, always smaller than unity. Furthermore, transforming to the new variable $x = 1/y$ in the integral I_2 ($a = 1$), and taking it into account that the arctan in front of the integral changes in the range $(0, \eta)$, we get

$$I_2(a=1)$$
$$=\frac{1}{\pi}\int_0^1\operatorname{arctg}\left\{\pi F_0x\Big/-2\left[1+F_0\left(1+\frac{x}{2}\ln\frac{1-x}{1+x}\right)\right]\right\}dx$$
$$<\frac{1}{\pi}\int_0^1\pi\,dx=1.$$

On the basis of the inequalities

$$0<x_0,\qquad I_2(a=1)<1$$

and Eq. (24), we conclude that $P > 0$ for normal incidence ($\vartheta = 0$). For grazing incidence, ($\vartheta = \pi/2$), i.e., for $a = 0$, $P = -1 < 0$, in accord with Eq. (23). Moreover, the expression on the right side of Eq. (23) is real and continuous. Consequently, in the change of the angle of incidence from 0 to $\pi/2$, an angle can be found for which $P = 0$.

On the basis of what has been said above, we can make the following general conclusions.

1. Independent of the value of F_0 (and the value of x_0 determined by it), i.e., in our consideration, independent of the nature of the Fermi liquid, the reflection coefficient of zero sound from a rigid, immovable wall reaches a maximum, equal to zero, for a certain value of the angle of incidence ϑ_0 (the actual value of ϑ_0, which is found from the equation $P = 0$, obviously depends on F_0).

2. Independent of the value of F_0, the reflection coefficient is equal to unity for grazing incidence ($\vartheta \to \pi/2$).

3. Inasmuch as the ratio of the amplitude of the reflected wave to the amplitude of the incidence wave P changes sign at the point $\vartheta = \vartheta_0$, the phase shift for reflection is zero on one side of the point ϑ_0, and is equal to π on the other side. Thus the minimum value of the reflection coefficient, which is equal to zero, is achieved at the point where we have the transition from reflection without loss of a half wave to reflection with loss of a half wave. This result is also general for all values of F_0.

For illustration of the laws described, we shall carry out the numerical calculation for the case of He^3.1) The parameters entering into the formula (23) are determined in the general way [2] from experimental data on the specific heat of He^3 at low temperatures (the ratio m_{eff}/m) and the velocity c of ordinary sound in He^3. Using the latest data obtained in the literature ($m_{eff} = 2.8\, m_{He^3}$,[6] $c = 183$ m/sec), we get the following values of the parameters: $F_0 \approx 9$, $x_0 \approx 0.53$.

Graphs of the dependence of the ratio of the amplitudes P (curve 1) and the reflection coefficient R (curve 2) on the angle of incidence ϑ are plotted in Fig. 3 for the given values of F_0 and x_0. For normal incidence, the reflection coefficient increases, then falls off monotonically and, reaching a minimum at almost grazing incidence, rapidly increases upon subsequent increase in the angle of incidence. For comparison here, we have also plotted curves (3, 4) computed under the assumption $m_{eff} = m_{He^3}$ (here, $F_0 = 2.95$, $x_0 = 0.78$); as we see, the path of the curves depends on the value of m_{eff}.

1)Actually, the function f is not a constant for He^3. We note, however, that the calculation of the reflection coefficient with account of the first two terms of the expansion of $f(\mathbf{p}, \mathbf{p}')_{p_0}$ in terms of spherical harmonics, which can be carried out completely in the case of normal incidence,[5] gives a result which is in excellent agreement with that obtained here with account of only the first term of the expansion.

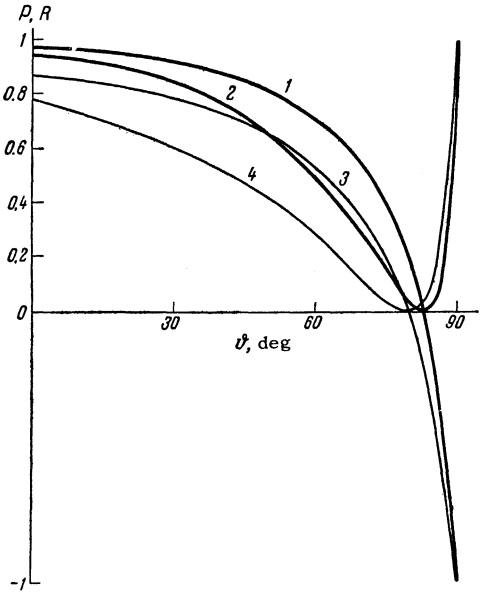

FIG. 3. Dependences of the ratio of the amplitude of the reflected wave to the amplitude of the incident wave P (curves 1, 3) and the reflection coefficient R (curves 2, 4) on the angle of incidence ϑ. The curves 1, 2 correspond to $m_{eff} = 0.8\, m_{He^3}$, curves 3, 4 are for $m_{eff} = m_{He^3}$.

