THE PRINCIPLE OF CORRESPONDENCE IN THE GENERAL THEORY OF RELATIVITY

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Submitted to JETP editor October 12, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 1741-1754 (May, 1964)

It is shown that in Newtonian theory the gravitational field can be described by means of a three-dimensional metrical form. This permits us to compare directly the relativistic gravitational theory with the Newtonian theory. A limiting process is described in which the gravitational fields of the general theory of relativity go over into the corresponding gravitational fields of the Newtonian theory. A calculation is made of the Newtonian representation of a gravitational field with a metric describing a space of the second type of maximum mobility (according to the classification of A. Z. Petrov). It is shown that by means of the limiting process referred to earlier a correspondence can be set up between quantities in the general theory of relativity and similar quantities in Newtonian gravitational theory.

1. G-SYSTEMS IN NEWTONIAN MECHANICS

We consider a gravitational field described by a potential Φ which is produced by masses of density ρ. Let x^i be rectangular cartesian coordinates in an inertial reference system, and ξ^i coordinates related to the coordinates x^i by the transformation[1)]

$$x^i = x^i(t, \xi^k), \qquad (1.1)$$

where t is Newtonian time. We shall refer to the coordinate system ξ^i as a G-system if it satisfies the following requirements:

1) the points $\xi^i = \text{const}$ move as freely falling particles, i.e.,[2)]

$$x^i{}_{,00} = \Phi_{,i}(t, x^k); \qquad (1.2)$$

2) the velocity field $x^i{}_{,0} = v_i(t, x^k)$ is irrotational, i.e., there exists a function $\varphi(t, x^i)$ such that

$$v_i = \varphi_{,i}(t, x^k). \qquad (1.3)$$

Moreover, one can assume without loss of generality that

$$x^i(0, \xi^k) = \xi^i. \qquad (1.4)$$

G-systems always exist. They can be determined in the following manner. According to condition 1) the functions $x^i(t, \xi^k)$ from the transformation (1.1) must satisfy the differential equations (1.2). Let the general solution of the system (1.2) be of the form

$$x^i = F^i(t, \xi^k, k^j). \qquad (1.5)$$

For the arbitrary constants we shall take the initial values

$$x^i(0, \xi^k) = \xi^i, \quad x^i_{,0}(0, \xi^k) = k^i. \qquad (1.6)$$

The relation

$$x^s{}_{,0k}x^s{}_{,i} - x^s{}_{,0i}\,x^s{}_{,k} = 0 \qquad (1.7)$$

is an integral of the system (1.2). Indeed, assuming (1.2) we have

$$(x^s{}_{,0k}x^s{}_{,i} - x^s{}_{,0i}x^s{}_{,k})_{,0} = x^s{}_{,00k}\,x^s{}_{,i} - x^s{}_{,00i}x^s{}_{,k}$$
$$= \Phi_{,sr}x^r{}_{,k}\,x^s{}_{,i} - \Phi_{,sr}x^r{}_{,i}\,x^s{}_{,k} = 0.$$

Consequently, if condition (1.7) is satisfied at the instant t = 0, then it will also be satisfied at any arbitrary time t. But Eq. (1.7) is the condition that the velocity field $x^i{}_{,0} = v_i$ should be irrotational, since (1.7) can be rewritten in the form $\bar{v}_{i,k} - \bar{v}_{k,i} = 0$, where $\bar{v}_i = v_s x^s{}_{,i} = x^s{}_{,0}x^s{}_{,i}$ are the components of the velocity v_i in the coordinate system ξ^i. From this it follows that it is sufficient to satisfy condition 2) only at the initial moment t = 0 by choosing the initial conditions (1.6) taking (1.3) into account:

$$k^i = \varphi_{,i}(0, \xi^k) = u_{,i}(\xi^k), \qquad (1.8)$$

[1)]Greek indices take on the values 0, 1, 2, 3, while Latin indices take on the values 1, 2, 3.

[2)]A comma in front of a subscript denotes ordinary differentiation, with the subscript zero indicating differentiation with respect to time. Thus, for example, $x^i{}_{,00} = \partial^2 x^i/\partial t^2$, $\Phi_{,i} = \partial\Phi/\partial x^i$, $x^i{}_{,k} = \partial x^i/\partial\xi^k$ etc. Covariant differentiation is denoted by a semi-colon.

where we have introduced the notation $u(\xi^i) = \varphi(0, \xi^i)$. By introducing expressions (1.8) into the right-hand side of (1.5) we obtain the transformation

$$x^i = F^i(t, \xi^k, u_{,j}), \tag{1.9}$$

which satisfies conditions 1) and 2), and which, consequently, defines the G-system ξ^i. The function $u(\xi^i)$ can, evidently, be chosen in an arbitrary manner. The set of G-systems is equivalent to the set of differentiable functions $u(\xi^i)$ of the three spatial coordinates ξ^i.

