HYDRODYNAMICS OF ROTATING HELIUM II IN AN ANNULAR CHANNEL

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An equation which yields the velocity field in rotating helium II in the general case is derived by a variational method. It is shown that formation of an internal irrotational region with a circulation much in excess of the circulation quantum h/m is energetically most favorable for an annular channel.

WE investigate here by a variational method the distribution of the velocities of helium II in a rotating annular channel formed by two coaxial cylinders. The peculiarities of the rotation of helium II in such a geometry were first considered by Bendt and Oliphant[1]. In an annular channel, the Feynman vortices are not uniformly distributed. The velocity distribution which is energetically more favored is one in which there are no Feynman vortices in a region bounded by the inside radius r_1 and some radius r_i, and the superfluid liquid rotates in irrotational fashion with some circulation Γ which is much larger than the circulation quantum h/m; outside the radius r_i there is formed a uniform system of Feynman vortices, which imitates the rotation of a solid [1)].

We begin the analysis of the problem with a derivation of an equation that determines in the general case the velocity field in the rotating helium. For the case of an annular channel this equation gives the velocity distribution picture given above.

As is well known, the energy of the vortex per unit length is equal to

$$\varepsilon = \frac{\rho_s \varkappa^2}{4\pi} \ln \frac{b}{a_0},$$

where ρ_s is the density of the superfluid component, κ—unit quantum of circulation, a_0—radius of the core of the vortex, and b—its effective radius. We assume that where there are vortices, they are distributed uniformly with density $2\omega/\kappa$. The energy and the momentum of the superfluid component are respectively

$$E = \frac{\rho_s}{2} \int_{r_1}^{r_2} v^2 2\pi r dr + \rho_s \frac{\varkappa}{4\pi} \int_{r_1}^{r_2} \omega \ln \frac{b}{a_0} 2\pi r dr,$$

$$M = \rho_s \int_{r_1}^{r_2} v r^2 \, 2\pi r dr + \frac{\rho_s \varkappa}{4\pi} \int_{r_1}^{r_2} 2\pi r dr.$$

Here r_2—outside radius of the channel: the first terms in these equations correspond to the energy and momentum of a liquid rotating with velocity v, and the second, to the corresponding contribution of the vortices. Bearing in mind that $b \sim \omega^{-1/2}$ and that $\omega = |\mathrm{curl}\, \mathbf{v}|$, with $\omega_z = r^{-1} d(vr)/dr$ and $\omega_r = \omega_\theta = 0$, the free energy of the liquid in the rotating coordinate system $F' = F - M\omega_0$ will have the form [2)]

$$F' = \pi\rho_s \int_{r_1}^{r_2} \Big[v^2 r + \frac{\varkappa}{2\pi} \Big| \frac{1}{r} \frac{d(vr)}{dr} \Big| \ln \frac{c}{|r^{-1} d(vr)/dr|^{1/2}} - 2\omega_0 v r^2 - \frac{\omega_0 \varkappa}{2\pi} r \Big] dr,$$

where ω_0—angular velocity of rotation of the vessel.

A solution of the variational problem leads to the following differential equation:

$$\frac{\varkappa}{4\pi}\Big(r^2 \frac{d^2 v}{dr^2} + r\frac{dv}{dr} - v\Big) + 2r(v - \omega_0 r)\Big(r\frac{dv}{dr} + v\Big) = 0. \qquad (1)$$

The equation obtained determines the equilibrium distribution of the velocities in rotating helium II. We can rewrite (1) in the form

$$(v - \omega_0 r) \frac{1}{r} \frac{d(vr)}{dr} + \frac{\varkappa}{8\pi} \frac{d}{dr} \frac{1}{r} \frac{d(vr)}{dr} = 0. \qquad (1')$$

This equation is satisfied by the solutions $v = \omega_0 r$ and $v = \Gamma/2\pi r$ (the latter corresponds to the absence of vortices). A mixture of the two is cer-

[1)]Strictly speaking, a vortex-free region is produced also near the outer wall of the channel. Estimates show that the dimensions of this region are sufficiently small and can be neglected.

[2)]The part of the free energy which is independent of the velocity of the liquid is omitted because it is of no importance in what follows.

tainly excluded; indeed, by putting in (1) $v = Ar + B/r$ we obtain either $B = 0$ and $A = \omega_0$ or else $A = 0$ and $B = \text{const}$. In this connection it is natural to assume that the space filled with the liquid breaks up into two regions, depending on the character of the velocity distribution: a region where the liquid rotates with velocity $v = \omega_0 r$, and a region where the liquid rotates with velocity $v = \Gamma/2\pi r$. [3)] It is obvious that the smallest free energy corresponds to the case when the irrotational region ($v = \Gamma/2\pi r$) is the internal region of the liquid.

