TRANSITION RADIATION FROM A UNIFORMLY MOVING CHARGE CROSSING A DIFFUSE BOUNDARY BETWEEN TWO MEDIA

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The high-frequency part of the spectrum of transition radiation is treated in the quasi-classical approximation in the case in which the characteristic length a in which there is a considerable change of the dielectric properties of the medium is much larger than the wavelength: $a \gg c/\omega\varepsilon^{1/2}$. For such frequencies the radiation is exponentially small, in analogy with the phenomenon of "over-the-barrier" reflection in quantum mechanics.[5]

1. INTRODUCTION

GINZBURG and Frank [1] have shown that a charge moving with a speed less than the phase velocity in the medium radiates not only when its velocity changes during the motion but also when the optical properties of the medium vary along its path, so that there is a change of the speed of the electromagnetic field carried along by the charge as it moves through the medium. The case treated by these authors, and also in several other articles on transition radiation, is that of media with sharp interfaces. This is a good approximation for radiation of wavelengths λ much larger than the distance a in which a considerable change of the dielectric properties of the medium occurs. For the short-wave part of the spectrum of transition radiation, however, it is very important to take into account the diffuseness of the boundary.

Of course the treatment of the problem in the general case of arbitrary wavelengths is complicated,1) and therefore we shall confine ourselves to the calculation of the transition radiation in the limit in which the wavelengths λ of the radiation in the medium are much smaller than the characteristic length a of the inhomogeneity of the medium. In this limit, which is a natural compliment to the treatment of sharp bounding surfaces, the transition radiation decreases exponentially as the length of the electromagnetic wave decreases, and the calculation consists of a separation of this exponentially small effect, in analogy with the way such effects are separated out in the theory of over-the-barrier reflection in quantum mechanics and in problems of the "nonconservation" of adiabatic invariants.

2. THE FUNDAMENTAL EQUATIONS

Let us consider a medium whose dielectric properties vary with the coordinate z. To determine the field produced by a particle moving uniformly along the z axis through the inhomogeneous medium, we use the Maxwell equations:

$$\operatorname{div} \mathbf{H} = 0, \qquad \operatorname{rot} \mathbf{E} = -c^{-1}\partial\mathbf{H}/\partial t,$$
$$\operatorname{div} \hat{\varepsilon}\mathbf{E} = 4\pi e\delta(\mathbf{r} - \mathbf{v}t),$$
$$\operatorname{rot} \mathbf{H} = \frac{1}{c}\frac{\partial \hat{\varepsilon}\mathbf{E}}{\partial t} + \frac{4\pi}{c} e\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t), \tag{1*}$$

where **E** and **H** are the electric and magnetic field strengths; e is the charge of the particle; **v** is the velocity of the particle: c is the speed of light in vacuum; and $\hat{\varepsilon}$ is the "dielectric constant" treated as an operator.[3]

We introduce the vector and scalar potentials

$$\mathbf{H} = \operatorname{rot} \mathbf{A}, \qquad \mathbf{E} = -c^{-1}\partial\mathbf{A}/\partial t - \operatorname{grad}\varphi \tag{2}$$

and subject them to the Lorentz supplementary condition in the medium:

$$\operatorname{div} \mathbf{A} + c^{-1}\partial\hat{\varepsilon}\varphi/\partial t = 0. \tag{3}$$

Then the first two equations of the system (1) are satisfied identically. Next it is helpful to expand the potentials in Fourier integrals with respect to the time t and the coordinates x and y:

$$\mathbf{A}(x, y, z, t) = \int \mathbf{A}(\omega, \mathbf{k}, z)\, e^{-i\omega t + i\mathbf{kr}}\, d\mathbf{k}\, d\omega. \tag{4}$$

1)This sort of treatment has been given by Amatuni and Korkhmazyan [2] for one particular form of the coordinate dependence of the dielectric constant, for which an exact solution can be obtained.

*rot = curl.

