THEORY OF NONSTATIONARY FINITE-AMPLITUDE WAVES IN A LOW-DENSITY PLASMA

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Nonstationary waves of finite (but small) amplitude propagating in a cold plasma both at an angle with respect to the magnetic field and perpendicular to the magnetic field are analyzed. Approximate equations for these waves are obtained. Nonstationary solutions of these equations are derived which describe waves at large distances from a source operating for a bounded time interval.

1. INTRODUCTION

THE propagation of waves of finite amplitude in a low-density plasma (in which the mean free path is appreciably greater than the dimensions over which the flow characteristics change) exhibits a number of distinguishing features as compared with ordinary gas dynamics. In gas dynamics the wave profile is determined by two factors: nonlinear effects which lead to the steepening of the wave profile, and dissipative processes, which cause "smearing." In a low-density plasma, in addition to the nonlinear steepening process of the wave profile, an important role is played by dispersion effects; these derive from the deviations from linear dispersion relations such as those that characterize conventional gas dynamics.

For example, consider waves propagating across the magnetic field in a cold plasma ($p \ll H^2/8\pi$). The dispersion relation for low-amplitude waves of this kind is given by

$$\frac{\omega}{k} = V_A \frac{\omega_{0e}}{\sqrt{k^2c^2 + \omega_{0e}^2}}, \qquad V_A = \frac{H}{\sqrt{4\pi n m_i}}, \qquad \omega_{0e} = \sqrt{\frac{4\pi n e^2}{m_e}}. \tag{1.1}$$

At small values of k and $\omega \ll \sqrt{\omega_{Hi}\omega_{He}}$) the dispersion can be regarded as linear. As the frequency increases the phase velocity starts to fall off; when $\omega = \sqrt{\omega_{Hi}\omega_{He}}$, in general, the phase velocity is zero. It follows from Eq. (1.1) that dispersion effects are most important when $k \sim \omega_{0e}/c$. (Following [1] we call c/ω_{0e} the dispersion length.)

It is well known that the dispersion relation in (1.1) allows the possibility of stationary solitary and periodic waves of finite amplitude; these have been considered by a number of authors.[1-5] The length scales of these waves are determined by the dispersion length. A similar situation holds for other cases of plasma flow (for greater detail see [1]).

Dispersion effects described by formulas such as (1.1) are characterized by the fact that the phase velocity is reduced as k increases. We call this negative dispersion. There are also cases in which the dispersion is positive, that is to say, the phase velocity of the linear waves increases with increasing k. In this case the nature of the nonlinear flow changes markedly; specifically, in addition to the solitary compression wave there can exist solitary refraction waves.[1,5] An example is the propagation of a wave at an angle with respect to the magnetic field in a cold plasma. The dispersion relation here is of the form (the magnetic-sound branch with $\omega \ll \sqrt{\omega_{Hi}\omega_{He}}$, ω_{0e})

$$\frac{\omega}{k} = \frac{V_A}{2}\left(\left[(1+\sin\theta)^2 + \frac{V_A^2}{\omega_{Hi}^2}k^2\sin^2\theta\right]^{1/2} + \left[(1-\sin\theta)^2 + \frac{V_A^2}{\omega_{Hi}^2}k^2\sin^2\theta\right]^{1/2}\right), \tag{1.2}$$

where θ is the angle between the perpendicular to the direction of propagation and the magnetic field (Fig. 1). At small angles θ the dispersion relation becomes

$$\frac{\omega}{k} = V_A\left(1 + \frac{V_A^2k^2}{2\omega_{Hi}^2}\theta^2\right). \tag{1.2a}$$

Stationary nonlinear rarefaction waves propagating at an angle with respect to the magnetic field have been analyzed in [5-7] (in [5,6] in addition to the analysis of solitary waves an analysis has been made of the oscillatory shock waves to which the solitary waves are transformed in the presence of small dissipation effects). In accordance with Eq. (1.2a) the length scale of these waves is deter-

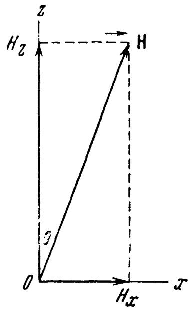

FIG. 1

mined by the dispersion length, which is of order $c\theta/\omega_{0i}$.

In the present work we investigate nonstationary waves of finite amplitude in a plasma. Because of the great mathematical difficulties, at the present time it has only been possible to consider certain kinds of flow of this kind. In [3] a qualitative analysis was made of waves propagating across the magnetic field in the linear approximation. In [2,8] numerical integration was used to study the propagation of waves of finite amplitude across a magnetic field; these waves were produced by a constant magnetic pressure acting at the boundary of the plasma starting at $t = 0$. In the present work we find the class of one-dimensional self-similar solutions (with respect to some variable) which describe the propagation of waves of finite (but small) amplitude in a cold plasma both across the magnetic field and at a small angle with respect to the field at large distances from a boundary; the waves are excited at the plasma boundary for a bounded time interval Δt.

