GEOMETRICAL OPTICS OF ELEMENTARY PARTICLES

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It is shown that the nonlocal potential which is obtained from quantum field theory can be replaced at large wave numbers by a local complex index of refraction.

1. INTRODUCTION

WE start with the equation for the single time wave function for two particles:

$$L\psi(x) = \int V(x, x', W)\psi(x')\,d^3x'. \qquad (1)$$

Here $x = x_1 - x_2$ is the relative coordinate of the two particles at the same time ($t_1 = t_2$); W is the energy of the stationary state; $V(x, x', W)$ is the nonlocal potential; the operator L can either be the Klein operator: $\epsilon^2/c^2 - K^2 = \nabla^2 + k^2$ (ϵ is the meson energy, μ is its mass, $K^2 = -\nabla^2 + \mu^2$), or the Dirac operator: $L = E/c - D(\nabla)$ (E is the nucleon energy $D = i\alpha\nabla + \beta mc^2$); we note that $W = E + \epsilon$. The equation given above was derived from the single time equations constructed with the aid of the ''elementary scattering matrix'' (cf., [1-3]). Recently the same equation was obtained in the momentum representation in [4]. Below we shall consider two limiting cases: the long wavelength case and the short wavelength case.

2. THE LONG WAVELENGTH CASE

Equation (1) can be rewritten in the form

$$L\psi(x) = U(x, W)\psi(x), \qquad (2)$$

where $U(x, W)$ is the local potential which, however, depends on the form of the wave function:

$$U(x, W) = \int \frac{V(x, x', W)\psi(x')}{\psi(x)}\,d^3x'. \qquad (3)$$

If the wave length $\lambdabar$ is much greater than the dimensions of the region a in which the nonlocal potential differs from zero (i.e., it is assumed that for $|x|, |x'| > aV(x, x', W) \sim 0$), then in this region the wave function $\psi(x)$ practically does not vary. Then within a we can set: $\psi(x')/\psi(x) \cong 1$ and, consequently, in this case the local potential

$$U(x, W) = \int V(x, x', W)\,d^3x' \qquad (4)$$

is simply the nonlocal potential $V(x, x', W)$ averaged over the x' space.

It might seem that in the case of shorter waves one could construct the local potential by means of the operation

$$U_n(x, W) = \int \frac{V(x, x', W)\psi_{n-1}(x')}{\psi_n(x)}\,d^3x', \qquad (5)$$

substituting each time into (5) in place of the ratio $\psi(x')/\psi(x)$ the preceding approximation:

$$L\psi_n(x) = U_n(x, W)\psi_n(x). \qquad (6)$$

This iteration process will not necessarily converge to the true solution under all circumstances: if the wave function of the (n-1)-th approximation has zeros at incorrect places, then the function of the n-th approximation will also have zeros at the same incorrect places. Indeed, it can be seen from (5) that at these points $U_n(x, W)$ will become infinite, and therefore $\psi_n(x)$ will vanish.

3. GEOMETRICAL OPTICS

We now consider the other limiting case when $\lambdabar \ll a$. We note that the cross section σ for elastic processes can be written in the form $\sigma = \pi a^2(1 - \beta)$ where a is the nucleon radius, while β is the transparency of the nucleon; therefore the condition for the applicability of geometrical optics can be written in the form

$$a/\lambdabar = [\sigma/\pi(1-\beta)\lambdabar^2]^{1/2} \qquad (7)$$

for $\lambdabar \to 0$. In this limiting case we represent the wave function in the form

$$\psi(x) = \exp\{ikS(x)\}, \qquad (8)$$

where $S(x)$ is the action function. In order not to complicate the subsequent discussion we restrict ourselves to the scalar equation $L = \nabla^2 + k^2$. Substituting (8) into (1) we obtain for $k \to \infty$

$$(\nabla S)^2 = n^2, \quad (9)$$

where n is the complex index of refraction defined by the equation

$$n^2 - 1 = k^{-2} \int V(x, x', W) \exp\{ik[S(x') - S(x)]\} d^3x'. \quad (10)$$

We note that

$$S(x') - S(x) = (x' - x)\nabla S + \cdots = n\rho\cos\theta,$$

$$\rho = |x' - x|.$$

We shall obtain the first approximation for $n^2 - 1$ if we set $n_0 = 1$ in the exponential in the integrand of (10). Then we have

$$n_1^2 - 1 = k^{-2} \int V(x, x + \rho, W) e^{ik\rho} d^3\rho, \quad (11)$$

i.e., in the first approximation the index of refraction n_1 is simply determined by the k-th Fourier component of the nonlocal potential.

Substituting n_1 obtained in this manner into the integral we shall obtain from formula (10) n_2 etc. On setting $n = \alpha + i\beta$ (α and β are functions of x and W, or k) we can easily see that a necessary condition for the convergence of the iteration process for the evaluation of n will be the condition $k\beta \to$ const (in particular, 0) for $k \to \infty$.

Otherwise the factor $\exp(-k\beta\rho\cos\theta)$ will appear in the integrand of (10) which in the region $\cos\theta < 0$ tends to ∞ as $k \to \infty$, and this would make the iteration process impossible.

We shall now show that as $k \to \infty$, $k\beta$ remains bounded. Indeed, from the optical theorem it follows that the imaginary part of the scattering amplitude

$$A(W, q) = ik \int_0^\infty bdb\,[1 - e^{2i\eta(b,k)}]\, J_0(bq) \quad (12)$$

(here b is the impact parameter, $\eta(b, k)$ is the phase of the scattered wave, q is the transferred momentum; $q = 2k\sin(\vartheta/2)$, $\eta = \delta(b, k) + i\gamma(b, k)$, $\gamma > 0$) for scattering angle $\vartheta = 0$ is related to the total cross section σ_t by the equation

$$\int bdb\,[1 - e^{-2\gamma}\cos 2\delta] = \sigma_t / 4\pi. \quad (13)$$

On the other hand

$$2\gamma(b, k) = k \int_0^\infty \beta(x, k)\, ds, \quad (14)$$

where the integral is taken over the path of the ray within the nucleon, for an impact parameter equal to b. If σ_t remains constant or decreases as $k \to \infty$, then $\gamma(b, k)$ must also be constant or decrease with increasing k. Then it can be seen from (14) that the product $k\beta$ remains bounded.

Thus, the concept of an index of refraction inside the particle as $k \to \infty$ acquires a simple physical meaning. This provides a theoretical basis for the application of geometrical optics to the description of the scattering of high energy particles as has been done in [1,6,7].

However, such a description of the scattering of particles is, of course, approximate and will not be valid for very large scattering angles, for example for backward scattering. As has been shown in [6], the backward scattering cross section is $\lesssim \lambda\!\!\!^{-2}$, and therefore condition (10) will not be satisfied.

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