Then when we use Eqs. (2) and (3) we get from Eq. (1) the equation for the Fourier transform of the vector potential **A**:

$$\frac{d^2\mathbf{A}}{dz^2} - \frac{\nabla\varepsilon}{\varepsilon}\operatorname{div}\mathbf{A} + \left[\varepsilon(z,\omega)\frac{\omega^2}{c^2} - \mathbf{k}^2\right]\mathbf{A} = -\frac{4\pi e}{c}\frac{\mathbf{v}}{v}e^{i\omega z/v}. \tag{5}$$

The only nonvanishing component of the vector potential is the component $A_z = A$ in the direction of motion of the particle, and after the substitution

$$A = \tilde{A}\varepsilon^{1/2} \tag{6}$$

we have the equation

$$\frac{d^2\tilde{A}}{dz^2} + \left[\varepsilon(z,\omega)\frac{\omega^2}{c^2} - \mathbf{k}^2 - \sqrt{\varepsilon}\left(\frac{1}{\sqrt{\varepsilon}}\right)''\right]\tilde{A} = -\frac{4\pi e}{c\sqrt{\varepsilon}}e^{i\omega z/v}. \tag{7}$$

When we have found the Fourier transform $A(\omega, \mathbf{k}, z)$ of the vector potential we can easily find the scalar potential $\varphi(\omega, \mathbf{k}, z)$ from Eq. (3) and the electric and magnetic fields of the radiation from Eq. (2).

3. THE SOLUTION OF THE EQUATIONS

The presence of the small parameter $c\varepsilon'/\omega\varepsilon^{3/2} < 1$ enables us to solve the equation (7) in the "quasi-classical approximation" (cf., e.g., [4]) The Green's function which satisfies the equation

$$\frac{d^2G(z,\xi)}{dz^2} + \left[\varepsilon(z,\omega)\frac{\omega^2}{c^2} - k^2 - \sqrt{\varepsilon}\left(\frac{1}{\sqrt{\varepsilon}}\right)''\right]G(z,\xi) = -\delta(z-\xi), \tag{8}$$

is obviously of the form

$$G(z,\xi) = \begin{cases}\psi^-(\xi)\psi^+(z)/w(\xi), & z > \xi\\ \psi^-(z)\psi^+(\xi)/w(z), & z < \xi\end{cases}, \tag{9}$$

where $\psi^+(z)$ is the solution of the homogeneous equation (8) which for $z \to +\infty$ becomes a plane wave travelling in the positive z direction, and for $z \to -\infty$ the function $\psi^-(z)$ is a wave travelling in the negative z direction: $w(z)$ is the Wronskian determinant, which in our case is a constant.

It must be noted that the solutions ψ^+ and ψ^- cannot be defined simply as the quasi-classical solutions with the corresponding exponentials [4]

$$\psi^{\pm}(z) = \frac{1}{\sqrt{p}}\exp\left\{\pm i\int_z^z p\,dz\right\}, \tag{10}$$

where

$$p^2 = \varepsilon(z,\omega)\omega^2/c^2 - k^2 - \sqrt{\varepsilon}(\varepsilon^{-1/2})'',$$

since Eq. (10) does not include the "over-the-barrier" effects, which in our case give the transition radiation. The over-the-barrier effects can be included by using the method developed in a paper Pokrovskiĭ and Khalatnikov,[5] since our homogeneous equation (8) is of the same form as the Schrödinger equation considered in that paper. To do this we must continue the solution $\psi^+(z)$, which has the form (10) for $z \to +\infty$, into the region $z \to -\infty$ not along the real axis, but along the so-called level lines, on which the phase $\int_z^z p\,dz$ of the quasi-classical function is real and the two solutions (10) are of the same order of magnitude. The solutions (10) are not valid near the complex turning points $\zeta_1 = z_1 + i\eta_1$, where $p(\zeta_1) = 0$, and also near poles of $p(z)$. At these points the asymptotic expressions (10) must be joined onto the exact solutions.

