GROUP PROPERTIES OF COMPLEX ANGULAR MOMENTUM

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The group-theoretical properties of complex angular momentum are analyzed. Properties of the eigenfunctions $f_{\alpha m}(\mathbf{n})$ of the operator $\mathbf{L}^2$ corresponding to complex eigenvalues $\alpha(\alpha+1)$ are discussed. The functions $f_{\alpha m}(\mathbf{n})$ possess nonintegrable singularities. A finite norm, invariant under rotations, is set up for them. The matrices $\mathfrak{D}^{(\alpha)}_{mn}$ are found which may be used to transform the functions $f_{\alpha m}(\mathbf{n})$ under finite rotations. It is shown that in a certain generalized sense these matrices provide a representation of the rotation group.

1. INTRODUCTION

In recent years a new method for the study of the asymptotic behavior of cross sections for various processes has been developed. It is based on investigations of analytic properties of partial wave amplitudes $f_l(E)$ for complex values of the angular momentum l.[1-6] For this reason the concept of complex angular momentum has been introduced in practice. However whereas for the usual (integer and half-integer) values of the angular momentum j there exist well-known irreducible representations $D^{(j)}$ of the rotation group in three dimentional space,[7] the group-theoretical interpretation of the complex values $j = \alpha$ has not been given until now.

A clarification of the group-theoretical meaning of complex angular momenta, aside from being of interest for its own sake, may also be important for the theory of unstable particles. Recently some qualitative considerations have been advanced in favor of the idea that the spin of an unstable particle may be complex.[8] 1) In that case a new mathematical apparatus is needed to describe the transformation properties of the "wave function" of the unstable particle under rotations of the frame of reference and to carry out addition of complex spins. The complexness of the spin of unstable particles would give rise to the appearance of specific terms in the angular distribution of the decay products of a polarized particle; comparison with experiment could then serve as a test of the hypothesis of a complex spin.

In this paper we construct irreducible representations $\mathfrak{D}^{(\alpha)}$ of the rotation group corresponding to an arbitrary complex value of the angular momentum α; these representations are multivalued and discontinuous. Functions $Y_{\alpha m}(\mathbf{n})$ ($m = 0, \pm 1, \pm 2, \ldots$) defined on the unit sphere are introduced. They are solutions of the equation $\mathbf{L}^2 Y_{\alpha m} = \alpha(\alpha+1) Y_{\alpha m}$, they transform under rotations according to the representation $\mathfrak{D}^{(\alpha)}$, and for $\alpha = l$ (integer) they go over into the conventional spherical harmonics $Y_{lm}(\mathbf{n})$. For $m \neq 0$ the functions $Y_{\alpha m}(\mathbf{n})$ have a nonintegrable singularity at $\vartheta = \pi$ and cannot therefore be considered as wave functions in the conventional sense. A new finite norm is defined for them, which is invariant under rotations and which goes over into the conventional one for $\alpha = l$; at that the multivaluedness of the representation $\mathfrak{D}^{(\alpha)}$ does not give rise to multivaluedness of this norm. Matrices for finite rotations for the representation $\mathfrak{D}^{(\alpha)}$ are found and addition theorems for the matrix elements $\mathfrak{D}^{(\alpha)}_{mn}(g)$ are obtained.

2. FORMULATION OF THE PROBLEM

Let us consider the eigenvalue problem for the square of the orbital angular momentum operator $\hat{\mathbf{L}}^2$:

$$\hat{\mathbf{L}}^2 f_{\alpha m}(\mathbf{n}) = -\left\{\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)\frac{\partial}{\partial\zeta}\right] + \frac{1}{1-\zeta^2}\frac{\partial^2}{\partial\varphi^2}\right\} f_{\alpha m}(\mathbf{n}) = \alpha(\alpha+1) f_{\alpha m}(\mathbf{n}),$$

$$\hat{L}_z f_{\alpha m}(\mathbf{n}) = -i\frac{\partial}{\partial\varphi} f_{\alpha m}(\mathbf{n}) = m f_{\alpha m}(\mathbf{n});$$

$$\mathbf{n} = \mathbf{r}/r, \quad \zeta = \cos\vartheta, \quad \hbar = 1. \tag{1}$$

In what follows we confine ourselves to the class of functions $f_{\alpha m}(\mathbf{n})$ singlevalued on the unit sphere. Then m may take on only integer values:

1) Private communication from I. S. Shapiro.

$m = 0, \pm 1, \pm 2, \ldots$ If one imposes on the eigenfunctions $f_{\alpha m}(\mathbf{n})$ the condition of boundedness then one arrives at the well-known results of quantum mechanics:[9]

1) The problem has a solution only for integer positive values $\alpha = l = 0, 1, 2, \ldots$ and they exhaust the spectrum of eigenvalues of the operator $\hat{\mathbf{L}}^2$.

2) The eigenfunctions $f_{lm}(\mathbf{n})$ have finite norm with the conventional definition of the scalar product

$$(f, g) = \int f^*(\mathbf{n})\, g(\mathbf{n})\, d\Omega_{\mathbf{n}}, \quad d\Omega_{\mathbf{n}} = \sin\vartheta d\vartheta d\varphi \tag{2}$$

and, when normalized to unity, coincide with the spherical harmonics $Y_{lm}(\vartheta, \varphi)$.

3) The set of functions f_{lm} ($m = -l, -l+1, \ldots, l$) provides a basis for an irreducible representation $D^{(l)}$ of weight l of the rotation group in three-dimensional space.

However Eqs. (1) have solutions for arbitrary complex values α.[10] Up to now these solutions have not been considered in quantum mechanics because they are not bounded on the unit sphere and because their norm, calculated according to (2), turns out to be infinite.