Let the expression for a line element in a G-system be of the form

$$ds^2 = \gamma_{rs}\, d\xi^r d\xi^s. \tag{1.10}$$

Then we have

$$\gamma_{ik} = x^s{}_{,i}\, x^s{}_{,k}, \tag{1.11}$$

$$\gamma^{rs} x^i{}_{,r} x^k{}_{,s} = \delta^{ik}. \tag{1.12}$$

We shall show that the coefficients γ_{ik} satisfy the system of equations

$$R_{00} = -4\pi G\rho, \qquad R_{0ijk} = 0, \qquad P_{ik} = 0, \tag{1.13}$$

where

$$R_{00} = \partial^2 \ln \sqrt{\gamma}/\partial t^2 + \Gamma^r_s \Gamma^s_r, \tag{1.14}$$

$$R_{0ijk} = \partial \Gamma_{ik}/\partial \xi^j - \partial \Gamma_{ij}/\partial \xi^k + \Gamma^s_{ik}\Gamma_{js} - \Gamma^s_{ij}\Gamma_{ks}, \tag{1.15}$$

$$P_{ik} = \partial^2 \ln \sqrt{\gamma}/\partial \xi^i \partial \xi^k - \partial \Gamma^s_{ik}/\partial \xi^s + \Gamma^s_{ir}\Gamma^r_{sk} - \Gamma^s_{ik}\Gamma^r_{sr}. \tag{1.16}$$

Here Γ^k_{ij} are Christoffel symbols constructed from γ_{ik}, $\Gamma_{ik} = \frac{1}{2}\gamma_{ik,0}$, $\Gamma^k_i = \gamma^{ks}\Gamma_{is}$, $\gamma = \det(\gamma_{ik})$; G is the gravitational constant. Since (1.10) is the metric form for a Euclidean space, while (1.16) is the Ricci tensor for this space, then evidently $P_{ik} = 0$.

We now proceed to evaluate expression (1.14). Differentiating with respect to ξ^k the equation

$$x^i{}_{,0} = \varphi_{,i}(t, x^k) \tag{1.17}$$

which follows from (1.3), and having in mind that $\partial\varphi_{,i}/\partial\xi^k = \varphi_{,is}\, x^s{}_{,k}$ we obtain

$$x^i{}_{,0k} = \varphi_{,is} x^s{}_{,k}. \tag{1.18}$$

Taking into account this expression we shall obtain from (1.11)

$$\Gamma_{ik} = \varphi_{,rs} x^r{}_{,i}\, x^s{}_{,k} \tag{1.19}$$

and further

$$\Gamma_{ij,k} = x^s{}_{,ij}\, x^s{}_{,k}. \tag{1.20}$$

But if we differentiate expression (1.17) with respect to t, then in virtue of (1.2) we shall obtain

$$\Phi_{,i} = \varphi_{,i0} + \varphi_{,is}\varphi_{,s}, \tag{1.21}$$

from which it follows that

$$\Delta\Phi = \varphi_{,ss0} + \varphi_{,ssr}\varphi_{,r} + \varphi_{,rs}\varphi_{,rs}. \tag{1.22}$$

We now have in accordance with (1.12) and (1.19)

$$\Gamma^r_s \Gamma^s_r = \varphi_{,rs}\varphi_{,rs} \tag{1.23}$$

and $\partial \ln \sqrt{\gamma}/\partial t = \Gamma^s_s = \varphi_{,ss}$, from which it follows, on taking (1.22) into account, that

$$\partial^2 \ln \sqrt{\gamma}/\partial t^2 = \Delta\Phi - \varphi_{,rs}\varphi_{,rs}. \tag{1.24}$$

Adding (1.23) and (1.24) we obtain

$$R_{00} = \Delta\Phi. \tag{1.25}$$

From this it can be seen that the first equation of (1.13) is the Poisson equation for the potential Φ.

Finally, simple calculations show that on the basis of (1.12), (1.19) and (1.20) R_{0ijk} is equal to zero, so that the second equation of (1.13) will also be satisfied.

We now prove the converse: if the coefficients γ_{ik} satisfy the system (1.13), then (1.10) is a metric form written in the G-system, with the gravitational field being determined by masses of density ρ.

In order to prove this we consider the system

$$\lambda^k_{i,j} = \Gamma^s_{ij}\lambda^k_s, \qquad \lambda^k_{i,0} = \Gamma^s_i \lambda^k_s. \tag{1.26}$$

The integrability conditions for this system have the form

$$\partial\Gamma^k_{ij}/\partial\xi^h - \partial\Gamma^k_{ih}/\partial\xi^j + \Gamma^s_{ij}\Gamma^k_{sh} - \Gamma^s_{ih}\Gamma^k_{sj} = 0, \tag{1.27}$$

$$\partial\Gamma^k_{ij}/\partial t - \partial\Gamma^k_i/\partial\xi^j + \Gamma^s_{ij}\Gamma^k_s - \Gamma^s_i\Gamma^k_{sj} = 0. \tag{1.28}$$

The left-hand side of (1.27) contains the Riemann-Christoffel tensor composed of γ_{ik}. It is equal to zero in virtue of the third equation of the system (1.13), since in the case of a three-dimensional space the vanishing of the Ricci tensor results also in the vanishing of the curvature tensor. The left-hand side of (1.28) contains the expression $\gamma^{ks}R_{0jis}$, which vanishes as a result of the second equation of the system (1.13). Thus, the integrability conditions for the system (1.26) are satisfied.

