CENTRALLY-SYMMETRIC EINSTEIN SPACES AND BIRKHOFF'S THEOREM

A. P. RYABUSHKO

Belorussian State University

Submitted to JETP editor June 11, 1963; resubmitted January 13, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 46, 2046-2048 (June, 1964)

It is shown that the generally accepted formulation of Birkhoff's theorem [1] is not exact. In particular, attention is drawn to centrally-symmetric Einstein spaces which do not possess any nonremovable singularities of the Schwarzschild type (at $R = 0$) and which are equivalent only to a part of the Schwarzschild space. A more precise formulation of Birkhoff's theorem is given.

THERE exists a widely known theorem [1] of Birkhoff which states that every centrally-symmetric solution of Einstein equations in vacuo

$$R_{ik} = 0, \qquad i, k = 0, 1, 2, 3 \tag{1}$$

is equivalent to the Schwarzschild solution

$$ds^2 = (1 - \alpha / R) dT^2 - (1 - \alpha / R)^{-1} dR^2 - R^2 (d\theta^2 + \sin^2\theta \, d\varphi^2), \qquad \alpha = \text{const.} \tag{2}$$

More precisely, the time dependence of the metric tensor g_{ik}, which is the solution of (1) possessing spherical symmetry and satisfying certain conditions at infinity, can always be removed with the aid of a transformation of coordinates so that solution (2) is obtained as a result. The correctness of Birkhoff's theorem is accepted by many authors [2-6]. In this paper we draw attention to examples of Einstein spaces whose metric satisfies equations (1), possesses spherical symmetry and depends on the time coordinate, but which cannot in any manner be reduced to the static Schwarzschild space by any transformation of coordinates permissible [1)] in the general theory of relativity.

1. A centrally-symmetric solution of equations (1) is the metric

$$ds^2 = dt^2 - dl^2,$$

$$dl^2 = \frac{4A^2 r_0'^2 dr^2}{9(t - r_0)^{2/3}} + A^2 (t - r_0)^{4/3} (d\theta^2 + \sin^2\theta d\varphi^2), \tag{3}$$

where $A \neq 0$ is an arbitrary constant, r_0 is an arbitrary function of r, the prime denotes a derivative with respect to r. Solution (3) is obtained from Tolman's general solution which is discussed in detail in the book of Landau and Lifshitz [8] (Sec. 97, problems 4 and 5), if in it we set the density ϵ of the dust-like matter equal to zero ($f = 0$, $F = \text{const} \neq 0$). This solution has singularities which fill the hypersurface $t - r_0 = 0$ on which the metric becomes degenerate: $g \equiv \det |g_{ik}| = 0$.

These singularities cannot be removed by any transformation of coordinates. Indeed, the scalar function calculated for the case of (3)

$$J_{\mathrm{I}} \equiv R_{ijkl} R^{ijkl} = 64 / 27 (t - r_0)^4 \tag{4}$$

has singularities on the same hypersurface. The analogous scalar function J_0 for the Schwarzschild metric (2) is given by the expression

$$J_0 = 12\, \alpha^2 / R^6. \tag{5}$$

By means of direct calculations one can demonstate a curious property of Einstein spaces with the metric (3): their three-dimensional spatial part corresponding to dl^2 is flat. The Schwarzschild field does not have such a property. The most general transformations of coordinates are known which transform the metrics (2) and (3) into each other [9]. These transformations are discontinuous on the two regular hypersurfaces $t - r_0 \pm 2\alpha/3$ of the Einstein space (3). The image of these hypersurfaces in the Schwarzschild space is the special hypersphere $R = \alpha$, considered respectively at the instants of "time" $T = \mp\infty$. These transformations are also not mutually unique. The regions $0 < t - r_0 < 2\alpha/3$ and $-2\alpha/3 < t - r_0 < 0$ correspond to the same region in the Schwarzschild field $0 < R < \alpha$, $|T| < \infty$. At the same time the regions $t - r_0 > 2\alpha/3$ and

1)Only continuous and differentiable transformations of coordinates with a nonvanishing Jacobian are considered permissible[2]. Moreover, opinions have been expressed[7] that the structure of space-time in the general theory of relativity is determined by a group of continuous and twice continuously differentiable transformations of coordinates $x^i = x^i(x'^k)$ (i.e., x^i form the class of functions C^2).

$t - r_0 < -2\alpha/3$ transform into the region $R > \alpha$, $|T| < \infty$. The world points of the Schwarzschild field which have the coordinates $R = \alpha$ and arbitrary T, θ, φ generally do not have images in the space (3). In other words, the Schwarzschild space is obtained from the Einstein space (3) by means of a "superposition" of two deformed parts of the latter one on top of another.

