RELAXATION TIME IN A TWO-TEMPERATURE MIXTURE OF CLASSICAL GASES

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A new form of the Boltzmann kinetic equation, which takes the conservation laws explicitly into account, is proposed and used to derive an equation for the temperature relaxation in a two-temperature mixture of classical gases. An approximate estimate is given for the relaxation time of this process.

LANDAU[1] derived an expression for the relaxation time in a system of electrons and ions in the case in which the electron and ion subsystems have a quasi-equilibrium Maxwell distribution with differing moduli (temperatures). The basis of this derivation is that the mass of the particles of one of the components of the mixture is negligibly small in comparison with the mass of the particles of the second component, and that, because of the Coulomb law of interaction between the charged particles, "collisions" at large target distances, for which a change in the absolute value of the momenta of the "colliding" particles is small, predominate. Subsequently, the problem of Landau has been considered in more detail by Dougal and Goldstein[2] and by Kihara[3]. In their researches, important use was also made of the characteristic features of the system under consideration (the strong nonequilibrium character of the masses of the particles of the components of the mixture and the long range character of the Coulomb interaction).

Recently, interest has developed in the consideration of the temperature relaxation in a two-temperature system of classical gases, in which the assumptions made above would be inapplicable (for example, a two-temperature mixture of neutrons). It is obvious that if the Landau conditions are applied in the solution of such a problem, then solution of the problem must lead to the Landau solution.

Such an investigation was completed by Deslog.[4] However, in the transition to the Landau conditions his results do not agree with the results of the former (a difference in the dependence of the relaxation time on the mass of the particles). This is evidently connected with the fact that the initial kinetic equation of Deslog was taken in the form suggested in [5]. This form of the equation raises objections, since the conservation laws were incorrectly used in its derivation.

In the present research, another method is proposed for the derivation of the relaxation equation for classical systems of two kinds of neutral particles, which differ only in mass, and each subsystem of which (each kind of particles) has a quasi-equilibrium Maxwell distribution with its own modulus (temperature). An initial mathematical representation introduced by Kihara[3] was used in the research. The resultant expression for the relaxation time differs from the result of Deslog and agrees with the expression of Landau.

We note that there is an essential inaccuracy in the work of Kihara.[3] To be precise, in his derivation he uses the expression

$$c_1^2 - c_1'^2 = \frac{2m_2}{m_1 + m_2} \mathbf{G}\,(\mathbf{g} - \mathbf{g}'), \tag{1}$$

where $\mathbf{c}_1$ and $\mathbf{c}_1'$ are the velocities of the electron before and after the "collision" with the ion ($\mathbf{c}_2$ and $\mathbf{c}_2'$ are the corresponding velocities of the ion); m_1 and m_2 are the masses of the electron and ion, respectively; $\mathbf{G} = (m_1\mathbf{c}_1 + m_2\mathbf{c}_2)/(m_1 + m_2)$; $\mathbf{g}$ and $\mathbf{g}'$ are the relative velocities before and after the "collision": $\mathbf{g} = \mathbf{c}_1 - \mathbf{c}_2$, $\mathbf{g}' = \mathbf{c}_1' - \mathbf{c}_2'$. However, this relation is incorrect. Actually, we have from (1)

$$(\mathbf{c}_1 + \mathbf{c}_2)\,(m_1\mathbf{c}_1 - m_1\mathbf{c}_1' + m_2\mathbf{c}_1 - m_2\mathbf{c}_1') = 2m_2\mathbf{G}\,(\mathbf{g} - \mathbf{g}').$$

With account of the conservation of momentum $m_1\mathbf{c}_1 - m_1\mathbf{c}_1' = m_2\mathbf{c}_2' - m_2\mathbf{c}_2$, this relation reduces to the form $\mathbf{c}_1 + \mathbf{c}_1' = 2\mathbf{G}$. Substituting therein $\mathbf{G}$ in the form

$$\mathbf{G} = \tfrac{1}{2}(m_1\mathbf{c}_1 + m_2\mathbf{c}_2 + m_1\mathbf{c}_1' + m_2\mathbf{c}_2')/(m_1 + m_2),$$

and after elementary transformations we get

$$\mathbf{g} = -\mathbf{g}'. \tag{2}$$

However, as is known from the general collision theory, only the absolute value of g is conserved, $|\mathbf{g}| = g'$, while the direction of $\mathbf{g}'$ can be arbitrary relative to $\mathbf{g}$ (this property was also employed by Kihara). The condition (2) is more rigid and does not correspond to reality.