The relation

$$\gamma^{rs}\lambda^i_r\lambda^k_s = \delta^{ik} \tag{1.29}$$

is an integral of the system (1.26). Indeed, it can be easily shown that in virtue of (1.26) and of the formulas

$$\gamma^{ik}{}_{,j} = -\gamma^{is}\Gamma^k_{js} - \gamma^{ks}\Gamma^i_{js}, \qquad \gamma^{ik}{}_{,0} = -\gamma^{is}\Gamma^k_s - \gamma^{ks}\Gamma^i_s \tag{1.30}$$

we have

$$(\gamma^{rs}\lambda_r^i\lambda_s^k)_{,\mu} = 0.$$

Consequently, we can determine nine functions $\lambda_i^k(t, \xi^j)$ which satisfy the system (1.26) and also the relation (1.29). For this it is sufficient to choose the initial values $(\lambda_i^k)_0$ at some fixed point $(\xi^i)_0$ at the instant t_0 in agreement with condition (1.29). We note that the functions λ_i^k are defined up to an arbitrary orthogonal transformation with constant coefficients.

As can be seen from (1.26), the functions λ_i^k also satisfy the condition $\lambda^k_{i,j} - \lambda^k_{j,i} = 0$ which is the integrability condition for the equations

$$x^k{}_{,i} = \lambda_i^k. \tag{1.31}$$

Therefore, the system (1.31) has solutions $x^i(t, \xi^k)$ which are defined up to arbitrary additive functions $f^i(t)$.

We now consider the transformation of coordinates

$$x^i = x^i(t, \xi^k). \tag{1.32}$$

It follows from (1.29) that

$$\gamma_{ik} = \lambda_i^s\lambda_k^s. \tag{1.33}$$

Consequently, we have

$$ds^2 = \gamma_{rs}d\xi^r d\xi^s = \lambda_r^p\lambda_s^p d\xi^r d\xi^s = dx^p dx^p,$$

from which it can be seen that x^i are orthogonal cartesian coordinates. Moreover, it can be easily shown that on the basis of (1.26) and (1.33) the functions (1.32) satisfy the relation (1.7), i.e., the condition that the motion of the points ξ^i = const should be irrotational with respect to the axes of the cartesian coordinates x^i.

Differentiation of (1.7) yields

$$x^s{}_{,00k}\, x^s{}_{,i} - x^s{}_{,00i}\, x^s{}_{,k} = 0. \tag{1.34}$$

Assuming that

$$x^i{}_{,00} = \Phi_i(t, x^k) \tag{1.35}$$

and noting that

$$x^i{}_{,00k} = \partial\Phi_i/\partial\xi^k = \Phi_{i,s}\, x^s{}_{,k},$$

we can write (1.34) in the form

$$(\Phi_{s,r} - \Phi_{r,s})\, x^s{}_{,i}\, x^r{}_{,k} = 0,$$

from which it follows that

$$\Phi_{i,k} - \Phi_{k,i} = 0.$$

Consequently, there exists a function $\Phi(t, x^i)$ such that $\Phi_i = \Phi_{,i}$. As a result of this (1.35) transforms into equation (1.2). Since the functions $x^i(t, \xi^k)$ are defined up to arbitrary additive functions $\ddot{f}^i(t)$, then the quantities Φ_i, are in accordance with (1.35) determined up to arbitrary additive functions $\ddot{f}^i(t)$, and the function Φ is consequently determined up to an additive term of the form $\ddot{f}^s x^s + f(t)$. This ambiguity can be removed with the aid of limiting conditions, for example by requiring that $\Phi = 0$ at infinity.

Finally, by repeating the calculations carried out above it can be shown that Eq. (1.25) holds. From this it follows in virtue of the first equation of the system (1.13) that Φ satisfies the Poisson equation

$$\Delta\Phi = -4\pi G\rho. \tag{1.36}$$

Thus, Φ is the potential of the gravitational field produced by masses of density ρ, the points ξ^i = const move in accordance with (1.2) as freely falling particles, and their motion is irrotational. The converse theorem stated above is, therefore, proved.

We see that to each gravitational field there corresponds a metric form of the form (1.10), (1.13) and, conversely, to each such metric form there corresponds a gravitational field produced by material bodies of density ρ. This means that in Newtonian theory the gravitational field can be described by means of the metric form (1.10), the coefficients of which satisfy the system of equations (1.13). The latter equations can be regarded as the fundamental equations of the Newtonian theory of the gravitational field.

2. G-SYSTEMS IN THE GENERAL THEORY OF RELATIVITY

In the general theory of relativity a coordinate system is said to be semigeodesic if in these coordinates the four-dimensional metric form is of the form

$$ds^2 = c^2\,dt^2 - \gamma_{rs}\,d\xi^r\,d\xi^s. \tag{2.1}$$

Here the time-like coordinate lines are geodesic, i.e., they are world lines of freely falling particles, while space-like coordinate hypersurfaces intersect the congruence of the parametric lines of t orthogonally. We shall say that a semigeodesic coordinate system is a G-system if as $c \to \infty$ the coefficients γ_{ik} and their derivatives have finite limiting values.