2. The metric

$$ds^2 = \frac{2\xi^2(1+\xi-\xi^2)}{1-\xi^2} \times \left[\frac{2\xi(1+\xi^2)}{1-\xi^2}(dt+dr)^2 - dt^2 + dr^2\right] - \frac{2\xi^3}{1-\xi^2}(dt^2-dr^2) + \frac{\xi^2}{1+\xi^2}(dt-dr)^2 - (1+\xi^2)^2(d\theta^2+\sin^2\theta d\varphi^2), \quad (6)$$

$\xi = \exp[(t+r)/2]$ which is a centrally-symmetric solution of equation (1) cannot be brought by any transformation of coordinates into the Schwarzschild metric (2). Indeed, the scalar function J_{II} calculated for (6) which is analogous to the scalar J_0 and J_I

$$J_{II} = 12/(1+e^{t+r})^6 \quad (7)$$

does not become infinite for any values of t and r. Further detailed calculations show that with the aid of discontinuous (at $t + r = 0$) transformations of coordinates the whole space (6) can be transformed only into that part of the Schwarzschild field for which $R \leq 1$.

3. We exhibit one more centrally-symmetric solution of (1):

$$ds^2 = \frac{(3+\operatorname{th}\eta)\operatorname{ch}^2\eta + 2\eta(2+\operatorname{th}\eta)(1-\ln^2\eta)}{(2+\operatorname{th}\eta)(1-\ln^2\eta)^2\operatorname{ch}^2\eta} \left(dt+\frac{dr}{r}\right)^2 - (2+\operatorname{th}\eta)^2(d\theta^2+\sin^2\theta d\varphi^2), \quad (8)^*$$

where $\eta \equiv re^t$. Assuming the existence of a transformation of coordinates which transforms the metric (8) into (2), from the equations

$$g_{ik} = (\partial x^{i'}/\partial x^i)(\partial x^{k'}/\partial x^k)\, g_{i'k'}$$

we obtain the equation $R = 2 + \tanh\eta$, which requires that R should be bounded, i.e., $2 \leq R \leq 3$. Further, we arrive at the conclusion that the whole space (8) is transformed only into the layer $2 \leq R \leq 3$ of the space (2). These transformation of coordinates are discontinuous at the two hypersurfaces $t + \ln r = \pm 1$.

Thus, if only permissible (cf. the footnote) transformations of coordinates are used, then it is impossible to prove Birkhoff's theorem. Consequently its formulation must be made more precise: if we consider as permissible discontinuous and mutually nonunique transformations of coordinates, then any centrally-symmetric Einstein space whose metric satisfies (1) is equivalent to the Schwarzschild field or to a part of this field.

We note, that in our view this equivalence is of a purely formal mathematical nature. Discontinuous and mutually nonunique transformations of coordinates radically violate the topological structure of space-time, as a result of which the physical equivalence of all centrally-symmetric spaces becomes doubtful. Further investigations in this direction are necessary.

Until recently there have been no investigations containing a criticism of Birkhoff's theorem. Among the first of such investigations is apparently included the paper by Unt [10] in which it is shown that when the continuity conditions of Lichnerowicz [7] are satisfied it is impossible to prove Birkhoff's theorem. Defects in the proof of Birkhoff's theorem were noted by Bonnor [11]. We call particular attention to the interesting paper by Petrov [12] which contains a critical analysis of the proof of Birkhoff's theorem, and in which it is shown that the latter is valid only under certain additional conditions associated with shock waves. The results of the present note do not overlap with the results of the papers cited above in which all the investigations are carried out under the assumption that the transformations of coordinates are carried out with the aid of functions which are at least once continuously differentiable (the class C^1).

*th = tanh, ch = cosh.

[1] G. D. Birkhoff, Relativity and Modern Physics, Cambridge, Harvard University Press, 1923, ch. 15.

[2] P. G. Bergmann, An Introduction to the Theory of Relativity, Prentice-Hall, 1942 (Russ. Transl. IIL, 1947).

[3] F. L. Pirani, Phys. Rev. **105**, 1089 (1957).

[4] J. Weber, General Theory of Relativity and Gravitational Waves, (Russ. Transl. IIL, 1962).

[5] M. A. Tonnela, The Fundamentals of Electromagnetism and of the Theory of Relativity (Russ. Transl. IIL, 1962).

[6] J. L. Synge, The General Theory of Relativity, (Russ. Trans. IIL, 1963).

[7] A. Lichnerowicz, Théories relativistes de la gravitation et de l'électromagnétisme, Paris, 1955.

[8] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Fizmatgiz, 1960.

[9] Yu. A. Rylov, JETP 40, 1755 (1961), Soviet Phys. JETP 13, 1235 (1961).

[10] V. A. Unt, Publ. of the Institute of Physics and Astronomy, Academy of Sciences, Estonian S.S.R., Investigations in Theoretical Physics, 19, Tartu, 1962, p. 54.

[11] W. B. Bónner, On Birkhoff's Theorem. Recent Developments in General Relativity, Warsaw, PWN; Oxford-London-New York-Paris, Pergamon Press, 1962, p. 167.

[12] A. Z. Petrov, JETP 44, 1525 (1963), Soviet Phys. JETP 17, 1026 (1963).

Translated by G. Volkoff
305