In setting up the problem, it is advantageous to transform the Boltzmann equation to a form which takes the conservation laws into explicit account. The general form of the Boltzmann equation is

$$\frac{\partial w(t, \mathbf{p}_1)}{\partial t} = \frac{1}{v} \int\limits_0^{4\pi} \int u_{12} S_{\text{дифф}} \{w(t, \mathbf{p}_3) w(t, \mathbf{p}_4) - w(t, \mathbf{p}_1) w(t, \mathbf{p}_2)\} d\Omega d\mathbf{p}_2,$$

where $\mathbf{p}_3$ and $\mathbf{p}_4$ are the momenta of the two particles before the collision, $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta after collision, $u_{12} = |\mathbf{p}_1 - \mathbf{p}_2|/m$, $v^{-1} = N/V$ (N is the number of particles in the system, V is the volume of the system). S_{diff} is the differential effective cross section, which depends on the relative velocity u_{12} of the colliding particles and the scattering angle $d\Omega = \sin\theta \, d\theta \, d\varphi$ (φ is the azimuthal angle).

The method of our account of the law of conservation of momentum is evident. The dimensionless factor $\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) d\mathbf{p}_3$ (where $\delta(\mathbf{p}) = \delta(p_x)\delta(p_y)\delta(p_z)$, $\delta(p_x)$ is the usual Dirac delta function), is introduced in the collision integral; in the integration over momentum space $\mathbf{p}_3$ those states are eliminated from consideration which do not obey the law of conservation of momentum. To account for the law of conservation of energy, we multiply the collision integral by the dimensionless factor $\delta(E_1 + E_2 - E_3 - E_4) dE_4$ and integrate over the energy scale E_4. For classical systems ($E = p^2/2m$), we have the following obvious relations:

$$dE_4 d\Omega = m^{-1} p_4 dp_4 d\Omega = p_4^2 dp_4 d\Omega / m p_4 = d\mathbf{p}_4 / m p_4,$$
$$\delta(E_1 + E_2 - E_3 - E_4) = 2m\delta(p_1^2 + p_2^2 - p_3^2 - p_4^2).$$

We then obtain the desired form of the kinetic equation:

$$\frac{\partial w(t, \mathbf{p}_1)}{\partial t} = \frac{1}{v} \iiint u_{12} S_{\text{diff}} \{w(t, \mathbf{p}_3) w(t, \mathbf{p}_4) - w(t, \mathbf{p}_1) w(t, \mathbf{p}_2)\} \frac{2}{p_4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \times \delta(p_1^2 + p_2^2 - p_3^2 - p_4^2) d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4. \tag{3}$$

Let us now consider a system of two types of uncharged particles in the volume V. The number of particles of the first and second types are denoted by N_I and N_{II}, respectively, their masses by m_I and m_{II}, their mean densities by $v_I^{-1} = N_I/V$ and $v_{II}^{-1} = N_{II}/V$. For simplicity, we consider the distribution of particles to be an equilibrium one in space so that the probability functions of the single particle distributions depend only on the time t and the momentum: $w(t, \mathbf{p}^{(I)})$ and $w(t, \mathbf{p}^{(II)})$.

The set of kinetic equations in the form (3) becomes

$$\frac{\partial w(t, \mathbf{p}_1^{(i)})}{\partial t} = \frac{1}{v_i} \iiint u_{12}^{(i,i)} S_{34\to12}^{(i,i)} \{w(t, \mathbf{p}_3^{(i)}) w(t, \mathbf{p}_4^{(i)}) - w(t, \mathbf{p}_1^{(i)}) w(t, \mathbf{p}_2^{(i)})\} \delta_{\mathbf{p}}^{(i,i)} \delta_{p^2}^{(i,i)} \frac{2}{p_4^{(i)}} d\mathbf{p}_2^{(i)} d\mathbf{p}_3^{(i)} d\mathbf{p}_4^{(i)}$$
$$+ \frac{1}{v_j} \iiint u_{12}^{(i,j)} S_{34\to12}^{(i,j)} \{w(t, \mathbf{p}_3^{(i)}) w(t, \mathbf{p}_4^{(j)}) - w(t, \mathbf{p}_1^{(i)}) w(t, \mathbf{p}_2^{(j)})\} \delta_{\mathbf{p}}^{(i,j)} \sum\nolimits^{(i,j)} d\mathbf{p}_2^{(j)} d\mathbf{p}_3^{(i)} d\mathbf{p}_4^{(j)}, \tag{4}$$