The coefficients γ_{ik} satisfy the equations of the gravitational field

$$R_{\mu\nu} = -\varkappa\left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T\right), \tag{2.2}$$

$$\varkappa = 8\pi G/c^2, \tag{2.3}$$

$$g_{00} = c^2, \quad g_{i0} = 0, \quad g_{ik} = -\gamma_{ik}. \tag{2.4}$$

Here R_{00} is given by formula (1.14),

$$R_{i0} = \gamma^{rs} R_{0ris}, \quad (2.5)$$

where R_{0ijk} is defined by formula (1.15),

$$R_{ik} = P_{ik} + c^{-2} Q_{ik}, \quad (2.6)$$

$$Q_{ik} = -\partial\Gamma_{ik}/\partial t - \Gamma^s_s\Gamma_{ik} + \Gamma_{si}\Gamma^s_k + \Gamma_{sk}\Gamma^s_i, \quad (2.7)$$

P_{ik} is the Ricci tensor composed of γ_{ik}, $T_{\mu\nu}$ is the mass tensor,

$$T = c^{-2} T_{00} - \gamma^{rs} T_{rs}. \quad (2.8)$$

The mass tensor and the coefficients (2.4) can be written in the following general form[1]:

$$T_{\mu\nu} = \rho_0 V_\mu V_\nu + \frac{1}{c^2} \sum_{s=1}^{3} p_s V^{s|}{}_\mu V^{s|}{}_\nu, \quad (2.9)$$

$$V_0^2 - V^{s|}{}_0 V^{s|}{}_0 = c^2, \quad (2.10)$$

$$V_i V_0 - V^{s|}{}_i V^{s|}{}_0 = 0, \quad (2.11)$$

$$V_i V_k - V^{s|}{}_i V^{s|}{}_k = -\gamma_{ik}, \quad (2.12)$$

where ρ_0 is the invariant material density, p_i are the proper partial pressures, V^μ is the four-velocity of matter;

$$V_0 = c/\alpha, \qquad V_i = -v_i/c\alpha, \quad (2.13)$$

$$\alpha^2 = 1 - w^2/c^2, \qquad w^2 = \gamma^{rs} w_r w_s \quad (2.14)$$

w_i is the velocity of matter in the three-dimensional space of the coordinates ξ_i, $V^{i|}{}_\mu$ are three mutually orthogonal space-like unit vectors perpendicular to the unit vector V_μ.

3. TRANSITION TO THE NEWTONIAN LIMIT

The world-line of an arbitrary material point (including photons) satisfies the condition $ds^2 \geq 0$ or, in accordance with (2.1), the condition $v^2 \leq c^2$, where v is the value of the three-dimensional velocity of the particle. Consequently, in a G-system the constant c has the meaning of an upper limit on the velocities of material objects.

We assume that the constant c increases without limit, i.e., following the example of Newtonian mechanics, we assume the possibility in principle of arbitrarily large velocities of material objects. Further, we shall assume that as $c \to \infty$ all the ratios of the form v/c, where v is the velocity of some material particle, tend to zero. If the limiting value of the constant c is taken to be infinite the last assumption does not impose any limitation on the magnitude of the velocities of material bodies; even infinite velocities are not excluded.

Under the assumptions made above we have the following limiting values. From (2.13) and (2.14) it follows that[3)]

$$\alpha \to 1, \qquad V_0/c \to 1, \qquad cV_i \to -w_i. \quad (3.1)$$

Taking into account that in accordance with (2.10) and (2.13) we have $V^{s|}{}_0 V^{s|}{}_0 = w^2/\alpha^2$, we obtain then

$$V^{s|}{}_0 V^{s|}{}_0 \to w^2, \quad (3.2)$$

and from this we conclude that

$$c^{-1} V^{i|}{}_0 \to 0. \quad (3.3)$$

From (2.12) it follows in virtue of (3.1) that

$$\gamma_{ik} \to V^{s|}{}_i V^{s|}{}_k. \quad (3.4)$$

Now, having in mind (3.1) and (3.3) we obtain

$$\varkappa T_{00} \to 8\pi G\rho_0, \quad (3.5)$$

$$T_{i0} \to -\rho_0 w_i, \quad (3.6)$$

$$c^2 T_{ik} \to \rho_0 w_i w_k + p_{ik}, \quad (3.7)$$

where we have introduced the notation

$$p_{ik} = \sum_{s=1}^{3} p_s V^{s|}{}_i V^{s|}{}_k, \quad (3.8)$$

and further

$$\varkappa T_{i0} \to 0, \qquad T_{ik} \to 0, \qquad c^2 \varkappa T \to 8\pi G\rho_0. \quad (3.9)$$

The expression Q_{ik} contains only the coefficients γ_{ik} and their derivatives which, in accordance with the properties of a G-system, have finite limiting values as $c \to \infty$. Therefore

$$Q_{ik}/c^2 \to 0. \quad (3.10)$$

We now proceed to investigate the transformation of coordinates

$$\xi^i = \xi^i(t, \bar{\xi}^k), \qquad \bar{t} = \bar{t}(t, \xi^k), \quad (3.11)$$

which gives the relation between two G-systems (cf., Appendix). We shall refer to the set of G-systems in which their velocities with respect to one another satisfy the requirement $\lim_{c\to\infty} (v^i/c) = 0$ as a G-family. It can be easily shown that in the case of a G-family as $c \to \infty$ the relativistic equations for the relative velocity (A.11) and (A.12) go over into the Newtonian equations (A.4) and (A.5). This means that as $c \to \infty$ the relativistic G-family goes over into the set of coordinate systems interrelated by the same transformations as the New-