The Eqs. (1) have two linearly independent solutions, which we shall take for noninteger α to be of the form

$$Y_{\alpha m}(\mathbf{n}) = \left(\frac{2\alpha+1}{4\pi}\right)^{1/2} \omega(\alpha, m)\, P_\alpha^m(\zeta)\, e^{im\varphi},$$

$$Z_{\alpha m}(\mathbf{n}) = e^{i\pi(\alpha+m)} \left(\frac{2\alpha+1}{4\pi}\right)^{1/2} \omega(\alpha, m)\, P_\alpha^m(-\zeta)\, e^{im\varphi}$$

$$= e^{i\pi\alpha}\, Y_{\alpha m}(-\mathbf{n}). \tag{3}$$

Here $P_\alpha^m(\zeta)$ is the associated Legendre function[10] and we have introduced the notation

$$\omega(\alpha, m) = [\Gamma(\alpha + 1 - m)/\Gamma(\alpha + 1 + m)]^{1/2}, \tag{4}$$

where $\Gamma(x)$ stands for the gamma function. Since

$$Y_{\alpha m}(\mathbf{n}) = iY_{-\alpha-1, m}(\mathbf{n}), \quad Z_{\alpha m}(\mathbf{n}) = ie^{2\pi i\alpha} Z_{-\alpha-1, m}(\mathbf{n}),$$

one may assume everywhere that $\mathrm{Re}\,\alpha > -\frac{1}{2}$. The behavior of $Y_{\alpha m}(\mathbf{n})$ and $Z_{\alpha m}(\mathbf{n})$ at the singular points of the differential equation (1) $\zeta = \pm 1$ is given by the following formulas

$$Y_{\alpha m}(\mathbf{n}) \sim e^{im\varphi} a_{\alpha m} (1-\zeta)^{|m|/2} \quad \text{as } \zeta \to 1,$$

$$Y_{\alpha m}(\mathbf{n}) \sim e^{im\varphi} b_{\alpha m} \begin{cases} (1+\zeta)^{-|m|/2} |m|^{-1}, & m \neq 0 \\ -\ln(1+\zeta), & m = 0 \end{cases} \quad \text{as } \zeta \to -1$$

$$Z_{\alpha m}(\mathbf{n}) \sim e^{im\varphi} c_{\alpha m} \begin{cases} (1-\zeta)^{-|m|/2} |m|^{-1}, & m \neq 0 \\ -\ln(1-\zeta), & m = 0 \end{cases} \quad \text{as } \zeta \to 1,$$

$$Z_{\alpha m}(\mathbf{n}) \sim e^{im\varphi} d_{\alpha m} (1+\zeta)^{|m|/2} \quad \text{as } \zeta \to -1;$$

$$a_{\alpha m} = \left(\frac{2\alpha+1}{4\pi}\right)^{1/2} \frac{\varepsilon_{-m}\, \omega(\alpha, -|m|)}{2^{|m|/2}(|m|)!},$$

$$b_{\alpha m} = -\frac{\sin\pi\alpha}{\pi} \left(\frac{2\alpha+1}{4\pi}\right)^{1/2} \varepsilon_m 2^{|m|/2} (|m|)!\, \omega(\alpha, |m|),$$

$$c_{\alpha m} = (-)^m e^{i\pi\alpha} b_{\alpha m},$$

$$d_{\alpha m} = (-)^m e^{i\pi\alpha} a_{\alpha m},$$

$$\varepsilon_m = \begin{cases} 1, & m \geqslant 0 \\ (-)^m, & m \leqslant 0 \end{cases}. \tag{5}$$

The effect of the operators $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$, $\hat{L}_z$ on the functions $Y_{\alpha m}(\mathbf{n})$ ($Z_{\alpha m}(\mathbf{n})$) is as follows:

$$\hat{L}_\pm Y_{\alpha m}(\mathbf{n}) = [(\alpha \mp m)(\alpha + 1 \pm m)]^{1/2} Y_{\alpha, m\pm 1}(\mathbf{n}),$$

$$\hat{L}_z Y_{\alpha m}(\mathbf{n}) = m Y_{\alpha m}(\mathbf{n}). \tag{6}$$

Formulas (6) are analogous to the corresponding formulas for the angular momentum operators $\hat{L}_i$ of quantum mechanics[9] and differ from them only in the replacement of the integer l by an arbitrary complex number α. In view of this analogy, the question naturally arises as to whether it might not be possible to interpret the complex angular momentum from the point of view of the theory of representations of the rotation group. It is then necessary to answer the following questions.

A) Is it possible to define for the functions $f_{\alpha m}(\mathbf{n})$ with complex α a norm $\|f_{\alpha m}\|$ which fulfills the following requirements:

1) the norm is invariant under rotations;

2) the norm is finite and nonzero for arbitrary α and m;

3) the correspondence is satisfied, i.e., for $\alpha = l$ the new norm coincides with the conventional one (2).

B) Does there exist an irreducible representation of the rotation group with complex weight α and do the functions $f_{\alpha m}(\mathbf{n})$ for fixed α provide a basis for this representation?

3. DEFINITION OF THE INVARIANT NORM

We shall look for an invariant norm for the functions $Y_{\alpha m}(\mathbf{n})$, satisfying conditions 1)–3), in the form

$$\|Y_{\alpha m}\|^2 = \int [\hat{N} Y_{\alpha^* m}(\mathbf{n})]^* Y_{\alpha m}(\mathbf{n})\, d\Omega_{\mathbf{n}}, \tag{7}$$

where $\hat{N}$ is some linear operator. Complex conjugation over α in $Y_{\alpha^* m}$ ensures analyticity of $\|Y_{\alpha m}\|$ in the angular momentum α. Under a rotation g the function $Y_{\alpha m}(\mathbf{n})$ transforms as follows:

$$\hat{T}_g Y_{\alpha m}(\mathbf{n}) = Y_{\alpha m}(g^{-1}\mathbf{n}). \tag{8}$$

From the requirement of invariance of the norm [condition 1)] follow the commutation relations

$$\hat{T}_g \hat{N} = \hat{N} \hat{T}_g, \tag{9}$$

$$\hat{L}_i \hat{N} = \hat{N} \hat{L}_i, \quad i = x, y, z. \tag{10}$$

Setting $\tilde{Y}_{\alpha m}(\mathbf{n}) = \hat{N} Y_{\alpha m}(\mathbf{n})$ we obtain from (10)