where i, j = I, II ($i \neq j$) and for brevity we use the notation

$$\delta_{\mathbf{p}}^{(i,j)} = \delta(\mathbf{p}_1^{(i)} + \mathbf{p}_2^{(j)} - \mathbf{p}_3^{(i)} - \mathbf{p}_4^{(j)}),$$
$$\delta_{p^2}^{(i,i)} = \delta(p_1^{(i)2} + p_2^{(i)2} - p_3^{(i)2} - p_4^{(i)2})$$
$$\sum\nolimits^{(i,j)} = \delta(m_j p_1^{(i)2}/m_i + p_2^{(j)2} - m_j p_3^{(i)2}/m_i - p_4^{(j)2})/p_4^{(j)} + \delta(p_1^{(i)2} + m_i p_2^{(j)2}/m_j - p_3^{(i)2} - m_i p_4^{(j)2}/m_j)/p_3^{(i)}$$

(such a symmetric form of the equations is brought about by the evident equivalence of the integration variables $\mathbf{p}_3^{(i)}$ and $\mathbf{p}_4^{(j)}$).

Let the first collision integrals in Eqs. (4) be larger than the second. Then, as the result of collisions only between "their own" particles in each subset, a quasi-equilibrium Boltzmann distribution is established with its own characteristic temperature:

$$w_0(p^{(i)}) = (2\pi m_i k T_i)^{-3/2} \exp(-p^{(i)2}/2m_i k T_i). \tag{5}$$

The quasi-equilibrium in these expressions lies in the fact that $w_0(p^{(i)})$ depends implicitly on the time t through its dependence on the time of the temperature T_i. The process of equalizing the temperatures is determined by the equations

$$\frac{\partial w(p_1^{(i)})}{\partial t} = \frac{1}{v_j} \iiint u_{12}^{(i,j)} S_{34\to12}^{(i,j)} [w_0(p_3^{(i)}) w_0(p_4^{(j)}) - w_0(p_1^{(i)}) w_0(p_2^{(j)})] \delta_p^{(i,j)} \sum\nolimits^{(i,j)} d\mathbf{p}_2^{(j)} d\mathbf{p}_3^{(i)} d\mathbf{p}_4^{(j)}. \tag{6}$$

Differentiating the left hand side of Eq. (6) with account of the explicit expressions (5) for w_0, we get

$$\left(\frac{p_1^{(i)2}}{2m_i k T_i} - \frac{3}{2}\right) \frac{w_0(p_1^{(i)})}{T_i} \frac{dT_i}{dt} = A_{ij},$$
$$i, j = \text{I, II}, \qquad i \neq j, \tag{7}$$

where, for brevity, the corresponding collision integrals are denoted by A_{ij}.

We multiply Eq. (7) on the right and on the left by $p_1^{(i)2}/2m_ikT_i - 3/2$ and integrate over the momentum space $\mathbf{p}_1^{(i)}$. Taking into account the exclusive form of w_0, we get

$$\frac{4}{\pi^{1/2}T_i}\frac{dT_i}{dt}\int_0^\infty (\alpha_i^{7/2}p_1^{(i)6} - 3\alpha_i^{5/2}p_1^{(i)4} + \frac{9}{4}p_1^{(i)2})\exp(-\alpha_i p_1^{(i)2})\,dp_1^{(i)} = \int\left(\alpha_i p_1^{(i)2} - \frac{3}{2}\right)A_{ij}\,d\mathbf{p}_1^{(i)}, \tag{8}$$

where $\alpha_i = (2m_ikT_i)^{-1}$. The integrals on the left hand sides are easily carried out. As a result we get

$$\frac{3}{2T_i}\frac{dT_i}{dt} = \int\left(\alpha_i p_1^{(i)2} - \frac{3}{2}\right)A_{ij}\,d\mathbf{p}_1^{(i)}. \tag{9}$$

We now transform the right side of Eq. (9) for i = I, j = II. From the principle of detailed balancing we have for the quasi-equilibrium expressions in w

$$\int A_{ij}\,d\mathbf{p}_1^{(i)} = 0, \tag{10}$$

$$\int p_1^{(i)2}A_{ij}\,d\mathbf{p}_1^{(i)} = -\int p_3^{(i)2}A_{ij}\,d\mathbf{p}_1^{(i)} = \frac{1}{2}\int(p_1^{(i)2} - p_3^{(i)2})\,A_{ij}\,d\mathbf{p}_1^{(i)}. \tag{11}$$