2. BASIC EQUATIONS

As in [2-8] we consider motion with characteristic frequencies much smaller than the electron Larmor frequency ω_{He} with $\omega_{He} \ll \omega_{0e} = \sqrt{4\pi ne^2/m_e}$. Under these conditions deviations from neutrality can be neglected, i.e., we assume that

$$n_e = r_i = n, \tag{2.1}$$

where n_e and n_i are the electron and ion densities respectively. We assume further that the gas kinetic pressure of the plasma is much smaller than the magnetic pressure $p \ll H^2/8\pi$. The equations for the electron and ion components of the plasma and M Maxwell's equations then assume the form

$$m_i d\mathbf{v}_i/dt = e\mathbf{E} + c^{-1}e\,[\mathbf{v}_i\mathbf{H}], \qquad (2.2a)^*$$

$$m_e d\mathbf{v}_e/dt = -e\mathbf{E} - c^{-1}e\,[\mathbf{v}_e\mathbf{H}], \qquad (2.2b)$$

$$\text{rot}\,\mathbf{H} = 4\pi\,nec^{-1}\,(\mathbf{v}_i - \mathbf{v}_e), \qquad (2.3a)$$

$$\partial\mathbf{H}/\partial t = -c\,\text{rot}\,\mathbf{E}, \qquad (2.3b)^*$$

$$\partial n/\partial t + \text{div}\,(n\mathbf{v}_i) = 0, \quad \partial n/\partial t + \text{div}\,(n\mathbf{v}_e) = 0. \tag{2.4}$$

Adding Eqs. (2.2a) and (2.2b) and using Eq. (2.3a) we find

$$\frac{d\mathbf{U}}{dt} = \frac{1}{4\pi nM}\,[\text{rot}\,\mathbf{H},\,\mathbf{H}], \tag{2.5}$$

where $\mathbf{U}$ is the mass velocity of the plasma, given by

$$\mathbf{U} = (m_i\mathbf{v}_i + m_e\mathbf{v}_e)/M \approx \mathbf{v}_i, \qquad M = m_i + m^e \approx m_i. \tag{2.6}$$

To obtain an equation for the magnetic field we must eliminate $\mathbf{E}$ from Eq. (2.3b). Expressing $\mathbf{E}$ from Eq. (2.2a) and taking account of Eq. (2.6) we find (for waves propagating at an angle with respect to the magnetic field)

$$\frac{\partial\mathbf{H}}{\partial t} = \text{rot}\,[\mathbf{U}\mathbf{H}] - \frac{Mc}{e}\,\text{rot}\,\frac{d\mathbf{U}}{dt}\,. \tag{2.7}$$

The second term on the right side of Eq. (2.7) is responsible for the dispersion effects; in particular it leads to the appearance of the second term in the dispersion equation (1.2a).

We now write Eqs. (2.4), (2.5) and (2.7) in component form assuming that the motion is one-dimensional, that is to say that all quantities depend only on t and x:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nU_x) = 0; \tag{2.8}$$

$$\frac{\partial U_x}{\partial t} + U_x\frac{\partial U_x}{\partial x} = -\frac{1}{8\pi nM}\frac{\partial}{\partial x}(H_z^2 + H_y^2),$$
$$\frac{\partial U_y}{\partial t} + U_x\frac{\partial U_y}{\partial x} = \frac{1}{4\pi nM}H_x\frac{\partial H_y}{\partial x},$$
$$\frac{\partial U_z}{\partial t} + U_x\frac{\partial U_z}{\partial x} = \frac{1}{4\pi nM}H_x\frac{\partial H_z}{\partial x}; \tag{2.9}$$

$$\frac{\partial H_y}{\partial t} = -\frac{\partial}{\partial x}(U_xH_y - U_yH_x) + \frac{c}{4\pi e}\frac{\partial}{\partial x}\left(\frac{1}{n}H_x\frac{\partial H_z}{\partial x}\right),$$
$$\frac{\partial H_z}{\partial t} = \frac{\partial}{\partial x}(U_zH_x - U_xH_z) - \frac{c}{4\pi e}\frac{\partial}{\partial x}\left(\frac{1}{n}H_x\frac{\partial H_y}{\partial x}\right),$$
$$H_x = \text{const}. \tag{2.10}$$

In the case of motion perpendicular to the magnetic field the last term in Eq. (2.7) and the corresponding terms in Eq. (2.10), which are responsible for the important dispersion effects, vanish. This is related to the fact that under present conditions the dispersion is due to the inertia of the electrons, which we have neglected up to this time. To obtain equations that replace Eq. (2.10) in this case we

*$[\mathbf{v}_i\mathbf{H}] = \mathbf{v}_i \times \mathbf{H}$.