In conclusion, the author expresses his gratitude to Prof. I. M. Khalatnikov for suggesting the problem and for valuable advice.

APPENDIX I

Before undertaking the solution of the integral equation (19), we transform it by using the value $\Phi(z = 0, \mu, \chi)$ for $\mu > 0$ from (17) to the form

$$\varphi_0(u)\Delta(u) - \frac{1}{F_0}[(1+F_0)\Delta(u) - 1]\left[\frac{1}{s(as_0 - u)} - \frac{\xi u}{c}\right] = Q(u) + \frac{1}{(as_0 - u)F_0 s}; \tag{I.1}$$

where

$$Q(u) = \frac{1}{4\pi}\int_0^{2\pi} d\chi \times \int_{-1}^{0}\left\{\frac{\Phi(z=0, \mu, \chi) - \xi}{1 - i\omega\tau - \sqrt{1-\mu^2}\cos\chi s_0\sqrt{1-a^2} + \mu u} + s_0(1 - i\omega\tau - \sqrt{1-\mu^2}\cos\chi s_0\sqrt{1-a^2} + \mu u)^{-1} \times (1 - i\omega\tau - \sqrt{1-\mu^2}\cos\chi s_0\sqrt{1-a^2} + \mu a s_0)^{-1}\right\}\mu\, d\mu.$$

$\Phi(z = 0, \mu, \chi)$ for $\mu < 0$ can be expressed in terms of $\varphi_0(u)$, where u will be some function of μ and χ; consequently, Eq. (I.1) is nothing else than the inhomogeneous integral equation relative to $\varphi_0(u)$. We shall solve it by the Wiener-Hopf method. [7]

The function $Q(u)$ is analytic for $\mathrm{Re}\, u < \sqrt{1 - x_0^2(1 - a^2)}$, $x_0 \equiv 1/s$. The function $\Delta(u)$ is analytic in the interval $|\mathrm{Re}\, u| < \sqrt{1 - x_0^2(1 - a^2)}$ and has two roots in that interval; [2] $u = \pm a s_0$. We introduce the function

$$G(u) = \Delta(u)\frac{u^2 - (1 - i\omega\tau)^2[1 - x_0^2(1 - a^2)]}{u^2 - a^2 s_0^2},$$

which is analytic in the interval $|\mathrm{Re}\, u| < \sqrt{1 - x_0^2(1 - a^2)}$ does not have any zeros in it and tends to unity as $u \to \infty$ in this interval.

Applying the Cauchy formula to $\ln G(u)$, we can write

$$G(u) = \frac{g_+(u)}{g_-(u)};\quad g_\pm(u) = \exp\left\{\frac{1}{2\pi i}\int_{\pm b - i\infty}^{\pm b + i\infty}\frac{\ln G(s)}{s - u}ds\right\},$$

$$0 < b < \sqrt{1 - x_0^2(1 - a^2)};$$

$g_+(u)$ is analytic for $\mathrm{Re}\, u < b$, $g_-(u)$ for $\mathrm{Re}\, u > -b$.

Expressing $\Delta(u)$ in Eq. (I.1) by $g_+(u)$ and $g_-(u)$ and multiplying both sides of the equation by $[\mu - (1 - i\omega\tau)\sqrt{1 - x_0^2(1 - a^2)}]$ $(a - i\sqrt{1 - a^2 s_0}) \times [g_+(u)(as_0)]^{-1}$, we get

$$\frac{1}{g_-(u)}\frac{u + as_0}{u + (1 - i\omega\tau)\sqrt{1 - x_0^2(1 - a^2)}} \times \left\{\varphi_0(u)(u - i\sqrt{1-a^2}\, s_0) - \frac{1+F_0}{F_0}\frac{\xi u}{u + i\sqrt{1-a^2}\, s_0} - \frac{1+F_0}{F_0}\frac{u - i\sqrt{1-a^2}\, s_0}{s(as_0 - u)}\right\}$$

$$= \frac{[u - (1 - i\omega\tau)\sqrt{1 - x_0^2(1 - a^2)}](u - i\sqrt{1-a^2}\, s_0)}{g_+(u)(u - as_0)} \times \left\{Q(u) - \frac{1}{F_0}\frac{\xi u}{u^2 + s_0^2(1 - a^2)}\right\}.$$