Thus the right hand side of Eq. (9) reduces to the form (for i = I, j = II)

$$\frac{\alpha_I}{2v_{II}}\iiiint(p_1^{(I)2} - p_3^{(I)2})\,u_{12}^{(I,II)}S_{34\to12}^{(I,II)}[w_0(p_3^{(I)})\,w_0(p_4^{(II)}) - w_0(p_1^{(I)})\,w_0(p_2^{(II)})]\;\delta_{\mathbf{p}}^{(I,II)}\textstyle\sum^{(I,II)}d\mathbf{p}_1^{(I)}\,d\mathbf{p}_2^{(II)}\,d\mathbf{p}_3^{(I)}\,d\mathbf{p}_4^{(II)}. \tag{12}$$

In accord with the theory of elastic collisions, $u_{12}^{(I,II)} = u_{34}^{(I,II)}$; moreover $S_{34\to12}^{(I,II)} = S_{12\to34}^{(I,II)}$. Therefore, with account of the conservation laws, we have

$$\frac{\alpha_I}{v_{II}}\iiiint(p_1^{(I)2} - p_3^{(I)2})u_{12}^{(I,II)}S_{34\to12}^{(I,II)}w_0(p_3^{(I)})w_0(p_4^{(II)})\,\delta_{\mathbf{p}}^{(I,II)}\times\textstyle\sum^{(I,II)}d\mathbf{p}_1^{(I)}\,d\mathbf{p}_2^{(II)}\,d\mathbf{p}_3^{(I)}\,d\mathbf{p}_4^{(II)}$$
$$= -\frac{\alpha_I}{v_{II}}\iiiint(p_1^{(I)2} - p_3^{(I)2})\,u_{12}^{(I,II)}S_{34\to12}^{(I,II)}\times w_0(p_1^{(I)})\,w_0(p_2^{(II)})\;\delta_{\mathbf{p}}^{(I,II)}\textstyle\sum^{(I,II)}d\mathbf{p}_1^{(I)}\,d\mathbf{p}_2^{(II)}\,d\mathbf{p}_3^{(I)}\,d\mathbf{p}_4^{(II)}. \tag{13}$$

As a result, Eq. (9) can be written in the form

$$k\frac{dT_I}{dt} = -\frac{2}{3v_{II}}\iiiint\left(\frac{p_1^{(I)2}}{m_I} - \frac{p_3^{(I)2}}{m_I}\right)u_{12}^{(I,II)}S_{34\to12}^{(I,II)}w_0(p_1^{(I)})\times w_0(p_2^{(II)})\;\delta_{\mathbf{p}}^{(I,II)}\textstyle\sum^{(I,II)}d\mathbf{p}_1^{(I)}\,d\mathbf{p}_2^{(II)}\,d\mathbf{p}_3^{(I)}\,d\mathbf{p}_4^{(II)}. \tag{14}$$

By virtue of the law of conservation of energy, one can also write

$$k\frac{dT_I}{dt} = -\frac{1}{3v_{II}}\iiiint\left[\left(\frac{p_1^{(I)2}}{m_I} - \frac{p_3^{(I)2}}{m_I}\right) - \left(\frac{p_2^{(II)2}}{m_{II}} - \frac{p_4^{(II)2}}{m_{II}}\right)\right]u_{12}^{(I,II)}S_{34\to12}^{(I,II)}\times w_0(p_1^{(I)})\,w_0(p_2^{(II)})\;\delta_{\mathbf{p}}^{(I,II)}\textstyle\sum^{(I,II)}d\mathbf{p}_1^{(I)}\,d\mathbf{p}_2^{(II)}\,d\mathbf{p}_3^{(I)}\,d\mathbf{p}_4^{(II)}. \tag{15}$$

Similarly,

$$k\frac{dT_{II}}{dt} = -\frac{1}{3v_I}\iiiint\left\{\left(\frac{p_1^{(II)2}}{m_{II}} - \frac{p_3^{(II)2}}{m_{II}}\right) - \left(\frac{p_2^{(I)2}}{m_I} - \frac{p_4^{(I)2}}{m_I}\right)\right\}u_{12}^{(II,I)}S_{34\to12}^{(II,I)}\times w_0(p_1^{(II)})\,w_0(p_2^{(I)})\;\delta_{\mathbf{p}}^{(I,II)}\textstyle\sum^{(I,II)}d\mathbf{p}_1^{(II)}\,d\mathbf{p}_2^{(I)}\,d\mathbf{p}_3^{(II)}\,d\mathbf{p}_4^{(I)}. \tag{16}$$