Assume that this region extends from r_1 to a certain radius r_i, the value of which is determined from the condition of minimum free energy

$$F' = -\frac{\pi\rho_s}{4}\omega_0^2(r_2^4 - r_i^4) + \pi\rho_s\frac{\Gamma^2}{(2\pi)^2}\ln\frac{r_i}{r_1} + \frac{\rho_s N\varkappa^2}{4\pi}\ln\frac{b}{a_0} - \pi\rho_s\omega_0\frac{\Gamma}{2\pi}(r_i^2 - r_1^2) - \frac{\rho_s\varkappa}{4}\omega_0(r_2^2 - r_i^2). \quad (2)$$

Minimization of the free energy with respect to r_i yields

$$\omega_0^2 r_i^4 - 2\left[\frac{\Gamma}{2\pi} + \frac{\varkappa}{2\pi}\left(\ln\frac{b}{a_0} - \frac{1}{2}\right)\right]\omega_0 r_i^2 + \left(\frac{\Gamma}{2\pi}\right)^2 = 0, \quad (3)$$

while minimization with respect to Γ yields:

$$\Gamma = \frac{2\pi\omega_0(r_i^2 - r_1^2)}{\ln(r_i^2/r_1^2)}. \quad (4)$$

A simultaneous solution of (3) and (4) yields r_i/r_1 and Γ as functions of $\omega_0 r_1^2$. Let $F'_{\min}$ be the energy of the liquid corresponding to the obtained distribution of the velocities with equilibrium values r_i and Γ, and let F'_{rig} be the free energy corresponding to the rotation of the liquid as a "rigid body." If we calculate with the aid of (2), (3), and (4) the difference $\Delta F' = F'_{\min} - F'_{\text{rig}}$, then we find that for the obtained velocity distribution the difference is negative, so that this velocity distribution is energetically favored. When $r_2 = r_i$, the entire liquid rotates in irrotational fashion with velocity $\Gamma/2\pi r$. From (3) and (4) we can calculate the corresponding angular velocity ω_{0k}. For example, for the case when $r_2/r_1 = 1.1$ we have $\omega_{0k} = 4.62 \times 10^{-1}\ \text{sec}^{-1}$. Completely irrotational rotation of helium was realized in [2].

As regards the shape of the meniscus with the obtained velocity distribution, it differs little from plane, since, as can be seen from (3), with increasing angular velocity, when the shape of the liquid surface changes noticeably, r_i becomes small and the meniscus becomes nearly parabolic. The equation of the free surface with velocity distribution v is of the form

$$\frac{\sigma z''}{(1 + z'^2)^{3/2}} + \frac{\sigma z'}{r(1 + z'^2)^{1/2}} + \rho g z - \rho\int^r \frac{v^2}{r}\,dr = 0. \quad (5)$$

Obviously when $r \gg (\sigma/\rho g)^{1/2}$ the first two terms in (5) can be neglected. In the case when $r \geq r_1 \approx 1$ cm this condition is satisfied in the entire region of the liquid, so that for the regions $(r_1 r_i)$ and for $(r_i r_2)$ we have respectively

$$z_1 = -\frac{\Gamma^2}{8\pi^2 g}\frac{1}{r^2} + c_1, \quad z_2 = \frac{\omega_0^2}{2g}r^2 + c_2. \quad (6)$$

Here we can determine c_2 from the boundary condition on the outer wall of the channel, however, neglecting surface tension, this constant can be set equal to zero. For c_1 the boundary condition on r_i yields

$$c_1 = \frac{\omega_0^2 r_i^2}{2g} + \frac{\Gamma^2}{8\pi^2 g}\frac{1}{r_i^2}. \quad (7)$$

It follows from (6) and (7) that the presence of an irrotational region leads to a decrease in the level of the meniscus in this region.

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3)Equation (1) is of second order, so that strictly speaking it would be necessary to make the solutions continuous on the boundary between the corresponding regions. However, a detailed analysis shows that the dimension of the region where the continuity is effected is of the order of $\sqrt{\varkappa/\omega}$, which is $\sqrt{\ln(b/a_0)}$ times smaller than the region of irrotational rotation. Consequently, in view of the smallness of the former we can neglect its contribution to all the processes and not join the solutions. The undetermined coefficient Γ, as will be shown later, is determined from the condition that the free energy be a minimum.

[1] P. Bendt and T. A. Oliphant, Phys. Rev. Lett. 6, 213 (1961).

[2] P. Bendt, Phys. Rev. 127, 1441 (1962).

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