$$\hat{\mathbf{L}}^2\tilde{Y}_{\alpha m} = \alpha(\alpha+1)\tilde{Y}_{\alpha m},$$
$$\hat{L}_z\tilde{Y}_{\alpha m} = m\tilde{Y}_{\alpha m},$$
$$\hat{L}_\pm\tilde{Y}_{\alpha m} = [(\alpha\mp m)(\alpha+1\pm m)]^{1/2}\tilde{Y}_{\alpha, m\pm 1}, \qquad (11)$$

hence

$$\tilde{Y}_{\alpha m}(\mathbf{n}) = \left(\frac{4\pi}{2\alpha+1}\right)^{1/2}[A_\alpha Y_{\alpha m}(\mathbf{n}) + B_\alpha Y_{\alpha m}(-\mathbf{n})] \qquad (12)$$

and, therefore, the operator $\hat{N}$, satisfying condition 1), is of the form

$$\hat{N} = A(\alpha)\hat{I} + B(\alpha)\hat{P}; \qquad (13)$$

where $\hat{I}$ is the identity operator, $\hat{P}$ is the operator for spatial inversion, and $A(\alpha)$ and $B(\alpha)$ are arbitrary functions of α. In order that the norm be finite [condition 2)] we must have $A(\alpha) = 0$. The norm (7) takes on the form

$$\|Y_{\alpha m}\|^2 = B(\alpha)\int [Y_{\alpha^* m}(-\mathbf{n})]^* Y_{\alpha m}(\mathbf{n})\, d\Omega_{\mathbf{n}}. \qquad (14)$$

From condition 3) it follows that

$$B(l) = (-1)^l \qquad (15)$$

for integer $\alpha = l$. Since the analytic continuation of the function $B(\alpha)$ from the discrete set of values (15) is not unique, the requirement of analyticity of the norm in the angular momentum does not uniquely determine the form of the function $B(\alpha)$. Choosing $B(\alpha) = e^{i\pi\alpha}$ we obtain

$$\|Y_{\alpha m}\|^2 = e^{i\pi\alpha}\int [Y_{\alpha^* m}(-\mathbf{n})]^* Y_{\alpha m}(\mathbf{n})\, d\Omega_{\mathbf{n}}. \qquad (16)$$

With the help of simple, but lengthy, calculations we obtain from (16) an explicit expression for the norm:

$$\|Y_{\alpha m}\|^2 = e^{i\pi\alpha}\Big\{\cos\pi\alpha - \frac{\sin\pi\alpha}{\pi}[\psi(\alpha+1+|m|) - \psi(\alpha+1-|m|)]\Big\}$$
$$= e^{i\pi\alpha}\Big\{\cos\pi\alpha - \frac{\sin\pi\alpha}{\pi}\sum_{k=-|m|}^{|m|-1}\frac{1}{\alpha-k}\Big\}. \qquad (17)$$

($\psi(x)$ is the logarithmic derivative of the Γ function[10]).

We note the following properties of the norm $\|Y_{\alpha m}\|$:

1. For complex α the norm is a complex number and does not satisfy the condition of positive-definiteness.

2. For noninteger α the norm depends on m, and as $|m| \to \infty$ the norm tends to zero:

$$\|Y_{\alpha m}\|^2 = -\frac{(2\alpha+1)e^{i\pi\alpha}\sin\pi\alpha}{\pi|m|}(1 + O(|m|^{-2})). \qquad (18)$$

However for an arbitrary finite value of m for $\mathrm{Im}\,\alpha \neq 0$ the norm vanishes nowhere.

3. The properties 1 and 2 show that the space of the functions $Y_{\alpha m}(\mathbf{n})$ differs substantially from the Hilbert space conventionally used in quantum mechanics. In the Hilbert space the following conditions are satisfied: the norm of an arbitrary nonzero vector is a positive number and if $\|\psi\| \to 0$ then the vector $\psi(x)$ tends to zero at almost all points x.[11] As can be seen from properties 1 and 2 both these conditions are violated for the function $Y_{\alpha m}(\mathbf{n})$.

4. All the zeros of $\|Y_{\alpha m}\|$ lie on the real axis $\mathrm{Im}\,\alpha = 0$. The number of zeros is infinite and they are distributed symmetrically with respect to the point $\alpha = -1/2$, which itself is a zero. For $-1/2 < \alpha \leq |m|$ the zeros lie at the points $\alpha = 0, 1, \ldots, |m| - 1$, and for $\alpha > |m|$ there is one zero in each interval $(n, n+1)$ where $n = |m|, |m|+1, \ldots$.

Analogous to (16) one may introduce the scalar product of any two functions $f_\alpha(\mathbf{n})$ and $h_{\alpha'}(\mathbf{n})$ that satisfy the equations $\hat{\mathbf{L}}^2 f_\alpha = \alpha(\alpha+1)f_\alpha$, $\hat{\mathbf{L}}^2 h_{\alpha'} = \alpha'(\alpha'+1)h_{\alpha'}$:

$$\langle f_\alpha, h_{\alpha'}\rangle = e^{i\pi\alpha}\int [f_{\alpha^*}(-\mathbf{n})]^* h_{\alpha'}(\mathbf{n})\, d\Omega_{\mathbf{n}}. \qquad (19)$$

The scalar product (19) is invariant under rotations and coincides for integer α and α' with the conventional scalar product (2).

We note that a certain amount of care is needed in the evaluation of scalar products of the type $\langle Y_{\alpha m}, Y_{\alpha m'}\rangle$ for $m \neq m'$. If one introduces into (19) $Y_{\alpha m}(\mathbf{n})$ from (3) one obtains an indeterminate expression of the form $0 \cdot \infty$, since the integral over φ gives zero and the integral over ϑ diverges at one of the limits of integration $\vartheta = 0$ or $\vartheta = \pi$. To resolve this indeterminacy it is necessary to first regularize the divergent integrals. Making use to this end of the method of Abel (see [12]) we find

$$\langle Y_{\alpha m}, Y_{\alpha' m'}\rangle = e^{i\pi\alpha}\int [Y_{\alpha^* m}(-\mathbf{n})]^* Y_{\alpha' m'}(\mathbf{n})\, d\Omega_{\mathbf{n}}$$
$$= \delta_{m,m'}\, e^{i\pi\alpha}\frac{[(2\alpha+1)(2\alpha'+1)]^{1/2}}{\pi(\alpha+\alpha'+1)(\alpha-\alpha')}$$
$$\times [\omega(\alpha,|m|)\,\omega(\alpha',-|m|)\sin\pi\alpha - \omega$$
$$\times (\alpha,-|m|)\,\omega(\alpha',|m|)\sin\pi\alpha'], \qquad (20)$$

from which it follows that the functions $Y_{\alpha m}$ and $Y_{\alpha' m}$ are, generally speaking, not orthogonal for $\alpha \neq \alpha'$, in contrast to the spherical harmonics Y_{lm} and $Y_{l'm}$ for $l \neq l'$.