Noting that $S_{34\to12}^{(II,I)} = S_{34\to12}^{(I,II)}$, $u_{12}^{(II,I)} = u_{12}^{(I,II)}$, one can easily establish the fact that the integrals in Eqs. (15) and (16) are equal in magnitude but opposite in sign. We thus obtain

$$k\frac{d(T_{II} - T_I)}{dt} = \frac{1}{3}\left(\frac{1}{v_I} + \frac{1}{v_{II}}\right)\iiiint\left[\left(\frac{p_1^{(I)2}}{m_I} - \frac{p_3^{(I)2}}{m_I}\right) - \left(\frac{p_2^{(II)2}}{m_{II}} - \frac{p_4^{(II)2}}{m_{II}}\right)\right]u_{12}^{(I,II)}S_{34\to12}^{(I,II)}w_0(p_1^{(I)})\,w_0(p_2^{(II)})\times\delta_{\mathbf{p}}^{(I,II)}\textstyle\sum^{(I,II)}d\mathbf{p}_1^{(I)}\,d\mathbf{p}_2^{(II)}\,d\mathbf{p}_3^{(I)}\,d\mathbf{p}_4^{(II)}. \tag{17}$$

We note that

$$\left(\frac{p_1^{(I)2}}{m_I} - \frac{p_3^{(I)2}}{m_I}\right) - \left(\frac{p_2^{(II)2}}{m_{II}} - \frac{p_4^{(II)2}}{m_{II}}\right) = (\mathbf{p}_1^{(I)} + \mathbf{p}_2^{(II)})(\mathbf{u}_{12}^{(I,II)} - \mathbf{u}_{34}^{(I,II)}). \tag{18}$$

If we represent $u_{34}^{(I,II)}$ in the form

$$\mathbf{u}_{34}^{(I,II)} = \mathbf{u}_{12}^{(I,II)}\cos\theta + \mathbf{j}u_{12}\sin\theta\cos\varphi + \mathbf{k}u_{12}\sin\theta\sin\varphi,$$

where **j** and **k** are mutually perpendicular unit vectors in the plane perpendicular to the vector $\mathbf{u}_{12}^{(I,II)}$, then it is evident that after integration on the right hand side of Eq. (17) over the spatial momenta $\mathbf{p}_3$ and $\mathbf{p}_4$, only the integral in which the factor $\mathbf{u}_{12}^{(I,II)}\cos\theta$ remains different from zero.

We thus obtained the equation

$$k\frac{d(T_{II} - T_I)}{dt} = \frac{1}{3}\left(\frac{1}{v_I} + \frac{1}{v_{II}}\right)\iint u_{12}^{(I,II)}Q^{(I,II)}\mathbf{u}_{12}^{(I,II)}\times(\mathbf{p}_1^{(I)} + \mathbf{p}_2^{(II)})\,w_0(p_1^{(I)})\,w_0(p_2^{(II)})\,d\mathbf{p}_1^{(I)}\,d\mathbf{p}_2^{(II)}, \tag{19}$$

$$Q^{(\mathrm{I},\,\mathrm{II})} = \iint S^{(\mathrm{I},\,\mathrm{II})}_{34\to 12}(1-\cos\theta)\,\delta_{\mathbf{p}}^{(\mathrm{I},\,\mathrm{II})}\sum\nolimits^{(\mathrm{I},\,\mathrm{II})} d\mathbf{p}_3^{(\mathrm{I})}\,d\mathbf{p}_4^{(\mathrm{II})}. \qquad (20)$$