*rot E = curl E.

eliminate the electric field from Eqs. (2.2b) and (2.3b)

$$\frac{\partial H}{\partial t} = -\frac{\partial}{\partial x}(UH) + \frac{m_e c}{e}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x}\right), \quad (2.11)$$

where we have introduced the notation $v_{ey} = v$, $H_z = H$ and made use of the fact that [from the equation of continuity (2.4)]

$$v_{ex} = v_{ix} = U_x = U$$

(i.e., the current flows in the direction perpendicular to the plasma motion).

Equation (2.11) must be supplemented by one additional equation for v. Such an equation can be obtained if we take the component of Eq. (2.3a) in the y direction:

$$\partial H/\partial x = 4\pi nec^{-1}v. \quad (2.12)$$

Equations (2.8) and (2.9) (in which we write $U_y = U_z = 0$, $U_x = U$) and Eqs. (2.11) and (2.12) describe one-dimensional motion of a plasma across a magnetic field. We note finally that Eq. (2.10) refers to motion at an angle with respect to the magnetic field and applies only for angles θ (Fig. 1) that satisfy the condition $\theta \gg \sqrt{m_e/m_i}$.

We now consider the possibility of simplifying the equations that have been obtained. First let us introduce the dimensionless variables

$$h_z = \frac{H_z}{H_{0z}}, \quad h_y = \frac{H_y}{H_{0z}}, \quad \alpha = \operatorname{tg}\theta = \frac{H_x}{H_{0z}},$$
$$k = \frac{n_0}{n}, \quad u_x = \frac{U_x}{V_A}, \quad u_y = \frac{U_y}{V_A}, \quad u_z = \frac{U_z}{V_A},$$
$$\tilde{x} = \frac{x}{\delta}, \quad \tau = \frac{V_A}{\delta}t, \quad (2.13)$$

where H_{0z} and n_0 are the component of the magnetic field transverse to the direction of plasma motion and the density of the unperturbed plasma respectively

$$V_A = \frac{H_{0z}}{\sqrt{4\pi n_0 M}}, \quad \delta = \frac{c}{\omega_{0i}} = \frac{c}{\sqrt{4\pi n_0 e^2/M}}. \quad (2.14)$$

It is now convenient to transform to the Lagrangian coordinate $\xi = \xi(\tilde{x}, \tau)$ defined by the relations

$$(\partial\xi/\partial\tilde{x})_\tau = n/n_0 = k^{-1}, \quad (\partial\xi/\partial\tau)_{\tilde{x}} = -k^{-1}u_x. \quad (2.15)$$

Then the equations for waves propagating at an angle with respect to the magnetic field become

$$\partial k/\partial\tau = \partial u_x/\partial\xi, \quad (2.16)$$

$$\frac{\partial u_x}{\partial\tau} = -\frac{1}{2}\frac{\partial}{\partial\xi}(h_z^2 + h_y^2), \quad \frac{\partial u_y}{\partial\tau} = \alpha\frac{\partial h_y}{\partial\xi},$$
$$\frac{\partial u_z}{\partial\tau} = \alpha\frac{\partial h_z}{\partial\xi}, \quad (2.17)$$

$$\frac{\partial}{\partial\tau}(kh_z) = \alpha\frac{\partial u_z}{\partial\xi} - \alpha\frac{\partial^2 h_y}{\partial\xi^2},$$
$$\frac{\partial}{\partial\tau}(kh_y) = \alpha\frac{\partial u_y}{\partial\xi} + \alpha\frac{\partial^2 h_z}{\partial\xi^2}. \quad (2.18)$$

The initial conditions are

$$k(\xi, 0) = 1, \; u_x(\xi, 0) = u_y(\xi, 0) = u_z(\xi, 0) = h_y(\xi, 0) = 0,$$
$$h_z(\xi, 0) = 1. \quad (2.19)$$

The boundary conditions (for $\xi = 0$) are:[1)]

$$h_z(0, \tau) = 1 + g(\tau), \quad (2.20)$$

where $g(\tau)$ is some function of time which differs from 0 over a specified time interval $\Delta\tau$ ($g(\tau) \neq 0$ when $0 < \tau < \Delta\tau$, $g(\tau) = 0$ when $\tau > \Delta\tau$, $\tau < 0$). These conditions correspond to the application of a magnetic piston at the plasma boundary over a bounded time interval.