4. IRREDUCIBLE REPRESENTATION OF THE ROTATION GROUP IN THE NEIGHBORHOOD OF THE IDENTITY

Let us now pass to the question formulated at the end of Sec. 2. Let there be three operators

$L_+ = L_x + iL_y$, $L_- = L_x - iL_y$ and L_z, which act on the infinite set of vectors $f_{\alpha m}$ ($m = 0, \pm 1, \pm 2, \ldots$) according to formulas analogous to (6):

$$L_\pm f_{\alpha m} = [(\alpha \mp m)(\alpha + 1 \pm m)]^{1/2} f_{\alpha, m\pm 1},$$
$$L_z f_{\alpha m} = m f_{\alpha m}. \quad (21)$$

Let us denote by R_α the space consisting of linear combinations of the basis vectors $f_{\alpha m}$ of the form

$$f_\alpha = \sum_{m=-\infty}^{\infty} c_m f_{\alpha m}$$

with complex coefficients c_m. From (21) follow the relations

$$[L_i, L_k] = i\varepsilon_{ikl} L_l,$$
$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2 = \alpha(\alpha + 1).$$

Since the commutation relations for the L_i coincide with the commutation relations for the infinitesimal operators of the rotation group, it follows from the theory of representations of Lie groups [13] that the L_i operators define in the space R_α a local representation $\mathfrak{D}^{(\alpha)}$ of the rotation group which certainly exists in the neighborhood of the identity rotation.

The totality of the vectors $f_{\alpha m}$, on which the L_i act according to the formulas (21), will be referred to as the canonical basis of the representation $\mathfrak{D}^{(\alpha)}$; the number α will be called the weight of the representation $\mathfrak{D}^{(\alpha)}$. It is seen from (21) that for noninteger α the space R_α contains no subspaces invariant with respect to the operators L_i. Therefore $\mathfrak{D}^{(\alpha)}$ is in this case an infinite-dimensional irreducible representation.[2] Either of the sets of functions $Y_{\alpha m}(\mathbf{n})$ or $Z_{\alpha m}(\mathbf{n})$ introduced in Sec. 2 may serve as a concrete example of a canonical basis for the representation $\mathfrak{D}^{(\alpha)}$.

The infinite dimensionality of the representation $\mathfrak{D}^{(\alpha)}$ for noninteger α makes the question of its existence for an arbitrary finite rotation nontrivial, since the expansion of the rotated vector $T_g f_{\alpha n}$ in terms of the initial vectors $f_{\alpha m}$ is given by an infinite series and questions arise regarding the convergence of this expansion (see Sec. 5). We emphasize that according to our definition of the canonical basis $f_{\alpha m}$ and according to Eq. (21), the matrices of the operators T_g for finite rotations are independent of the normalization of $f_{\alpha m}$ and have the same form in all canonical bases.

[2]Various non-equivalent definitions of irreducibility exist for infinite-dimensional representations.[14] We use the conventional definition consisting of the requirement of absence of invariant subspaces.

5. FINITE ROTATION MATRICES

For the determination of the finite rotation matrices it is convenient to consider the realization of the representation $\mathfrak{D}^{(\alpha)}$ of the rotation group on analytic functions $f(z)$ of a complex variable z, rather than on the functions defined on a sphere of the type of $Y_{\alpha m}(\mathbf{n})$. This method is a natural generalization of the method of generating polynomials of Cartan (see [15,16]).

Since to each rotation g, specified by the Euler angles $(\varphi_1, \vartheta, \varphi_2)$[3], there correspond two unitary matrices u:

$$g \to \pm u, \quad u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a = \cos\frac{\theta}{2} e^{-i(\varphi_1+\varphi_2)/2},$$
$$b = -i\sin\frac{\theta}{2} e^{-i(\varphi_1-\varphi_2)/2}, \qquad c = -b^*, \qquad d = a^*, \quad (22)$$

one may discuss instead of the representations of the rotation group the representations of the group of unitary matrices u.

Corresponding to each matrix u we introduce a linear transformation T_u on the function $f(z)$:

$$T_u f(z) = (bz + d)^{2\alpha} f\left(\frac{az + c}{bz + d}\right). \quad (23)$$

This transformation possesses the group property

$$T_{u_1} \cdot T_{u_2} = T_{u_1 u_2} \quad (24)$$

and defines a certain linear representation $u \to T_u$ on the analytic functions $f(z)$. Employing the usual method [16] we obtain the infinitesimal operators corresponding to the transformation (23):

$$\hat{L}_+ = L_x + iL_y = -z^2 d/dz + 2\alpha z,$$
$$\hat{L}_- = L_x - iL_y = d/dz, \qquad \hat{L}_z = z d/dz - \alpha. \quad (25)$$

Since in view of (25) the operator $\hat{\mathbf{L}}^2 = \alpha(\alpha + 1)$, the representation (23) coincides with the representation $\mathfrak{D}^{(\alpha)}$ of the rotation group defined in Sec. 4.