The rules for passing around such points in typical cases have been formulated previously.[5] The result is that on the level line nearest to the real axis and connecting $z \to -\infty$ and $z \to +\infty$ (for example, in Fig. 1 the line $L_1 + L_2$ passing through the turning point ζ_1) we get the correct solution of the homogeneous equation (8). We then have to continue it from the level line to the real axis of z. We can then write the Fourier transform $\tilde{A}$ in the form

$$\tilde{A}(\omega,k,z) = \frac{-4\pi e}{c}\left\{\int_{-\infty}^{z}\frac{\psi^-(\xi)\psi^+(z)}{w(\xi)\sqrt{\varepsilon}}e^{i\omega\xi/v}d\xi - \int_{\infty}^{z}\frac{\psi^-(z)\psi^+(\xi)}{w(\xi)\sqrt{\varepsilon}}e^{i\omega\xi/v}d\xi\right\}. \tag{11}$$

This formula is not very suitable for calculating the field near the diffuse boundary, since in this region the solutions ψ^+ and ψ^- are quite different from the quasi-classical solutions (10), and in order to include the exponentially small effect we need a very accurate continuation of $\psi^+(\zeta)$, $\psi^-(\zeta)$ from the level lines onto the z axis and the calculation of an integral of a rapidly oscillating function. Therefore we consider the field $A(z)$ in the radiation zone, where

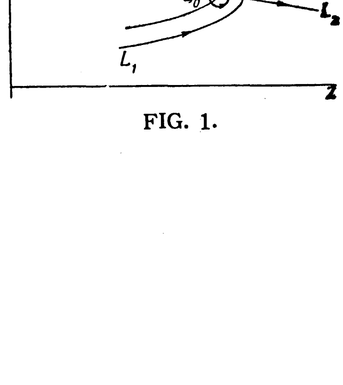

FIG. 1.

the medium can be regarded as completely homogeneous and the dielectric constant $\varepsilon(z, \omega)$ as equal to constant values $\varepsilon^{\pm}(\omega)$ for $z \to \pm\infty$. Then for $z \to +\infty$ we get

$$\widetilde{A} = \frac{2\pi e}{ic}\Big\{\psi^+(z)\int_{-\infty}^{+\infty}\frac{\psi^-(\xi)}{\sqrt{\varepsilon}}e^{i\omega\xi/v}\,d\xi - \psi^+(z)\int_z^{\infty}\frac{\psi^-(\xi)}{\sqrt{\varepsilon^+}}e^{i\omega\xi/v}d\xi + \psi^-(z)\int_z^{\infty}\frac{\psi^+(\xi)}{\sqrt{\varepsilon^+}}e^{i\omega\xi/v}\,d\xi\Big\}, \quad (12)$$

where, in accordance with Eq. (10), we have set $w = -2i$.

Since in the range of integration of the last two integrals we can use $\varepsilon = \varepsilon^+ = \text{const}$, $p = p^+ = \text{const}$, we can rewrite this expression in the form[2)]:

$$\widetilde{A} = \frac{2\pi e}{ic}\psi^+(z)\int_{L_1+L_2}\frac{\psi^-(\zeta)}{\sqrt{\varepsilon(\zeta)}}e^{i\omega\zeta/v}\,d\zeta - \frac{4\pi e}{c\sqrt{\varepsilon^+}}\frac{e^{i\omega z/v}}{(\omega/v)^2 - p_+^2}.$$

Here we have displaced the path of integration in the first integral from the real axis to the nearest level line $L_1 + L_2$ (see Fig. 1). We note that in the two last integrals in Eq. (12) the over-the-barrier effects cancel each other, and the remainder describes the radiation of a uniformly moving charge in a homogeneous medium.[3] In what follows we shall omit this part of the radiation

Accordingly all of the transition radiation is described by the integral

$$\widetilde{A} = \frac{2\pi e}{ic}\psi^+(z)\int_{L_1+L_2}\frac{\psi^-(\zeta)}{\sqrt{\varepsilon(\zeta)}}e^{i\omega\zeta/v}d\zeta, \qquad z \to +\infty \quad (13)$$

Precisely similarly we have in the region $z \to -\infty$:

$$\widetilde{A} = \frac{2\pi e}{ic}\psi^-(z)\int_{L_1+L_2}\frac{\psi^+(\zeta)}{\sqrt{\varepsilon(\zeta)}}e^{i\omega\zeta/v}d\zeta. \quad (13a)$$