The system of equations (2.16)—(2.18) is rather complicated for our purposes. For this reason we introduce some simplifying assumptions that allow us to neglect a number of terms while retaining the basic features. To understand the method of simplifying the system we note that the dispersion effects of interest appear only in second-order in $\alpha \approx \theta$ ($\theta \ll 1$) [cf. Eq. (1.2a)]. Hence we assume that the angle θ is small and limit ourselves to terms of order θ^2. However it should be recalled that when the angle θ becomes too small ($\theta \ll \sqrt{m_e/m_i}$) the original equation (2.18) no longer holds. Hence, in order for the equations (2.17) and (2.18) to retain the dispersion effects we require simply that

$$m_e/m_i \ll \theta^2 \; (\approx \alpha^2) \ll 1. \quad (2.21)$$

Further

$$h_z = 1 + b, \quad k = 1 + \varkappa. \quad (2.22)$$

It will be assumed that b and κ are small quantities; only second-order terms in these quantities are considered. It is easy to show that in this case we do not loose the nonlinear effects in the equations that lead to the steepening of the front.

[1)]These boundary conditions correspond to the excitation of a wave of the magnetic sound branch (when $\omega \ll \omega_{Hi}$, in the linear approximation this gives the "magnetic sound"). It is possible to use other boundary conditions, for example $h_y(0, \tau) = g(\tau)$, $h_z(0, \tau) = 1$ corresponding to excitation of Alfvén waves. However this branch is characterized by negative dispersion as for a wave propagating perpendicularly to the magnetic field. Hence the analysis of this wave does not lead to any fundamentally new results compared with the case of a wave propagating across the magnetic field.

(Terms ~ $\alpha^2 b^2$ etc. can be neglected.) As a result the following equations are obtained:

$$\frac{\partial^2 \varkappa}{\partial \tau^2} + \frac{\partial^2 b}{\partial \xi^2} = -\frac{1}{2}\frac{\partial^2}{\partial \xi^2}(b^2) - \frac{1}{2}\frac{\partial^2}{\partial \xi^2}(h_y^2), \quad (2.23)$$

$$\frac{\partial}{\partial \tau}(\varkappa + b + \varkappa b) = -\alpha \frac{\partial^2 h_y}{\partial \xi^2},$$

$$\frac{\partial}{\partial \tau}(h_y + \varkappa h_y) = \alpha \frac{\partial^2 b}{\partial \xi^2}. \quad (2.24)$$

If quadratic terms and dispersion terms in this equation (the latter are proportional to α) are neglected we obtain wave equations for κ, b and the other quantities:

$$\frac{\partial^2 f}{\partial \tau^2} - \frac{\partial^2 f}{\partial \xi^2} = 0, \qquad f = b, \varkappa, h_y, u_x, u_y, u_z. \quad (2.25)$$

In accordance with the initial values and the boundary conditions (2.19) and (2.20) we are interested in waves moving in the positive x direction. It follows from Eq. (2.25) that for these waves $\partial f/\partial \xi = -\partial f/\partial \tau$. This relation holds if terms of order f^2 and α^2 are neglected. Inasmuch as the right side of Eq. (2.24) already contains the quantity α we can make the substitution

$$\partial/\partial \xi \to -\partial/\partial \tau, \quad (2.26)$$

as a result of which

$$\varkappa + b + \varkappa b = -\alpha \partial h_y/\partial \tau, \qquad h_y + \varkappa h_y = \alpha \partial b/\partial \tau. \quad (2.27)$$

Thus, expressing the quantities κ and h_y in terms of b and substituting in Eq. (2.23) with the same degree of accuracy we obtain the following equation:

$$\frac{\partial^2 b}{\partial \xi^2} - \frac{\partial^2 b}{\partial \tau^2} = -\frac{\partial^2}{\partial \tau^2}(b^2) - \frac{1}{2}\frac{\partial^2}{\partial \xi^2}(b^2) + \alpha^2 \frac{\partial^4 b}{\partial \tau^4}. \quad (2.28)$$

Here again, we can make the substitution (2.26) in the second term on the right side. Furthermore, for the accuracy in which we are interested the right side of Eq. (2.28) can be written in the following form when the substitution (2.26) is made:

$$\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \tau}\right)\left(\frac{\partial b}{\partial \xi} + \frac{\partial b}{\partial \tau}\right) = -2\frac{\partial}{\partial \tau}\left(\frac{\partial b}{\partial \xi} + \frac{\partial b}{\partial \tau}\right). \quad (2.29)$$

As a result the order of Eq. (2.28) is reduced and we find

$$\frac{\partial b}{\partial \xi} + \frac{\partial b}{\partial \tau} = \frac{3}{2} b \frac{\partial b}{\partial \tau} - \frac{1}{2}\alpha^2 \frac{\partial^3 b}{\partial \tau^3}. \quad (2.30)$$