Let us introduce the functions

$$f_{\alpha m}(z) = C_{\alpha m} z^{\alpha + m},$$
$$C_{\alpha m} = C_\alpha [\Gamma(\alpha + 1 + m)\Gamma(\alpha + 1 - m)]^{-1/2},$$
$$m = 0, \pm 1, \pm 2, \ldots, \quad (26)$$

forming a canonical basis of the representation $\mathfrak{D}^{(\alpha)}$. From (23) we find the general form of the operator T_e corresponding to the identity rotation:

$$T_e f(z) = e^{2\pi i \alpha q} f(e^{i\pi(p-q)} z). \quad (27)$$

In particular

$$T_e f_{\alpha m}(z) = e^{2\pi i \alpha k} f_{\alpha m}(z), \quad (28)$$

[3]The Euler angles are defined as in Lyubarskiĭ's book.[13]

where $k = (p+q)/2$ is an integer. Therefore all operators T_u of the form $e^{2\pi i \alpha k}\hat{I}$ ($\hat{I}$ —the unity operator) correspond to the identity rotation. If α is a complex or real irrational number, then an infinite number of operators T_u corresponds to the identity rotation and the representation $\mathfrak{D}^{(\alpha)}$ is infinite-valued. This multivaluedness cannot be eliminated by a special choice of phases of the elements of the matrix u, however it does not give rise to multivaluedness in the invariant norm.

The matrix elements $\mathfrak{D}^{(\alpha)}_{mn}(u)$ of the operator T_u are defined as the coefficients of the convergent expansion of the transformed vector $T_u f_{\alpha n}(z)$ in terms of the canonical basis $f_{\alpha m}(z)$:

$$T_u f_{\alpha n}(z) = \sum_{m=-\infty}^{\infty} \mathfrak{D}^{(\alpha)}_{mn}(u) f_{\alpha m}(z). \tag{29}$$

As can be seen from (23), for noninteger α the function $T_u f_{\alpha n}(z)$ has two branch points:

$$z_1 = -\frac{c}{a} = i \operatorname{tg}\frac{\theta}{2} e^{i\varphi_1}, \qquad z_2 = -\frac{d}{b} = -i \operatorname{ctg}\frac{\theta}{2} e^{i\varphi_1}, \tag{29a*}$$

$z_1 z_2^* = -1$. Therefore, in contrast to finite-dimensional representations, the expansion $T_u f_{\alpha n}(z)$ in the basis $f_{\alpha m}(z)$ does not converge in the entire z plane but only in the ring K(u):

$$\operatorname{tg}\frac{\theta}{2} < |z| < \operatorname{ctg}\frac{\theta}{2} \quad \text{for} \quad 0 \leqslant \theta < \frac{\pi}{2},$$
$$\operatorname{ctg}\frac{\theta}{2} < |z| < \operatorname{tg}\frac{\theta}{2} \quad \text{for} \quad \frac{\pi}{2} < \theta \leqslant \pi. \tag{30}$$

The $\mathfrak{D}^{(\alpha)}_{mn}(u)$ are the coefficients in the Laurent series of the function $z^{-\alpha} T_u f_{\alpha n}(z)$ in the ring K(u) and are therefore determined by the formula

$$\mathfrak{D}^{(\alpha)}_{mn}(u) = \frac{1}{2\pi i}\frac{C_{\alpha n}}{C_{\alpha m}} \oint_{\Gamma} \frac{(az+c)^{\alpha+n}(bz+d)^{\alpha-n}}{z^{\alpha+m+1}} dz, \tag{31}$$

where Γ is a closed contour enclosing in a positive direction the branch points at $z = 0$, $z = z_1$ for $\theta < \pi/2$ and the branch points $z = 0$, $z = z_2$ for $\theta > \pi/2$. Making use of the expressions for a, b, c, d in terms of the Euler angles (22) and the representation of the hypergeometric function in terms of a contour integral [17] one may reduce $\mathfrak{D}^{(\alpha)}_{mn}(u)$ to the following form:

$$\mathfrak{D}^{(\alpha)}_{mn}(u) = \begin{cases} D^{(\alpha)}_{mn}(u) & \text{for} \quad 0 \leqslant \theta < \pi/2 \\ \Delta^{(\alpha)}_{mn}(u) & \text{for} \quad \pi/2 < \theta \leqslant \pi \end{cases}. \tag{32}$$

Here $D^{(\alpha)}_{mn}(u)$ and $\Delta^{(\alpha)}_{mn}(u)$ are two different[4)] analytic functions of $\zeta = \cos\theta$, defined by

$$D^{(\alpha)}_{mn}(u) = A^{\alpha}_{mn}(1-\tau)^p \tau^q F$$
$$\times(-\alpha+p+q,\ \alpha+1+p+q;\ 2q+1;\ \tau)\, e^{-i(m\varphi_1+n\varphi_2)},$$
$$\Delta^{(\alpha)}_{mn}(u) = B^{\alpha}_{mn}(1-\tau)^p \tau^q F$$
$$\times(-\alpha+p+q,\ \alpha+1+p+q;\ 2p+1;\ 1-\tau)\, e^{-i(m\varphi_1+n\varphi_2)},$$
$$A^{\alpha}_{mn} = (-i)^{2q}\,\omega(\alpha, -p-q)\,\omega(\alpha, p-q)/(2q)!,$$
$$B^{\alpha}_{mn} = e^{-i\pi\alpha} i^{2p}\,\omega(\alpha, -p-q)\,\omega(\alpha, -p+q)/(2p)!,$$
$$p = \tfrac{1}{2}|m+n|, \qquad q = \tfrac{1}{2}|m-n|,$$
$$\tau = (1-\zeta)/2 = \sin^2(\theta/2). \tag{33}$$

The functions $D^{(\alpha)}_{mn}(u)$ and $\Delta^{(\alpha)}_{mn}(u)$ satisfy the boundary conditions

$$D^{(\alpha)}_{mn}(0,0,0) = \delta_{mn}, \qquad \Delta^{(\alpha)}_{mn}(0,\pi,0) = e^{-i\pi\alpha}\delta_{m,-n} \tag{33a}$$

and their behavior at the points $\zeta = 1$ and $\zeta = -1$ is given by

$$D^{(\alpha)}_{mn}(u) \sim e^{-i(m\varphi_1+n\varphi_2)}(-i)^{2q} f^{\alpha}_{pq}(1-\zeta)^q \quad \text{for} \quad \zeta \to 1,$$