Bearing in mind that the desired equation serves to estimate the relaxation time of the process under consideration, which is determined only in order of magnitude, we give an approximate estimate for the value of $Q^{(\mathrm{I},\mathrm{II})}$. For this purpose, we neglect the factor $\cos\theta$ in the integrand and transform from the variables $p_3^{(\mathrm{I})}$, $p_4^{(\mathrm{II})}$ to $\mathbf{P}_{34} = p_3^{(\mathrm{I})} + p_4^{(\mathrm{II})}$ and $u_{34} = p_3^{(\mathrm{I})}/m_{\mathrm{I}} - p_4^{(\mathrm{II})}/m_{\mathrm{II}}$. The Jacobian of the transformation is equal to μ^2, where μ is the reduced mass ($\mu^{-1} = m_{\mathrm{I}}^{-1} + m_{\mathrm{II}}^{-1}$). In these variables, we have

$$\delta_{\mathbf{p}}\sum\nolimits^{(\mathrm{I},\,\mathrm{II})} = \frac{1}{\mu}\,\delta(\mathbf{P}_{12}-\mathbf{P}_{34})\,\delta(u_{12}^2-u_{34}^2) \times\left\{\frac{1}{m_{\mathrm{I}}}\left(\frac{\mathbf{P}_{34}^2}{m_{\mathrm{I}}^2}+u_{34}^2+\frac{2\mathbf{P}_{34}\mathbf{u}_{34}}{m_{\mathrm{I}}}\right)^{-1/2} + \frac{1}{m_{\mathrm{II}}}\left(\frac{\mathbf{P}_{34}^2}{m_{\mathrm{II}}^2}+u_{34}^2-\frac{2\mathbf{P}_{34}\mathbf{u}_{34}}{m_{\mathrm{II}}}\right)^{-1/2}\right\}.$$

Integration over $\mathbf{P}_{34}$ is trivial because the presence of the delta function. In integration over $\mathbf{u}_{34}$, we transform to spherical coordinates with the latitude measured from the vector $\mathbf{P}_{12}$. Then integration over u_{34}^2 is trivial because of the presence of the delta function, while integration over the angular variables does not present any difficulty if it is assumed that $S^{(\mathrm{I},\mathrm{II})}_{34\to 12}$ does not depend on these variables. We then get

$$Q^{(\mathrm{I},\,\mathrm{II})} = 2\pi\mu\overline{S^{(\mathrm{I},\,\mathrm{II})}}\,u_{12}/P_{12}. \qquad (21)$$

Keeping in mind the estimate of the order of magnitude of the relaxation time, $Q^{(\mathrm{I},\mathrm{II})}$ can be approximated in the following fashion:[6]

$$Q^{(\mathrm{I},\,\mathrm{II})} \approx 2\pi\frac{\mu}{m_{\mathrm{I}}+m_{\mathrm{II}}}\overline{S^{(\mathrm{I},\,\mathrm{II})}} \qquad (22)$$

(for a large difference in masses, $\mu/(m_{\mathrm{I}}+m_{\mathrm{II}})$ reduces to $m_{\mathrm{I}}/m_{\mathrm{II}}$ for $m_{\mathrm{II}} \gg m_{\mathrm{I}}$).

As usual, one can approximate as follows in Eq. (19):

$$k\,d(T_{\mathrm{II}}-T_{\mathrm{I}})/dt = (v_{\mathrm{I}}^{-1}+v_{\mathrm{II}}^{-1})\,Q^{(\mathrm{I},\,\mathrm{II})}\overline{u_{12}^{(\mathrm{I},\,\mathrm{II})}}\,(kT_{\mathrm{I}}-kT_{\mathrm{II}}) \qquad (23)$$

($\overline{u^{(\mathrm{I},\mathrm{II})}}$ is the mean value), since in the integral

$$\iint(\mathbf{p}_1^{(\mathrm{I})}+\mathbf{p}_2^{(\mathrm{II})})\,\mathbf{u}_{12}^{(\mathrm{I},\,\mathrm{II})}w_0(p_1^{(\mathrm{I})})\,w_0(p_2^{(\mathrm{II})})\,d\mathbf{p}_1^{(\mathrm{I})}\,d\mathbf{p}_2^{(\mathrm{II})}$$

after the substitution $u_{12}^{(\mathrm{I},\mathrm{II})} = u_1^{(\mathrm{I})} - u_2^{(\mathrm{II})}$, the integrals containing $p_1^{(\mathrm{I})}u_2^{(\mathrm{II})}$ and $p_2^{(\mathrm{II})}u_1^{(\mathrm{I})}$ vanish while the integrals containing $p_1^{(\mathrm{I})}u_1^{(\mathrm{I})}$ and $p_2^{(\mathrm{II})}u_2^{(\mathrm{II})}$ will be equal to $3kT_{\mathrm{I}}$ and $3kT_{\mathrm{II}}$, respectively. Furthermore, we set, approximately,[6]

$$\overline{u_{12}^{(\mathrm{I},\,\mathrm{II})}} \approx (kT_{\mathrm{I}}/m_{\mathrm{I}} + kT_{\mathrm{II}}/m_{\mathrm{II}})^{1/2} \approx (kT_{\mathrm{I}}/\mu)^{1/2}.$$