The case of waves propagating across the magnetic field can be treated in complete analogy with the above procedure. In Eqs. (2.28) and (2.9) (in which we write $U_y = U_z = 0$, $U_x = U$) and Eqs. (2.11) and (2.12) we transform to Lagrangian coordinates and dimensionless variables in accordance with Eqs. (2.13)—(2.15), thereby obtaining

$$\frac{\partial^2 k}{\partial \tau^2} = -\frac{1}{2}\frac{\partial^2}{\partial \xi^2}(h^2), \qquad \beta^2 \frac{\partial^2 h}{\partial \xi^2} = kh - 1, \quad (2.31)$$

where $\beta^2 = m_e/m_i$.

Writing $h = 1 + b$, $k = 1 + \kappa$ ($b, \kappa \ll 1$) and retaining second-order terms we have

$$\frac{\partial^2 b}{\partial \xi^2} - \frac{\partial^2 b}{\partial \tau^2} = -\frac{\partial^2}{\partial \tau^2}(b^2) - \frac{1}{2}\frac{\partial^2}{\partial \xi^2}(b^2) - \beta^2 \frac{\partial^4 b}{\partial \tau^2 \partial \xi^2}. \quad (2.32)$$

Transforming Eq. (2.32) in the same way as Eq. (2.28) we find

$$\frac{\partial b}{\partial \xi} + \frac{\partial b}{\partial \tau} = \frac{3}{2} b \frac{\partial b}{\partial \tau} + \frac{1}{2}\beta^2 \frac{\partial^3 b}{\partial \tau^3}. \quad (2.33)$$

Equations (2.30) and (2.33) differ only in the sign of the higher derivative and can be written in the common form

$$\frac{\partial b}{\partial \xi} + \frac{\partial b}{\partial \tau} - \frac{3}{2} b \frac{\partial b}{\partial \tau} + \frac{1}{2}\mu \frac{\partial^3 b}{\partial \tau^3} = 0, \quad (2.34)$$

$$\mu = \begin{cases} \alpha^2 \approx \theta^2, & \sqrt{m_e/m_i} \ll \theta \ll 1, \\ -\beta^2 = -m_e/m_i, & \theta = 0. \end{cases} \quad (2.35)$$

The quantity μ is a dimensionless parameter that describes the nature of the dispersion; its sign is the same as the sign of the dispersion.

In Eq. (2.34) we introduce the new variables

$$\zeta = \pm |\mu|^{-1/2} \xi, \qquad \eta = 2^{1/3} |\mu|^{-1/2} (\tau - \xi),$$

$$b = \pm \frac{2^{2/3}}{3} f, \quad (2.36)$$

where the ± sign corresponds to the sign of the dispersion. As a result we obtain the equation

$$\frac{\partial f}{\partial \zeta} - f \frac{\partial f}{\partial \eta} + \frac{\partial^3 f}{\partial \eta^3} = 0. \quad (2.37)$$

It is easily shown that Eq. (2.37) can be obtained for other kinds of plasma waves of low amplitudes,[2)] for example ion waves in a cold plasma for the case $\omega_{He} > \omega_{0e}$ (here it is important to take account of deviations from neutrality) (stationary solutions for these conditions are considered in greater detail in [1]).

It is easy to verify that Eqs. (2.30) and (2.33) [and consequently Eq. (2.37)] satisfy solutions that describe stationary solitary (and periodic) waves propagating at an angle to the magnetic field and across the field respectively. Actually, substitution in Eqs. (2.30) and (2.33) of a function of the form

$$b = b(\xi - w\tau), \quad (2.38)$$

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2)We note that an equation of the type in (2.37) describes waves at the surface of a heavy liquid in a channel of finite depth. The analogy between these waves and plasma waves has been pointed out in[9,10].

yields the following solutions:

$$b(z) = \mp |b_{max}| \operatorname{sech}^2 \left(\left| \frac{b_{max}}{\mu} \right|^{1/2} \frac{z}{2} \right), \qquad (2.39)$$

where $z = \xi - w\tau$, $w = 1 + b_{max}/2$ is the velocity of the solitary waves while μ is determined by Eq. (2.35). The minus sign is taken for a wave propagating at an angle to the magnetic field and the plus sign for a wave propagating across the magnetic field, that is to say, in the first case we obtain a rarefraction wave and in the second case a compressional wave; these coincide (to the accuracy used here) with the solutions obtained in [1-7]. Thus, the terms that have been neglected in the derivation of Eqs. (2.30) and (2.33) do not contain the interesting nonlinear and dispersive effects.