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3)Naturally w_i, w^2, $V^{i|}_\mu$, ρ_0, p_i contained in the right-hand sides of (3.1), (3.2), (3.4) – (3.8) should be interpreted as the limiting values of the corresponding relativistic quantities.

tonian G-systems in a certain gravitational field. These transformations are obtained from (3.11) by means of the transition to the limit $c \to \infty$ and have the form

$$\xi^i = \xi^i(t, \bar{\xi}^k), \qquad \bar{t} = t, \tag{3.12}$$

since on the basis of (A.8) $\bar{t}_{,0} \to 1$, $\bar{t}_{,i} \to 0$. In the limit $c \to \infty$ the metric tensor (2.4) can be decomposed into an infinite invariant g_{00} and the three-dimensional tensor with respect to the transformations (3.12)

$$\tilde{\gamma}_{ik} = \lim_{c\to\infty} \gamma_{ik}, \tag{3.13}$$

as can be seen from (A.9).

We now turn to the field equations (2.2) written in the G-system. As a result of the transition to the limit $c \to \infty$ they go over, in accordance with (3.5), (3.9) and (3.10), into the equations

$$R_{00} = -4\pi G\rho_0, \qquad R_{i0} = 0, \qquad P_{ik} = 0, \tag{3.14}$$

which are satisfied by the limiting values (3.13). The first and the third equations of the system (3.14) coincide with the first and the third equations of the system (1.13), while the second equation (3.14) differs from the second equation of the system (1.13). However, if the gravitational field has the property that in a G-system

$$\lim_{c\to\infty} R_{0ijk} = 0, \tag{3.15}$$

then as $c \to \infty$ Einstein's equations (2.2) written in the G-system go over into the Newtonian equations (1.13). The limiting values (3.13) which now satisfy the system (1.13) consequently define a certain gravitational field with the potential Φ in the sense of the Newtonian theory. This field corresponds to a distribution of matter of density ρ_0 and can be regarded as a Newtonian representation of the relativistic gravitational field γ_{ik} under consideration.

We shall show that the Newtonian representation of the gravitational field γ_{ik} does not depend on the choice of the G-system within some particular G-family and is therefore defined by means of the G-family. Let the system of coordinates $\bar{t}$, $\bar{\xi}^i$ be some other G-system from the given G-family and let

$$\lim_{c\to\infty} \bar{\gamma}_{ik} = \tilde{\tilde{\gamma}}_{ik}.$$

Since in the limit $c \to \infty$ the system of coordinates t, ξ^i goes over into a Newtonian G-system determined in a gravitational field of potential Φ, and, as has been pointed out earlier, in the limit the coordinate systems t, ξ^i and $\bar{t}$, $\bar{\xi}^i$ are related by the transformation (3.12) which satisfies conditions (A.4) and (A.5), then in accordance with the proof given in the Appendix the coordinate system $\bar{t}$, $\bar{\xi}^i$ also goes over into a Newtonian G-system defined in the gravitational field Φ. In the latter G-system the tensor (3.13) has the components $\tilde{\tilde{\gamma}}_{ik}$. From this it can be seen that the coefficients $\tilde{\tilde{\gamma}}_{ik}$ describe the same gravitational field as do the coefficients $\tilde{\gamma}_{ik}$.

The independence of the Newtonian representation of the gravitational field γ_{ik} on the choice of the G-family, i.e., the problem of the uniqueness of the Newtonian representation is not discussed in this paper. According to physical considerations the uniqueness of the Newtonian representation is probable. In any case we see that Einstein's gravitational theory contains the Newtonian theory within itself as a limiting case.

McVittie[2], in solving specific problems, repeatedly used in his book the limiting transition $c \to \infty$ in order to obtain formulas of the Newtonian gravitational theory from the formulas of the general theory of relativity derived for a weak gravitational field. We did not base our discussion on approximate relativistic formulas and on a specific form of the mass tensor.

Example 1. In a centrally-symmetric gravitational field produced by a mass m the transformation

$$c\tau = ct - 2k\sqrt{\rho} + k^2 \ln\left|\frac{\sqrt{\rho}+k}{\sqrt{\rho}-k}\right|, \tag{3.16}$$

$$\rho^{3/2} = r^{3/2} - \tfrac{3}{2}kct, \qquad k^2 = 2Gm/c^2 \tag{3.17}$$

relates the Schwarzschild coordinates τ, ρ, θ, φ to the G-coordinates t, r, θ, φ, in terms of which the metric form has the form

$$ds^2 = c^2dt^2 - \rho^{-1}rdr^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{3.18}$$

In this special case the coefficients γ_{ik} do not depend on the parameter c. Therefore, the expression for the three-dimensional Newtonian line element corresponding to the form (3.18) has the form

$$ds^2 = \rho^{-1}rdr^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3.19}$$

where ρ is defined, as before, by relation (3.17). With the aid of the transformation given by formula (3.17) the metric form (3.19) is transformed to the form

$$ds^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

from which it can be seen that ρ, θ, φ are the usual polar coordinates in Euclidean space.