Thus, the equation determining the process of temperature equalization has the following approximate form:

$$\frac{d(T_{\mathrm{II}}-T_{\mathrm{I}})}{dt} = -\left(\frac{1}{v_{\mathrm{I}}}+\frac{1}{v_{\mathrm{II}}}\right) \times\frac{2\pi\mu}{m_{\mathrm{I}}+m_{\mathrm{II}}}\left(\frac{kT_{\mathrm{I}}}{\mu}\right)^{1/2}\overline{S^{(\mathrm{I},\,\mathrm{II})}}\,(T_{\mathrm{II}}-T_{\mathrm{I}}). \qquad (24)$$

We then get for the relaxation time τ:

$$\frac{1}{\tau} = \left(\frac{1}{v_{\mathrm{I}}}+\frac{1}{v_{\mathrm{II}}}\right)\frac{2\pi\mu}{m_{\mathrm{I}}+m_{\mathrm{II}}}\overline{S^{(\mathrm{I},\,\mathrm{II})}}\left(\frac{kT_{\mathrm{I}}}{\mu}\right)^{1/2}. \qquad (25)$$

It is easy to establish the fact that one can get the Landau estimate from (25) for charged particles if one applies his condition:

$$m_{\mathrm{II}} \gg m_{\mathrm{I}}, \qquad S = A\frac{e_{\mathrm{I}}^2 e_{\mathrm{II}}^2}{(kT_{\mathrm{I}})^2}\ln\left[\left(\frac{kT}{b}\right)^3\left(\frac{1}{v_{\mathrm{I}}}+\frac{1}{v_{\mathrm{II}}}\right)^{-1}\right],$$

where $b = (e_{\mathrm{I}}^2/v_{\mathrm{I}} + e_{\mathrm{II}}^2/v_{\mathrm{II}})(1/v_{\mathrm{I}} + 1/v_{\mathrm{II}})^{-1}$, A is a numerical factor.

We also note that for comparable temperatures ($T_{\mathrm{II}} \approx T_{\mathrm{I}}$) our estimate is identical with the estimate of Kogan,[7] which was derived for the corresponding process in a plasma.

In Eq. (19), one can also, in place of the approximate estimates given above, carry out a calculation of the integral on the right hand side according to a scheme which is similar to the calculation scheme of Kogan. For this, in the integral

$$\frac{1}{3}\left(\frac{1}{v_{\mathrm{I}}}+\frac{1}{v_{\mathrm{II}}}\right)Q^{(\mathrm{I},\,\mathrm{II})}\iint u_{12}^{(\mathrm{I},\,\mathrm{II})}\mathbf{u}_{12}^{(\mathrm{I},\,\mathrm{II})}(\mathbf{p}_1^{(\mathrm{I})}+\mathbf{p}_2^{(\mathrm{II})})\,w_0(p_1^{(\mathrm{I})}) \times w_0(p_2^{(\mathrm{II})})\,d\mathbf{p}_1^{(\mathrm{I})}\,d\mathbf{p}_2^{(\mathrm{II})}$$

we transform from the variables $p_1^{(\mathrm{I})}$, $p_2^{(\mathrm{II})}$ to the variables $\mathbf{P}_{12}$ and $\mathbf{u}_{12}$. The integral will then have the form

$$\frac{1}{3}\left(\frac{1}{v_{\mathrm{I}}}+\frac{1}{v_{\mathrm{II}}}\right)Q^{(\mathrm{I},\,\mathrm{II})}\mu^3 C\iint u_{12}\mathbf{u}_{12}\mathbf{P}_{12} \times\exp(-\alpha P_{12}^2-\beta u_{12}^2-\gamma\mathbf{P}_{12}\mathbf{u}_{12})\,d\mathbf{P}_{12}\,d\mathbf{u}_{12};$$

$$C = (2\pi m_{\mathrm{I}}kT_{\mathrm{I}})^{-3/2}(2\pi m_{\mathrm{II}}kT_{\mathrm{II}})^{-3/2},$$