3. SELF-SIMILAR SOLUTIONS OF EQ. (2.37)

We shall first consider the solution of the linearized equation obtained from Eq. (2.37) when the $f\partial f/\partial\eta$ term is neglected in the initial conditions and the boundary conditions (2.19), (2.20). This solution is of the form

$$f(\zeta, \eta) = \int_{-\infty}^{\infty} G(\zeta, \eta - \eta') \varphi(\eta')\, d\eta', \qquad (3.1)$$

$$G(\zeta, \eta) = \frac{1}{\sqrt{\pi}(3\zeta)^{1/3}} \Phi\left[\frac{\eta}{(3\zeta)^{1/3}}\right], \qquad (3.1a)$$

$$\Phi(\chi) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \cos\left(u\chi + \frac{u^3}{3}\right) du. \qquad (3.1b)$$

Here, $\varphi(\eta) = f(0, \eta) = \pm(3/2^{2/3})\, g(2^{1/3}|\mu|^{-1/2}\tau)$ while $\Phi(\chi)$ is the Airy function, whose asymptote is given by

$$\Phi(\chi) \approx \begin{cases} (1/2\,\chi^{1/4}) \exp\{-{}^2/_3\,\chi^{3/2}\}, & \chi \to +\infty \quad (3.2a) \\ |\chi|^{-1/4} \sin({}^2/_3\,|\chi|^{3/2} + \pi/4), & \chi \to -\infty \quad (3.2b) \end{cases}.$$

If the wave is radiated for a bounded time interval $\Delta\tau$ and propagates at an angle to the magnetic field, at sufficiently large distances from a source located at the plasma boundary the solution (3.1) of the linearized equation can be written in the following form (going over to the old variables ξ and τ):

$$f(\xi, \tau) \approx G(\xi, \tau) \int_{-\infty}^{\infty} \varphi(\eta)\, d\eta$$

$$= \text{const} \cdot \xi^{-1/3} \Phi\left[(2/3\mu)^{1/3} \xi^{-1/3} (\tau - \xi)\right]. \qquad (3.3)$$

Thus, at large distances from the source the solution does not depend on the details of the variation of magnetic field at the plasma boundary and the profile of the magnetic field in the wave is of the form shown in Fig. 2a (for $\tau \gg \Delta\tau$). We see that when ξ is reduced the wave damps exponentially so that Eq. (3.3) applies formally for small ξ as well (but sufficiently large τ). The wave packet as a whole moves away from the plasma boundary with a mean velocity of the order of the Alfvén velocity. The packet spreads and its amplitude is reduced.

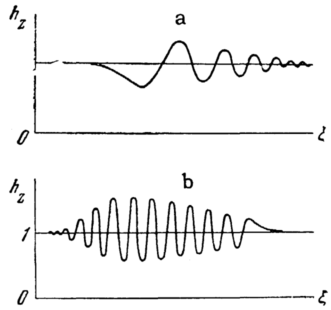

FIG. 2. Profile showing the change of magnetic field in a wave propagating at an angle with respect to the magnetic field, a, and across the magnetic field, b.

The wave that propagates across the magnetic field is characterized by a profile whose general form is shown in Fig. 2b. Because the Green's function (3.1a) oscillates rapidly in this case when $\xi \to 0$ (since $\zeta = -|\mu|^{-1/2}\xi$) it can only be taken outside the integral sign [as in (3.3)] at large ξ. For small values of ξ the wave profile depends on the details of the variation of the field at the plasma boundary during the time interval $\Delta\tau$.[3)]

It is reasonable to expect that the solution of the nonlinear equation (2.37), which describes the profile of a wave radiating in the plasma for a bounded time interval, should conserve the basic qualitative features of the solution (3.3). Hence, we seek a solution of Eq. (2.37) in the form $f(\zeta, \eta) = \zeta^n \psi(\zeta^{-1/3}\eta)$ where n is a number. To determine the index n we note that Eq. (2.37) (as well as the initial values and boundary conditions) are invariant under the transformation

$$\zeta \to \gamma\zeta, \qquad \eta \to \gamma^{1/3}\eta, \qquad f \to \gamma^{-2/3} f. \qquad (3.4)$$

It then follows that $n = -2/3$. Hence we take

$$f(\zeta, \eta) = -\zeta^{-2/3}\psi(-\zeta^{-1/3}\eta) \qquad (3.5)$$

and obtain from Eq. (2.37) the following ordinary

[3)]This is connected with the fact that for negative dispersion the shorter wavelengths lag behind the long wavelengths. When the nonlinear term is taken into account this effect is still more pronounced because the nonlinear steepening of the front produces more and more short-wave harmonics.