$$D^{(\alpha)}_{mn}(u) \sim -e^{-i(m\varphi_1+n\varphi_2)} i^{2p} \frac{\sin\pi\alpha}{\pi} g^{\alpha}_{pq}$$
$$\times \begin{cases} (1+\zeta)^{-p}/2p, & p \neq 0 \\ -\ln(1+\zeta), & p = 0 \end{cases} \quad \text{for} \quad \zeta \to -1;$$

$$\Delta^{(\alpha)}_{mn}(u) \sim -e^{-i(m\varphi_1+n\varphi_2)}(-i)^{2q} e^{-i\pi\alpha}\frac{\sin\pi\alpha}{\pi}(f^{\alpha}_{pq})^{-1}$$
$$\times \begin{cases} (1-\zeta)^{-q}/2q, & q \neq 0 \\ -\ln(1-\zeta), & q = 0 \end{cases} \quad \text{for} \quad \zeta \to 1,$$

$$\Delta^{(\alpha)}_{mn}(u) \sim e^{-i(m\varphi_1+n\varphi_2)} i^{2p} e^{-i\pi\alpha}(g^{\alpha}_{pq})^{-1}(1+\zeta)^p \quad \text{for} \quad \zeta \to -1;$$
$$f^{\alpha}_{pq} = 2^{-q}\omega(\alpha, -p-q)\,\omega(\alpha, p-q)/(2q)!,$$
$$g^{\alpha}_{pq} = 2^{p}\omega(\alpha, p+q)\,\omega(\alpha, p-q)\,(2p)!. \tag{34}$$

It follows from (32) that $\mathfrak{D}^{(\alpha)}_{mn}(u)$ is not analytic in the entire complex ζ plane and may be represented in the form

$$\mathfrak{D}^{(\alpha)}_{mn}(u) = D^{(\alpha)}_{mn}(u)\,\theta(\operatorname{Re}\zeta) + \Delta^{(\alpha)}_{mn}(u)\,\theta(-\operatorname{Re}\zeta),$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. Thus the line $\operatorname{Re}\zeta = 0$ is completely covered by singularities of the function $\mathfrak{D}^{(\alpha)}_{mn}(u)$. For example, in the simplest case $m = n = 0$ we have

$$\mathfrak{D}^{(\alpha)}_{00}(u) = \begin{cases} P_\alpha(\zeta) & \text{for} \quad \operatorname{Re}\zeta > 0 \\ e^{-i\pi\alpha}P_\alpha(-\zeta) & \text{for} \quad \operatorname{Re}\zeta < 0 \end{cases}.$$

Let us go on now to the derivation of addition theorems for the matrices $D^{(\alpha)}(u)$ and $\Delta^{(\alpha)}(u)$. We consider first the case when the rotations u_1 and u_2 and their product u_1u_2 correspond to Euler angles θ_1, θ_2, and θ_{12} less than $\pi/2$. According to (32), for such rotations $\mathfrak{D}^{(\alpha)}_{mn}(u) = D^{(\alpha)}_{mn}(u)$. Applying in succession the operators T_{u_2} and T_{u_1} to the basis vector $f_{\alpha n}(z)$ we obtain

$$T_{u_1}T_{u_2}f_{\alpha n}(z) = \sum_{k,\,m=-\infty}^{\infty} D^{(\alpha)}_{mk}(u_1)\,D^{(\alpha)}_{kn}(u_2)\,f_{\alpha m}(z). \tag{35}$$

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*tg = tan, ctg = cot.

4)For integer $\alpha = l$ the functions $D^{(\alpha)}_{mn}(u)$ and $\Delta^{(\alpha)}_{mn}(u)$ coincide with each other and with the conventional $D^{(l)}_{mn}(u)$ for finite-dimensional rotations.[13]

The series in question converges for values of z satisfying simultaneously the two conditions:

$$z \in K(u_1), \qquad w \in K(u_2), \tag{35a}$$

where

$$w = (a_1 z + c_1)/(b_1 z + d_1).$$

On the other hand

$$T_{u_1u_2} f_{\alpha n}(z) = \sum_{m=-\infty}^{\infty} D_{mn}^{(\alpha)}(u_1u_2) f_{\alpha m}(z), \tag{36}$$

where the series on the right converges provided

$$z \in K(u_1u_2). \tag{36a}$$

It is not hard to see that for $0 \le \theta_1, \theta_2, \theta_1+\theta_2 < \pi/2$ there exists in the complex z plane a certain two-dimensional region for whose points the conditions (35a) and (36a) are simultaneously satisfied and where the series (35) and (36) converge. Since $T_{u_1}T_{u_2}f_{\alpha n}(z) \equiv T_{u_1u_2}f_{\alpha n}(z)$ we deduce the following addition theorem

$$\sum_{k=-\infty}^{\infty} D_{mk}^{(\alpha)}(u_1) D_{kn}^{(\alpha)}(u_2) = D_{mn}^{(\alpha)}(u_1u_2). \tag{37}$$

Both sides of Eq. (37) are analytic functions of $\zeta_i = \cos\theta_i$, which coincide for $\theta_1+\theta_2 < \pi/2$. Since the series appearing on the left side converges in the larger region $\theta_1+\theta_2 < \pi$, it follows from the uniqueness of analytic continuation that the equality (37) holds also in this larger region of angles.

In an analogous fashion addition theorems may be obtained for other cases. We give the results:

$$\sum_{k=-\infty}^{\infty} D_{mk}^{(\alpha)}(u_1) D_{kn}^{(\alpha)}(u_2) = \begin{cases} D_{mn}^{(\alpha)}(u_1u_2), & \theta_1+\theta_2 < \pi \\ \infty, & \theta_1+\theta_2 > \pi \end{cases}, \tag{38a}$$

$$\sum_{k=-\infty}^{\infty} \Delta_{mk}^{(\alpha)}(u_1) \Delta_{kn}^{(\alpha)}(u_2) = \begin{cases} \infty, & \theta_1+\theta_2 < \pi \\ e^{-2\pi i\alpha} D_{mn}^{(\alpha)}(u_1u_2) & \theta_1+\theta_2 > \pi \end{cases} \tag{38b}$$

$$\sum_{k=-\infty}^{\infty} D_{mk}^{(\alpha)}(u_1) \Delta_{kn}^{(\alpha)}(u_2) = \begin{cases} \Delta_{mn}^{(\alpha)}(u_1u_2), & \theta_1 < \theta_2 \\ \infty, & \theta_1 > \theta_2 \end{cases}. \tag{38c}$$