Further, by compensating the remaining poles through the addition to both parts of the equation of the expression

$$\frac{1+F_0}{F_0}\left[\frac{\xi\beta}{2}\frac{u - i\sqrt{1-a^2}\, s_0}{u + i\sqrt{1-a^2}\, s_0} - \gamma\frac{u - i\sqrt{1-a^2}\, s_0}{u - as_0}\right],$$

where

$$\beta = \frac{1}{1+F_0}\frac{i\sqrt{1-a^2}\, s_0 + (1 - i\omega\tau)\sqrt{1 - x_0^2(1 - a^2)}}{g_+(-i\sqrt{1-a^2}\, s_0)(i\sqrt{1-a^2} + a)s_0}, \tag{I.2}$$

$$\gamma = \frac{2as_0x_0}{g_-(as_0)[as_0 + (1 - i\omega\tau)\sqrt{1 - x_0^2(1 - a^2)}]}, \tag{I.3}$$

we get an equation whose left-hand side is analytic for $\mathrm{Re}\, u > 0$ and is bounded (if we assume that $\varphi_0(u) \sim 1/u$ as $u \to \infty$), while the right-hand side is analytic for $\mathrm{Re}\, u < \min\{b, \mathrm{Re}\, as_0\}$, and is bounded as $u \to \infty$. Since there is an overlap band $0 < \mathrm{Re}\, u < \min\{b, \mathrm{Re}\, as_0\}$ in which the two parts coincide, one is the analytic continuation of the other, and, in accord with the Liouville theorem, the total function is equal to a constant (we note that our analysis is suitable for any $a \neq 0$). It is easy to find this constant by taking the value of the left side at the point $u = i\sqrt{1-a^2}\, s_0$. Then, equating the left side of the equation to this constant, we get the solution of $\varphi_0(u)$ in the form (21), where

$$\alpha = \frac{1}{g_-(i\sqrt{1-a^2}\,s_0)} \times \frac{(i\sqrt{1-a^2}+a)s_0}{i\sqrt{1-a^2}\,s_0 + (1-i\omega\tau)\sqrt{1-x_0^2(1-a^2)}}. \quad (I.4)$$

It is easy to prove that $\varphi_0(u)$ is analytic in the right half plane and, actually, $\sim 1/u$ as $u \to \infty$.

APPENDIX II

The existence of the dependence on a greatly complicates the problem of finding ξ; therefore, we shall consider this specifically:

$$\int \mu\Phi(z=0,\mu,\chi)\,d\Omega = \int_0^{2\pi} d\chi \int_{-1}^{0} \mu\Phi(z=0,\mu,\chi)\,d\mu + \int_0^{2\pi} d\chi \int_0^1 \mu\Phi(z=0,\mu,\chi)\,d\mu.$$

The difficulty lies in the calculation of the first integral. We first transform from $\varphi_0(u)$ to the original function

$$\Phi_0(z) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \varphi_0(u)\,e^{uz}du. \quad (II.1)$$

Starting from the form $\varphi_0(u)$, we can choose $\epsilon = 0$. Further, direct integration of Eq. (16) gives

$$\Phi_{\mu<0}(z=0,\mu,\chi) = -\frac{F_0}{1+F_0}(1-i\omega\tau)\frac{1}{\mu}\int_0^\infty \Phi_0(z') \times \exp\left\{\frac{z'}{\mu}\left[1-i\omega\tau-\sqrt{1-\mu^2}\cos\chi s_0\sqrt{1-a^2}\right]\right\}dz'.$$

We substitute $\Phi_0(z')$ from (II.1) and integrate over z'. Then, using the obtained result, we find

$$I \equiv \int_0^{2\pi} d\chi \int_{-1}^{0} \mu\Phi(z=0,\mu,\chi)\,d\mu = I' - I'',$$

$$I' = \frac{1-i\omega\tau}{1+F_0}\int_{-i\infty}^{i\infty} \varphi_0(u)\frac{2iu}{u^2+s_0^2(1-a^2)}du,$$

$$I'' = (1-i\omega\tau)\int_{-i\infty}^{i\infty} \varphi_0(u)G(u) \times \frac{2iu(u^2-a^2s_0^2)\,du}{[u^2+s_0^2(1-a^2)]\{u^2-(1-i\omega\tau)^2[1-x_0^2(1-a^2)]\}}.$$

In the integral $I'\varphi_0(u)$ is analytic in the right half-plane and the entire integrand approaches zero as $1/u^2$ as $u \to \infty$; therefore,

$$I' = 2\pi\frac{1-i\omega\tau}{1+F_0}\varphi_0(is_0\sqrt{1-a^2}).$$

In integral I'', for convenience in the continuation of the integrand in the left-hand plane, we write down $g_-(u)$ for $\operatorname{Re} u = 0$ in the form (see Fig. 2)

$$g_-(u) = \frac{1}{G(u)}\exp\left\{-\frac{1}{2\pi i}\int_{(MNP)} \frac{\ln G(s)}{s-u}ds\right\}. \quad (II.2)$$

Here the integral is taken over the sides of the cut, removing the ambiguity of the function G in the right half-plane.

We find the value of I'' by closing the contour of integration in the left half plane and again bypassing the cut, now over M′N′P′, to remove the ambiguity of $G(u)$ in the left half-plane.

As a result of a cumbersome computation of a similar type, and transformation of the integrals in the complex plane to integrals over the real variable, we find the relation for ξ by means of substitution of the computed values of I' and I'' in I, and then I in (18).

The final result is given in the text, Eq. (22).

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