differential equation for the function $\psi(z)$ $(z = -\xi^{-1/3}\eta)$:

$$3\psi''' + 3\psi\psi' + z\psi' + 2\psi = 0. \tag{3.6}$$

We seek solutions of Eq. (3.6) that decay as $z \to -\infty$. To determine the asymptotes of these solutions we neglect the nonlinear terms and obtain the equation

$$3\psi''' + z\psi' + 2\psi = 0. \tag{3.7}$$

Two linearly independent solutions of Eq. (3.7) will be the products of the solutions of the equation $3\varphi'' + z\varphi = 0$ (which reduces to the Airy equation). To be convinced of this we differentiate this equation twice and substitute $\varphi'(z) = \psi(z)$, as a result of which we obtain Eq. (3.7). It is found that the damping condition for $z \to -\infty$ is satisfied only by a solution of the form

$$\psi(z) = -C\Phi'(-3^{-1/3}z), \qquad C > 0, \tag{3.8}$$

where $\Phi(\chi)$ is the Airy function. Using Eq. (3.2a) we see that the (one-parameter) family of solutions of Eq. (3.6) in which we are interested has the following asymptotes as $z \to -\infty$: [4)]

$$\psi(z) = {}^{1}/{}_{2}C\,(3^{-1/3}\,|z|)^{1/4}\exp\{-2\,(|z|/3)^{3/2}\}. \tag{3.9}$$

The form of the solution for the remaining values of z can be determined by numerical integration of Eq. (3.6). The results of this integration are shown in Fig. 3. As initial conditions we have taken the values of $\psi(z)$, $\psi'(z)$ and $\psi''(z)$ for certain large negative values of z; these values are computed for the asymptote (3.8). If $C \le 1.6$ in (3.8) the solution $\psi(z)$ is oscillatory and the amplitude of the oscillations increases with increasing z (curves I and II in Fig. 3). If $C \ge 1.8$ the solution is represented by curves such as III and IV and does not have physical meaning.

The qualitative features of the solutions shown in Fig. 3 can be obtained as follows. We assume first that C in Eq. (3.8) is small. Then the solution of Eq. (3.8) will satisfy Eq. (3.6) approximately for small (in modulus) z. As $z \to +\infty$ the solution (3.8) starts to increase as $z^{1/4}$; hence for large z we must take account of the linear term. However, at small values of C the nonlinearity becomes important when $\Phi'(-3^{-1/3}z)$ is determined by the oscillating asymptote [in accordance with Eq. (3.2b)]. Hence, the solution of the nonlinear equation (3.6) for small values of C that ap-

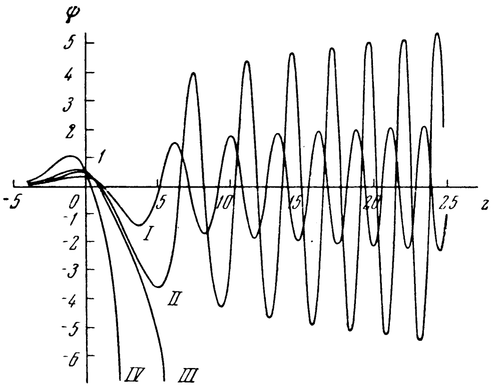

FIG. 3. Results of the numerical solution of Eq. (3.6). Curve I, C = 1.0; II, C = 1.6; III, C = 1.8; IV, C = 5.

proaches the asymptote in (3.8) as $z \to -\infty$ will have the asymptotic form given below as $z \to +\infty$:

$$\psi(z) = z^{n_1}\{C_1\cos[2(z/3)^{3/2}] + C_2\sin[2(z/3)^{3/2}]\} + z^{n_2}\{C_3\cos[4(z/3)^{3/2}] + C_4\sin[4(z/3)^{3/2}]\}; \tag{3.10}$$

because of the small nonlinearity $C_i(z)$ is a bounded, slowly varying (compared with z^{n_1} and z^{n_2}) function and $n_2 < n_1$. Substituting Eq. (3.10) in Eq. (3.6) we see that $n_1 = 1/4$, $n_2 = -1/2$, and that the asymptote of the solutions for $z \to +\infty$ is of the form

$$\psi(z) = z^{1/4}\{C_1\cos[2(z/3)^{3/2}] + C_2\sin[2(z/3)^{3/2}]\}. \tag{3.11}$$

This asymptote is found to agree with the asymptote for the numerical solutions when $C \le 1.6$. It can also be shown that the solutions for $C \ge 1.8$ diverge as $(z-\delta)^{-2}$, where δ is a constant determined by the value of C such that δ decreases as C increases.