For integer $\alpha = l$ the Eqs. (38) go over into the well known addition theorems for the D-functions of finite-dimensional representations, [7] and for $m = n = 0$ and arbitrary α they coincide with the addition theorems for the associate Legendre functions $P_\alpha^m(\zeta)$. [10]

It follows from Eqs. (32) and (38a) that in the neighborhood of the identity rotation consisting of the rotations with $0 \le \theta < \pi/2$, the matrices $\mathfrak{D}^{(\alpha)}(u)$ possess the group property and the representation $\mathfrak{D}^{(\alpha)}$ of the rotation group exists, is single-valued and is continuous. However the matrices $\mathfrak{D}^{(\alpha)}(u)$ do not constitute on the whole a representation of the rotation group in a rigorous sense [7] because for arbitrary rotations u_1 and u_2 the equality $\mathfrak{D}^{(\alpha)}(u_1)\mathfrak{D}^{(\alpha)}(u_2) = \mathfrak{D}^{(\alpha)}(u_1u_2)$ may be violated.

Let us consider, for example, rotations for which the Euler angles $\varphi_1 = \varphi_2 = 0$, and for brevity let us denote $\mathfrak{D}^{(\alpha)}(0, \theta, 0)$ by $\mathfrak{D}^{(\alpha)}(\theta)$. If $0 \le \theta_1, \theta_2 < \pi/2$ then, according to (32), $\mathfrak{D}^{(\alpha)}(\theta_i) = D^{(\alpha)}(\theta_i)$ and as long as $\theta_1+\theta_2 < \pi/2$ we have $\mathfrak{D}^{(\alpha)}(\theta_1)\mathfrak{D}^{(\alpha)}(\theta_2) = \mathfrak{D}^{(\alpha)}(\theta_1+\theta_2)$. As soon as $\theta_1+\theta_2$ exceeds $\pi/2$ this equality is violated: $\mathfrak{D}^{(\alpha)}(\theta_1)\mathfrak{D}^{(\alpha)}(\theta_2) = D^{(\alpha)}(\theta_1+\theta_2) \ne \mathfrak{D}^{(\alpha)}(\theta_1+\theta_2)$ since for $\theta_1+\theta_2 > \pi/2$ we have $\mathfrak{D}^{(\alpha)}(\theta_1+\theta_2) = \Delta^{(\alpha)}(\theta_1+\theta_2)$.

However the group property of the matrices $\mathfrak{D}^{(\alpha)}(u)$ may be restored if one considers along with the convergent expansions in the basis $f_{\alpha m}(z)$ on an equal footing also the divergent series, having given a prescription for their summation. The transformation (23) in the space of the functions $f_{\alpha m}(z)$ forms a group [see (24)], and the violation of the group property for the matrices $\mathfrak{D}^{(\alpha)}(u)$ is closely related to the fact that the $\mathfrak{D}_{mn}^{(\alpha)}(u)$ are defined, according to (29), as the coefficients in the convergent expansion of the rotated vector $T_u f_{\alpha n}(z)$ in terms of the basis $f_{\alpha m}(z)$.

Let us consider the series

$$\sum_{m=-\infty}^{\infty} D_{mn}^{(\alpha)}(u) f_{\alpha m}(z) \tag{39}$$

for complex values of $\zeta = \cos\theta$. Individual terms of this series are analytic functions of ζ. For $\operatorname{Re}\zeta > 0$ the series (39) converges uniformly in the ring $K(\zeta)$: $|z_1| < |z| < |z_2|$, where, according to (29),

$$z_1 = i\left(\frac{1-\zeta}{1+\zeta}\right)^{1/2} e^{i\varphi_1}, \quad z_2 = -i\left(\frac{1+\zeta}{1-\zeta}\right)^{1/2} e^{i\varphi_1}, \quad |z_1|\,|z_2| = 1.$$

For $\operatorname{Re}\zeta = 0$ the ring of convergence collapses into the circumference $|z| = 1$, and for $\operatorname{Re}\zeta < 0$ we have $|z_1| > |z_2|$ and there exists no region in the z plane in which the series (39) converges. But since for $\operatorname{Re}\zeta > 0$ the sum of the series (39) coincides with the function $T_u f_{\alpha n}(z)$, analytic in the entire plane, it is seen that if we utilize for the summation of the series (39), divergent for $\operatorname{Re}\zeta < 0$, the method of analytic continuation [12] (in the parameter ζ) then $T_u f_{\alpha n}(z)$ is the generalized sum of the series (39) also for $\operatorname{Re}\zeta < 0$.

Analogous considerations apply also to the summation of the series

$$\sum_{m=-\infty}^{\infty} \Delta_{mn}^{(\alpha)}(u) f_{\alpha m}(z)$$

in the half-plane $\operatorname{Re}\zeta > 0$. As a result we may associate with each rotation u two matrices $D^{(\alpha)}(u)$ and $\Delta^{(\alpha)}(u)$, which are the coefficients in the expansion of the rotated vector $T_u f_{\alpha n}(z)$ in terms of the basis $f_{\alpha m}(z)$:

$$T_u f_{\alpha n}(z) = \sum_{m=-\infty}^{\infty} D_{mn}^{(\alpha)}(u) f_{\alpha m}(z) \equiv \sum_{m=-\infty}^{\infty} \Delta_{mn}^{(\alpha)}(u) f_{\alpha m}(z), \quad (40$$

where one of these expansion converges in the conventional sense and the other diverges (except for rotations u with $\theta = \pi/2$). Since the generalized sums of both series coincide with the same function $T_u f_{\alpha n}(z)$, the matrices $D^{(\alpha)}(u)$ and $\Delta^{(\alpha)}(u)$ define the same operator $\hat{T}_u$ and are in this sense equivalent, which we shall denote by

$$D_{mn}^{(\alpha)}(u) \sim \Delta_{mn}^{(\alpha)}(u). \quad (41)$$

It follows from here that the group property of the matrices $\mathfrak{D}^{(\alpha)}(u)$ for arbitrary rotations u_1 and u_2 is expressed by the equivalence relation