Substitution of the quasi-classical solutions into these integrals would give an incorrect result. This is because the integral itself is exponentially small, whereas on the level lines the integrand is by no means small and oscillates rapidly, so that the small error in the asymptotic solutions changes the order of magnitude of the integral. We shall avoid this difficulty if we displace the path of integration so as to make the exponential factor as small as possible. When the path is displaced into the upper half of the ζ plane the integrand decreases exponentially everywhere owing to the cut-off factor $\exp(-\omega v^{-1}\,\mathrm{Im}\,\zeta)$. The possibility of such displacement is restricted, however, by the necessity of passing around the singularities of the integrand, i.e., of the solutions $\psi^+(\zeta)$ and $\psi^-(\zeta)$. Since the evaluation of the integrals (13) depends on the types and relative positions of the singularities, it is reasonable to break up our treatment of the problem into calculations of the integrals (13) in several typical cases.

a) Suppose the dielectric constant changes very little in magnitude, i.e.,

$$\frac{\Delta\varepsilon}{\varepsilon} = \frac{\max\varepsilon(z,\omega) - \min\varepsilon(z,\omega)}{\min\varepsilon(z,\omega)} < 1; \quad (14)$$

then, obviously, the entire region of rapid change of the function $p^2(\zeta)$ is concentrated near the singularities of $\varepsilon(\zeta, \omega)$. Suppose that we have as such a singularity a simple pole of $\varepsilon(\zeta)$ at the point ζ_0 with the residue ε_r. The point ζ_0 is a singular point of the differential equation (8), and consequently is a branch point of the solutions $\psi^{\pm}(\zeta)$. Therefore a cut must be drawn from ζ_0. We further deform the path of integration $L_1 + L_2$ in Eq. (13) into the contour C around the cut running from the point $\zeta = \zeta_0$ (Fig. 1). For nonrelativistic speeds $v < c\varepsilon^{-1/2}$ the integrand falls off rapidly along the path C as we go away from the pole ζ_0, so that only a region of integration $\zeta \sim v/\omega$ is important.

In this paper we confine ourselves to the case in which the residue at the pole is not very small, so that in the larger part of the region of integration we can use the quasi-classical approximation for the "field" $\sim \omega^2\varepsilon_r/c^2\zeta$:

$$\omega^2\varepsilon_r/c^2 \gg \omega/v. \quad (15)$$

Using the order-of-magnitude estimate $\varepsilon_r \sim a\Delta\varepsilon$ [a is the characteristic distance over which the important change of $\varepsilon(z)$ occurs], we rewrite the inequality (15) in the form

$$\omega a\Delta\varepsilon/c \gg c/v.$$

Under this condition we can neglect the term $\varepsilon^{1/2}(\varepsilon^{-1/2})''$ in the homogeneous equation (8) and write it approximately in the form

$$A'' + k_0^2\frac{\zeta - \zeta_1}{\zeta - \zeta_0}A = 0;$$

$$k_0^2 = c^{-2}\omega^2\varepsilon(\zeta_0) - k^2, \qquad \zeta_0 - \zeta_1 = \omega^2\varepsilon_r/k_0^2c^2. \quad (16)$$

With the change of variables

$$\xi = -2ik_0(\zeta - \zeta_0), \qquad \lambda = \tfrac{1}{2}ik_0(\zeta_0 - \zeta_1) \quad (17)$$

Eq. (16) reduces to the Whittaker equation

$$W'' + \left(-\frac{1}{4} + \frac{\lambda}{\xi} + \frac{1/4 - \mu^2}{\xi^2}\right)W = 0.$$

In the paper of Pokrovskiĭ and Khalatnikov [5]

2)Hereafter we shall everywhere suppose that $\omega < 0$, so that Jordan's lemma applies. Because $A(\mathbf{r}, t)$ is real, for negative frequencies $\omega < 0$ we have $A(-\omega) = A^*(\omega)$.

it is shown that for the solutions we need we can choose the Whittaker functions [6]:

$$\psi^{\pm}(\xi) = k_0^{-1/2} \exp\left\{\mp \lambda \ln \frac{\pm\lambda}{e}\right\} W_{\pm\lambda, 1/2}(\pm\xi). \quad (18)$$