We now consider the results that follow from the solutions that have been obtained. Using Eqs. (2.36) and (3.5) we find the magnetic field in the wave

$$h_z(\xi,\tau) = 1 + b(\xi,\tau) = 1 - (2^{2/3}/3)\mu^{1/3}\xi^{-2/3}\psi[-(2/\mu)^{1/3}\xi^{-1/3}(\tau-\xi)]. \tag{3.12}$$

Let us first consider a wave propagating at an angle to the magnetic field $(\mu > 0)$. When $\tau \gg 1$, $\xi \gg 1$, $\xi < \tau$ the deviation of the magnetic field from the equilibrium value is

$$b = -{}^{1}/{}_{3}C\,(\mu/2\cdot3^{1/3})^{1/4}\tau^{-3/4}(\tau-\xi)^{1/4} \times\exp\{-({}^{2}/{}_{3})^{3/2}\mu^{-1/2}\xi^{-1/2}(\tau-\xi)^{3/2}\}, \tag{3.13}$$

i.e., the quantity b falls off exponentially with distance from the point $\xi = \tau$ in the negative direction (for this reason we replace ξ by τ in front of the exponential). In the course of time the wave propagating at an angle with respect to the magnetic field damps as $\tau^{-3/4}$. When $\xi > \tau$ the pro-

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4)We note that the third solution is proportional to z^{-2} as $z \to -\infty$. The numerical solution of the nonlinear equation (3.6) for this asymptote leads to a solution that is proportional to z as $z \to +\infty$. Hence, it has no meaning.

file of the wave oscillates, damping with distance from the point $\xi = \tau$ as

$$b = -(2^{3/4}/3)\,\mu^{1/4}\xi^{-1/2}\{C_1 \cos[(2/3)^{3/2}\mu^{-1/2}\xi] + C_2 \sin[(2/3)^{3/2}\mu^{-1/2}\xi]\} \qquad (3.14)$$

(here we have used Eq. (3.11) and taken $\xi \gg \tau$).

We see that the general qualitative form of the profile is the same as in the linearized problem (Fig. 2a); quantitatively, however, the nature of the variation of b is different in Eqs. (3.13) and (3.14).

For a wave propagating across the magnetic field ($\mu < 0$), when $\tau \gg 1$, $\xi \gg 1$, and $\xi > \tau$ we have

$$b = {}^1/_3 C\,(|\mu|/2\cdot 3^{1/3})^{1/4}\,\tau^{-1/2}\exp\{-(2/3)^{1/2}|\mu|^{-3/2}\xi\}, \qquad (3.15)$$

i.e., the quantity b damps exponentially with increasing ξ (here we again replace ξ by τ in front of the exponential).

When $\xi < \tau$ the quantity b oscillates and is given by

$$b = (2^{3/4}/3)\,|\mu|^{1/4}\,\xi^{-3/4}(\tau - \xi)^{1/4} \times \{C_1 \cos[(2/3)^{3/2}|\mu|^{-1/2}\xi^{-1/2}(\tau - \xi)^{3/2}] + C_2 \sin[(2/3)^{3/2}|\mu|^{-1/2}\xi^{-1/2}(\tau - \xi)^{3/2}]\}. \qquad (3.16)$$

It is evident that the solution shown in Fig. 3 does not apply at very small ξ. This result is associated with the following circumstance noted in the investigation of the linearized problem: in the case of negative dispersion the solutions with small ξ depend strongly on the detailed nature of the excitation process. However, the general form of the profile will be the same as in Fig. 2b.

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[1] Vedenov, Velikhov, and Sagdeev, Nuclear Fusion **1**, 82 (1961).

[2] J. Adlam and J. Allen, Phil. Mag. **3**, 448 (1958).

[3] Gardner, Goertzel, Grad, Morawetz, Rose, and Rubin, Proceedings of the Conference on Peaceful Uses of Atomic Energy, Geneva, 1958, Vol. **31**, p. 230.

[4] Davis, Lüst, and Schlüter, Naturforsch **13a**, 916 (1958).

[5] R. Z. Sagdeev, Proceedings of the Symposium on Electromagnetics and Fluid Dynamics of Gaseous Plasma, Interscience, New York, 1961, p. 443.

[6] V. I. Karpman, ZhTF **33**, 959 (1963), Soviet Phys. Technical Phys. **8**, 715 (1964).

[7] A. P. Kazantsev, JETP **44**, 1283 (1963), Soviet Phys. JETP **17**, 865 (1963).

[8] Auer, Hurwitz, and Kilb, Phys. Fluids **4**, 1105 (1961).

[9] R. Z. Sagdeev, Voprosy magnitnoĭ gidrodinamiki i dinamiki plazmy (Problems of Magnetohydrodynamics and Plasma Dynamics) Riga, 1960, p. 275.

[10] C. S. Gardner and G. K. Morkawa, Preprint, New York University, 1960.

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271