The rectangular cartesian coordinates x, y, z are given as functions of t, r, θ, φ by the formula

$$x = \rho \sin\theta \cos\theta,\ y = \rho \sin\theta \sin\varphi,\ z = \rho \cos\theta,$$

where ρ is defined by relation (3.17). The potential of the gravitational field (3.19) is evaluated on the basis of (1.2) from the equation

$$d\Phi = x_{,00}dx + y_{,00}dy + z_{,00}dz = -Gm\rho^{-2}d\rho,$$

from which it follows that $\Phi = Gm/\rho$.

Thus, the Schwarzschild solution has for its Newtonian representation a gravitational field produced by a point mass m. We note that the limiting transition is valid not only for large values of ρ, where the field is weak, but also arbitrarily close to the central body.

Example 2. Let us consider the gravitational field with the metric form

$$ds^2 = 2d\tau dy^1 - \text{sh}^2\tau\,(dy^2)^2 - \sin^2\tau\,(dy^3)^2, \quad (3.20)^*$$

the coefficients of which satisfy the Einstein equations $R_{\mu\nu} = 0$. This (according to Petrov's classification) is a gravitational field of the second type (space of the second type of maximum mobility) [3]. We go over to a G-system by means of the transformation

$$\tau = t + \xi^1/c,\quad y^1 = \tfrac{1}{2}c\,(ct - \xi^1),\quad y^2 = \xi^2,\quad y^3 = \xi^3,$$

so that

$$ds^2 = c^2dt^2 - (d\xi^1)^2 - \text{sh}^2\,(t + \xi^1/c)\,(d\xi^2)^2 - \sin^2\,(t + \xi^1/c)\,(d\xi^3)^2. \quad (3.21)$$

It can be shown that the condition (3.15) is satisfied. Therefore, there exists a Newtonian representation with the metric form

$$ds^2 = (d\xi^1)^2 + \text{sh}^2\,t\,(d\xi^2)^2 + \sin^2\,t\,(d\xi^3)^2. \quad (3.22)$$

The rectangular cartesian coordinates x, y, z are related to the coordinates ξ^i by the transformation

$$x = \xi^1,\qquad y = \xi^2\,\text{sh}\,t,\qquad z = \xi^3 \sin t,$$

satisfying conditions (1.26) and (1.31). Proceeding in analogy with the preceding example we obtain for the potential of the gravitational field (3.22) the equation

$$d\Phi = ydy - zdz,$$

from which it follows that

$$\Phi = \tfrac{1}{2}\,(y^2 - z^2). \quad (3.23)$$

At infinity the field (3.23) does not vanish—a property which is also possessed by the gravitational field (3.21) [3].

The gravitational field (3.23) can be produced by infinite masses situated at infinity.

*sh = sinh.

4. THE PRINCIPLE OF CORRESPONDENCE

In this section we shall consider the limiting transition $c \to \infty$ for an unspecified form of the gravitational field γ_{ik}. With the aid of such a limiting transition which, in fact, is accomplished in a semigeodesic coordinate system of a general form, a correspondence is established between the theories of Newton and of Einstein. The general expressions and quantities of the general theory of relativity are placed in correspondence with certain general expressions and quantities of the Newtonian gravitational theory. More exactly: if in a semigeodesic coordinate system the relativistic quantity A has for $c \to \infty$ a limiting value different from zero or infinity, then this limiting value represents a quantity in the Newtonian theory which corresponds to the quantity A.

This principle of correspondence enables us to analyze quantities of the general theory of relativity with the aid of concepts of Newtonian theory and simplifies the determination of the physical meaning of certain expressions in the general theory of relativity.

Example 3. In a semigeodesic coordinate system the Riemann-Christoffel tensor has the following components:

$$R_{0i0k} = \partial\Gamma_{ik}/\partial t - \Gamma_i^s\Gamma_{ks}, \quad (4.1)$$

R_{0ijk}, defined by formula (1.15), and

$$R_{hijk} = -P_{hijk} + c^{-2}\,(\Gamma_{hk}\Gamma_{ij} - \Gamma_{hj}\Gamma_{ik}), \quad (4.2)$$

where P_{hijk} is the three-dimensional Riemann-Christoffel tensor composed of γ_{ik}. If there exists a Newtonian limit for the gravitational field γ_{ik}, then in accordance with (3.15) $R_{0ijk} \to 0$. Moreover, in view of the last equation of the system (3.14) $P_{hijk} \to 0$ and, consequently, in accordance with (4.2) $R_{hijk} \to 0$. However, the components R_{0i0k} in general do not tend to zero. Simple calculations show that in virtue of (1.12), (1.18), (1.19), and (1.21),

$$R_{i0k} \to \Phi_{,rs}x^r{}_{,i}\,x^s{}_{,k}.$$

We see that the Reimann-Christoffel tensor corresponds in the Newtonian gravitational theory to a second rank tensor whose components in a system of rectangular cartesian coordinates are equal to second derivatives of the potential Φ_{ik} [2,4].