$$\mathfrak{D}^{(\alpha)}(u_1)\,\mathfrak{D}^{(\alpha)}(u_2) \sim \mathfrak{D}^{(\alpha)}(u_1 u_2), \quad (42)$$

which replaces the equality $D^{(l)}(u_1)\,D^{(l)}(u_2) = D^{(l)}(u_1 u_2)$ valid for conventional finite-dimensional representations. It follows from (42) that

$$\mathfrak{D}^{(\alpha)}(u_1)\,\mathfrak{D}^{(\alpha)}(u_2) \ldots \mathfrak{D}^{(\alpha)}(u_n) \sim \mathfrak{D}^{(\alpha)}(u_1 u_2 \ldots u_n) \quad (43)$$

for arbitrary rotations $u_1, u_2, \ldots, u_n$. This relieves us of the need of investigating the convergence of series that arise in the process of calculations: in the final result one must introduce the matrix $\mathfrak{D}^{(\alpha)}(u_1 u_2 \ldots u_n)$ determined according to (32) and finite for any rotation.

We present formulas for the transformation of the functions $Y_{\alpha m}(\mathbf{n})$ under rotations. From Eqs. (3), (33), and (38) we obtain

$$D_{m0}^{(\alpha)}(\varphi_1, \theta, \varphi_2) = \left(\frac{4\pi}{2\alpha+1}\right)^{1/2} Y_{\alpha m}\left(\theta, \frac{\pi}{2} - \varphi_1\right),$$

$$\Delta_{m0}^{(\alpha)}(\varphi_1, \theta, \varphi_2) = e^{-2\pi i \alpha}\left(\frac{4\pi}{2\alpha+1}\right)^{1/2} Z_{\alpha m}\left(\theta, \frac{\pi}{2} - \varphi_1\right); \quad (44)$$

$$\hat{T}_g Y_{\alpha m}(\mathbf{n}) = Y_{\alpha m}(g^{-1}\mathbf{n})$$

$$= \begin{cases} \sum_{k=-\infty}^{\infty} D_{km}^{(\alpha)}(g)\, Y_{\alpha k}(\mathbf{n}) & \text{for} \quad 0 \leqslant \vartheta < \pi - \theta \\ \sum_{k=-\infty}^{\infty} \Delta_{km}^{(\alpha)}(g)\, Z_{\alpha k}(\mathbf{n}) & \text{for} \quad \pi - \theta < \vartheta \leqslant \pi \end{cases}, \quad (45)$$

where $\mathbf{n} = (\vartheta, \varphi)$. It is seen from (45) that in order to obtain a convergent expansion for $Y_{\alpha m}(g^{-1}\mathbf{n})$ one must use both canonical bases ($Y_{\alpha m}(\mathbf{n})$ and $Z_{\alpha m}(\mathbf{n})$) on the unit sphere.

The definition of the scalar product (19) may be generalized to functions on the rotations group. In particular for the functions $D_{mn}^{(\alpha)}(g)$ we have

$$\langle D_{mn}^{(\alpha)}, D_{m'n'}^{(\alpha')}\rangle = \int dg\, \{\hat{P} D_{mn}^{(\alpha^*)}(g)\, \hat{P}^{-1}\}^* D_{m'n'}^{(\alpha')}(g)$$

$$= \int dg\, \{\Delta_{mn}^{(\alpha^*)}(g)\}^* D_{m'n'}^{(\alpha')}(g)$$

$$= \delta_{m,m'}\delta_{n,n'}\, e^{i\pi\alpha}\, \frac{\omega(\alpha, -p+q)\,\omega(\alpha', p-q)}{\alpha(\alpha+1) - \alpha'(\alpha'+1)}$$

$$\times \left[\frac{\omega(\alpha, p+q)}{\omega(\alpha', p+q)}\,\frac{\sin \pi\alpha}{\pi} - \frac{\omega(\alpha', p+q)}{\omega(\alpha, p+q)}\,\frac{\sin \pi\alpha'}{\pi}\right],$$

$$dg = \tfrac{1}{8}\pi^{-2} \sin\theta\, d\theta\, d\varphi_1\, d\varphi_2. \quad (46)$$

Thus the functions $D_{mn}^{(\alpha)}(g)$ and $D_{mn}^{(\alpha')}(g)$ are, in general, not orthogonal for $\alpha \neq \alpha'$, in contrast to the functions $D_{mn}^{(l)}(g)$ for the finite-dimensional representations which form an orthonormal set of functions on the rotations group.

For $\alpha = \alpha'$ we obtain from (46) the invariant norm for the functions $D_{mn}^{(\alpha)}(g)$:

$$\| D_{mn}^{(\alpha)} \|^2 = \frac{e^{i\pi\alpha}}{2\alpha+1}\Big\{\cos \pi\alpha$$

$$- \frac{\sin \pi\alpha}{\pi}\left[\psi(\alpha+1+p+q) - \psi(\alpha+1-p-q)\right]\Big\}. \quad (47)$$

In conclusion we note that after this work was written there appeared in print the paper of Beltrametti and Luzatto[18] devoted to the analogous problem. They also obtain, Eqs. (38) and (45) of our Sec. 5, but by a different method. The above-mentioned authors do not, however, consider the question of an invariant norm for the functions $Y_{\alpha m}(\mathbf{n})$, and their interpretation of the group properties of the matrices $\mathfrak{D}^{(\alpha)}(u)$ differs from ours.

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Note added in proof (April 2, 1964). It is interesting to observe that the invariant norm $\|Y_{\alpha m}\|$ [see (16) and (17)] is positive definite not only at the integer points ($\alpha = l$), but also on the half-line $\alpha = -\frac{1}{2} + i\tau$, $0 < \tau < +\infty$. This property distinguishes the points $\alpha = -\frac{1}{2} + i\tau$ from all other arbitrary points in the complex plane and, most likely, will turn out to be important in the expansion of functions defined on the unit sphere in terms of the functions $Y_{\alpha m}(\mathbf{n})$ (the analogue of the Plancherel theorem in the theory of Fourier transforms).

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