$$\alpha = \frac{\mu^2}{2m_{\mathrm{I}}m_{\mathrm{II}}}\left(\frac{1}{m_{\mathrm{II}}kT_{\mathrm{I}}}+\frac{1}{m_{\mathrm{I}}kT_{\mathrm{II}}}\right),$$

$$\beta = \frac{\mu^2}{2}\left(\frac{1}{m_{\mathrm{I}}kT_{\mathrm{I}}}+\frac{1}{m_{\mathrm{II}}kT_{\mathrm{II}}}\right),$$

$$\gamma = \frac{\mu^2}{m_{\mathrm{I}}m_{\mathrm{II}}}\left(\frac{1}{kT_{\mathrm{I}}}-\frac{1}{kT_{\mathrm{II}}}\right) = \frac{\mu^2}{m_{\mathrm{I}}m_{\mathrm{II}}}\frac{1}{kT_{\mathrm{I}}}\left(\frac{T_{\mathrm{II}}-T_{\mathrm{I}}}{T_{\mathrm{II}}}\right).$$

In the integrations over both $\mathbf{P}_{12}$ and $\mathbf{u}_{12}$, we transform to spherical coordinates. After integration over the angular coordinates, we get

$$\frac{8\pi^2}{3\gamma}\left(\frac{1}{v_{\rm I}}+\frac{1}{v_{\rm II}}\right)Q^{(\rm I,\,II)}\mu^3C\int_0^\infty\int_0^\infty P_{12}^2u_{12}^3$$

$$\times\exp\left(-\alpha P_{12}^2-\beta u_{12}^2\right)f\left(\gamma P_{12}u_{12}\right)dP_{12}\,du_{12},$$

where $f(x) = \cosh x - x^{-1}\sinh x$. Taking it into account that $f(x)$ is an even function, the limits of integration over P_{12} can be extended to the range from $-\infty$ to $+\infty$; after this, integration over P_{12} and then over u_{12} does not present any difficulty. The final result has the form

$$\pi^{5/2}\left(v_{\rm I}^{-1}+v_{\rm II}^{-1}\right)Q^{(\rm I,\,II)}\mu^3C\gamma/3\alpha^{5/2}\left(\beta-\gamma^2/4\alpha\right)^3.$$

If $|T_{\rm II} - T_{\rm I}|/T_{\rm I} \ll 1$, then $\gamma^2/4\alpha \ll 4\beta$, and one can neglect the term $\gamma^2/4\alpha$ in the denominator. We will then have the following expression in extended form for τ:

$$\tau=\frac{8\,(2\pi)^{1/2}}{3}\left(\frac{1}{v_{\rm I}}+\frac{1}{v_{\rm II}}\right)\overline{S^{(\rm I,\,II)}}\left(k^2T_{\rm I}T_{\rm II}\right)^{-5/2}\frac{\mu}{m_{\rm I}+m_{\rm II}}$$

$$\times\left[\mu^{-6}\left(\frac{1}{kT_{\rm I}m_{\rm II}}+\frac{1}{kT_{\rm II}m_{\rm I}}\right)^{-5/2}\left(\frac{1}{kT_{\rm I}m_{\rm I}}+\frac{1}{kT_{\rm II}m_{\rm II}}\right)^{-3}\right]. \quad (26)$$

For comparable temperatures ($T_{\rm II} \approx T_{\rm I}$), we then get

$$\tau\approx\frac{8\,(2\pi)^{1/2}}{3}\left(\frac{1}{v_{\rm I}}+\frac{1}{v_{\rm II}}\right)\overline{S^{(\rm I,\,II)}}\frac{\mu}{m_{\rm I}+m_{\rm II}}\left(\frac{kT_{\rm I}}{\mu}\right)^{1/2}, \quad (27)$$

i.e., we get the previous result (the numerical factors are practically identical).

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[1] L. D. Landau, JETP **7**, 203 (1937).

[2] A. A. Dougal and L. Goldstein, Phys. Rev. **109**, 615 (1958).

[3] T. K. Kihara, J. Phys. Soc. Japan **14**, 402 (1959).

[4] E. A. Deslog, Phys. Fluids **5**, 1223 (1962).

[5] E. A. Deslog and S. W. Mathysse, Amer. J. Phys. **28**, 1 (1960).

[6] Ya. I. Frenkel', Statisticheskaya fizika (Statistical Physics) (Acad. of Sci. Press, 1948).

[7] V. I. Kogan, Fizika plasmy (Plasma Physics) **1**, 130 (1958).

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