Example 4. In the general theory of relativity the integrability conditions for Einstein's equations (2.2) have the form

$$T^{\sigma}_{\mu;\sigma} = 0. \quad (4.3)$$

Taking into account the fact that in a semigeodesic coordinate system the four-dimensional Christoffel brackets are given by the formulas

$$\left\{ {0 \atop 0\ 0} \right\} = 0, \quad \left\{ {0 \atop 0\ i} \right\} = 0, \quad \left\{ {i \atop 0\ 0} \right\} = 0,$$

$$\left\{ {0 \atop i\ k} \right\} = \frac{\Gamma_{ik}}{c^2}, \quad \left\{ {j \atop 0\ i} \right\} = \Gamma^j_i, \quad \left\{ {j \atop i\ k} \right\} = \Gamma^j_{ik}$$

and that on the basis of formulas (3.5)–(3.8) we have

$$T^0_0 \to \rho_0, \quad T^i_0 \to \rho_0 u^i, \quad c^2 T^0_i \to -\rho_0 w_i,$$

$$c^2 T^k_i \to -\rho_0 w_i w^k - p^k_i,$$

where $p^k_i = \gamma^{ks} p_{is}$, we can easily show that as $c \to \infty$ the equations $T^\sigma_{0;\sigma} = 0$ and $c^2 T^\sigma_{i;\sigma} = 0$ take on the form

$$\partial (\sqrt{\gamma}\, \rho_0)/\partial t + \partial (\sqrt{\gamma}\, \rho_0 w^s)/\partial \xi^s = 0,$$

$$\rho_0 (\partial w_i/\partial t + w^s w_{i;s}) + p^s_{i;s} = 0$$

(here a semicolon denotes three-dimensional covariant differentiation). These are the hydrodynamical equations of Newtonian mechanics written in a G-system, i.e., the Newtonian equations of motion of the sources of the gravitational field.

APPENDIX

Let the transformation $\xi^i = \xi^i(t, \bar{\xi}^k)$ specify the relation between two Newtonian G-systems defined in the same gravitational field described by the potential Φ. Let the transformations $x^i = x^i(t, \xi^k)$ and $x^i = \bar{x}^i(t, \bar{\xi}^k)$ relate these G-systems to the rectangular cartesian coordinates x^i. Since

$$\bar{x}^i (t, \bar{\xi}^k) = x^i [t, \xi^j(t, \bar{\xi}^k)],$$

we have

$$\bar{x}^i{}_{,0} = x^i{}_{,0} + x^i{}_{,s} \xi^s{}_{,0}. \qquad \text{(A.1)}$$

The derivative $\xi^i_{,0} = v^i(t, \xi^k)$ describes the velocity field generated in the G-system coordinates ξ^i by the motion of the points $\bar{\xi}^i = \text{const}$. We shall refer to v^i as the velocity of the $\bar{\xi}$-system with respect to the ξ-system. Taking into account the fact that in accordance with (1.3) we have $\bar{x}^i{}_{,0} = \bar{\varphi}_{,i}(t, x^k)$, $x^i{}_{,0} = \varphi_{,i}(t, x^k)$, we can write (A.1) in the form

$$x^i{}_{,s} v^s = \bar{\varphi}_{,i} - \varphi_{,i}. \qquad \text{(A.2)}$$

With the aid of (1.11) and (A.2) we evaluate

$$v_i = \gamma_{is} v^s = x^r{}_{,i} x^r{}_{,s} v^s = (\bar{\varphi}_{,r} - \varphi_{,r})\, x^r{}_{,i} \qquad \text{(A.3)}$$

and from this utilizing formulas (1.18), (1.20), (1.21), and (A.2), and having in mind that by assumption φ and $\bar{\varphi}$ satisfy equation (1.21) for the same Φ, we obtain

$$v_{i,0} = (\bar{\varphi}_{,s} - \varphi_{,s})\, \varphi_{,rs} x^r{}_{,i} - (\bar{\varphi}_{,s} - \varphi_{,s})\, \bar{\varphi}_{,rs} x^r{}_{,i},$$

$$v_{i,k} = (\bar{\varphi}_{,s} - \varphi_{,s})\, x^s{}_{,ik} + (\bar{\varphi}_{,rs} - \varphi_{,rs})\, x^r{}_{,i} x^s{}_{,k},$$

$$v^s v_{s;i} = v^s v_{s,i} - \Gamma_{ir,s} v^r v^s = (\bar{\varphi}_{,s} - \varphi_{,s})(\bar{\varphi}_{,rs} - \varphi_{,rs})\, x^r{}_{,i}.$$

From these equations it follows that

$$v_{i,0} + v^s v_{s;i} = 0, \qquad \text{(A.4)}$$

$$v_{i,k} - v_{k,i} = 0. \qquad \text{(A.5)}$$

These are necessary conditions for the relative velocity between the G-systems. But conditions (A.4) and (A.5) are also sufficient, i.e., if the $\bar{\xi}$-system moves with respect to the G-system coordinates ξ^i with a velocity v^i satisfying conditions (A.4) and (A.5), then the $\bar{\xi}$-system is a G-system. To prove this we have to repeat the calculations carried out above with the only difference that in place of the equation $\bar{x}^i{}_{,0} = \bar{\varphi}_{,i}$, which now has to be proved, we assume $\bar{x}^i{}_{,0} = \bar{\varphi}_i(t, x^k)$ and obtain

$$v_i = (\bar{\varphi}_r - \varphi_{,r})\, x^r{}_{,i}.$$

Substituting this expression into equations (A.4) and (A.5), we obtain for $\bar{\varphi}_i$ the following relations:

$$\bar{\varphi}_{i,k} - \bar{\varphi}_{k,i} = 0, \qquad \bar{\varphi}_{i,0} + \bar{\varphi}_{s,i} \bar{\varphi}_s = \Phi_{,i}.$$

From the first equation it follows that there exists a function $\bar{\varphi}(t, x^i)$ such that $\bar{\varphi}_i = \bar{\varphi}_{,i}$, and from the second equation, in virtue of the now already proved relation $\bar{x}^i{}_{,0} = \bar{\varphi}_{,i}$, the equation $\bar{x}^i{}_{,00} = \Phi_{,i}$ follows. From this it can be seen that the $\bar{\xi}$-system satisfies the requirements 1) and 2) of section 1 which are characteristic of a G-system.

We now consider in the general theory of relativity semigeodesic coordinate systems related to one another by the transformations

$$\xi^i = \xi^i (t, \bar{\xi}^k), \qquad \text{(A.6)}$$

$$\bar{t} = \bar{t}\, (t, \xi^k). \qquad \text{(A.7)}$$

Equations (A.6) describe the motion of the points $\bar{\xi}^i = \text{const}$ in the coordinate system t, ξ^i, while (A.7) introduces a new time coordinate $\bar{t}$. We shall also in this case refer to the derivative $\xi^i{}_{,0} = v^i(t, \xi^k)$ as the velocity of the $(\bar{t}, \bar{\xi}^i)$-system with respect to the (t, ξ^i)-system. From the equation

$$c^2 dt^2 - \gamma_{rs} d\xi^r d\xi^s = c^2 d\bar{t}^2 - \bar{\gamma}_{rs} d\bar{\xi}^r d\bar{\xi}^s$$

we deduce the following relations:

$$(\bar{t}_{,0} + \bar{t}_{,s} v^s)^2 = \mu^2,$$

$$c^2 (\bar{t}_{,0} + \bar{t}_{,s} v^s)\, \bar{t}_{,r} \xi^r{}_{,i} = \gamma_{rs} v^s \xi^r{}_{,i},$$

$$c^2 \bar{t}_{,r} \bar{t}_{,s} \xi^r{}_{,i} \xi^s{}_{,k} - \bar{\gamma}_{ik} = -\gamma_{rs} \xi^r{}_{,i} \xi^s{}_{,k}.$$

From this it follows that

$$\bar{t}_{,0} = 1/\mu, \quad \bar{t}_{,i} = -v_i/c^2\mu, \tag{A.8}$$

$$(\gamma_{rs} + v_r v_s/c^2\mu^2)\,\xi^r{}_{,i}\xi^s{}_{,k} = \bar{\gamma}_{ik}, \tag{A.9}$$

here we have introduced the notation

$$\mu^2 = 1 - v^2/c^2, \qquad v^2 = v_s v^s, \qquad v_i = \gamma_{is}v^s. \tag{A.10}$$

With the aid of (A.8) we obtain the relations

$$\frac{\partial}{\partial t}\left(\frac{v_i}{\mu}\right) + c^2\frac{\partial}{\partial \xi^i}\left(\frac{1}{\mu}\right) = 0, \tag{A.11}$$

$$\frac{\partial}{\partial \xi^k}\left(\frac{v_i}{\mu}\right) - \frac{\partial}{\partial \xi^i}\left(\frac{v_k}{\mu}\right) = 0, \tag{A.12}$$

which are the necessary conditions for the velocity of one semigeodesic coordinate system with respect to the other one. But conditions (A.11) and (A.12) are also sufficient: if the $(\bar{t}, \bar{\xi})$-system moves with respect to the semigeodesic coordinate system t, ξ^i with velocity v^i satisfying these conditions, then the time coordinate $\bar{t}$ can be chosen in such a way that the $(\bar{t}, \bar{\xi})$-system will be semigeodesic.

Indeed, (A.11) and (A.12) are the integrability conditions for the system (A.8). Consequently, there exists a function $\bar{t}(t, \xi^i)$, satisfying conditions (A.8). By taking the quantity $\bar{t}$ for the time coordinate we obtain the transformation equation (A.7). For the function (A.6) we have only the equation $\xi^i{}_{,0} = v^i$. But, assuming $\bar{\xi}^i = f^i(t, \xi^k)$, we have $\bar{\xi}^i \equiv f^i[t, \xi^k(t, \bar{\xi}^j)]$, so that

$$f^i{}_{,0} + f^i{}_{,s}v^s = 0. \tag{A.13}$$

Now, in accordance with the transformation law for the components of the metrical tensor, we have in virtue of (A.8) and (A.13)

$$\bar{g}^{00} = c^{-2}(\bar{t}_{,0})^2 - \gamma^{rs}\bar{t}_{,r}\bar{t}_{,s} = c^{-2},$$

$$\bar{g}^{0i} = c^{-2}\bar{t}_{,0}f^i{}_{,0} - \gamma^{rs}\bar{t}_{,r}f^i{}_{,s} = 0.$$

From this it can be seen that the $(\bar{t}, \bar{\xi})$-system is semigeodesic.

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Translated by G. Volkoff
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