The values of the function $W_{\pm\lambda,1/2}(\pm\xi)$ on the contour C on the two sides of the cut are connected by the formula [8]

$$W_{\lambda\mu}(e^{-2\pi i}\xi) = e^{-2\pi i\lambda} W_{\lambda,\mu}(\xi) - \frac{2\pi i e^{-i\pi\lambda}}{\Gamma(1/2+\mu-\lambda)\,\Gamma(1/2-\mu-\lambda)} W_{-\lambda,\mu}(-\xi), \quad (19)$$

so that the integral along the contour C reduces to the integral over ξ from 0 to ∞ of a Whittaker function, a power of the variable, and an exponential. The result we obtain is

$$I^{\pm} \equiv \int_C \frac{\psi^{\pm}(\zeta)}{\sqrt{\varepsilon}} e^{i\omega\zeta/v}\, d\zeta = \frac{\exp(i\omega\zeta_0/v + 3\pi i/4 \mp \lambda \ln \pm \lambda/e)}{2^{3/2}k_0^2\varepsilon_r^{1/2}}$$
$$\times\left\{(1 - e^{\mp 2\pi i\lambda}) \frac{4\Gamma(5/2)\,\Gamma(3/2)}{(\omega/k_0v \pm 1)^{5/2}\,\Gamma(5/2 \mp \lambda)} F\left(\frac{5}{2}, 1 \mp \lambda, \frac{5}{2} \mp \lambda; \frac{2\omega \mp k_0v}{2\omega \pm k_0v}\right)\right.$$
$$+ \frac{2\pi i e^{\mp i\pi\lambda}}{\Gamma(1 \mp \lambda)\,\Gamma(\mp\lambda)} \frac{4\Gamma(5/2)\,\Gamma(3/2)}{(\omega/k_0v \mp 1)^{5/2}\,\Gamma(5/2 \pm \lambda)}$$
$$\left.\times F\left(\frac{5}{2}, 1 \pm \lambda, \frac{5}{2} \pm \lambda; \frac{2\omega \pm k_0v}{2\omega \mp k_0v}\right)\right\}, \quad (20)$$

where $F(\alpha, \beta, \gamma, \xi)$ is the hypergeometric function of Gauss.

We can obtain simpler expressions for $I^{\pm}$ if the particle is nonrelativistic, $v \ll c/\varepsilon^{1/2}$. In fact, the main contribution to the integral then comes from the region $\xi \sim \lambda(v\varepsilon^{1/2}/c)^2 \ll \lambda$. Moreover, by Eq. (15), $\lambda \gg 1$, and in this range of variation of λ, ξ we can use the asymptotic expressions for $W_{\lambda,\mu}(\xi)$ for $\lambda \rightarrow \infty$. [6] For the situation shown in Fig. 1, when $\arg\lambda < \pi/2$ we have

$$I^+ = I_0\{[D_{-7/2}(-i\nu) + iD_{-7/2}(i\nu)]\, e^{-i\pi\lambda} - iD_{-7/2}(i\nu)\, e^{i\pi\lambda}\},$$
$$I^- = I_0\{D_{-7/2}(i\nu) + iD_{-7/2}(-i\nu)\}\, e^{i\pi\lambda}, \quad (21)$$

where

$$\nu^2 = \frac{2i\omega v\varepsilon_r}{c^2}, \qquad I_0 = \frac{e^{i\omega\zeta_0/v}}{2\sqrt{-2i\nu\varepsilon_r}} \left(\frac{v}{\omega}\right)^2 \Gamma\left(\frac{7}{2}\right) e^{\nu^2/4},$$

and $D_p(\nu)$ is the parabolic-cylinder function.

The first two terms in the expression for I^+ in Eq. (21) correspond to the quasi-classical solution $p^{-1/2}\exp(i\int p\, d\zeta)$ on the two sides of the cut, and the third term arises because on the left side of the cut there is an admixture of the exponential $p^{-1/2}\exp(-i\int p\, d\zeta)$, which appears when the Stokes line is crossed. For I^- there is no admixture of the "foreign" exponential $\exp(i\int p\, d\zeta)$, since we never cross the Stokes line for the solution $\psi^-(\zeta)$.

If the distance between the turning point ζ_1 and the pole ζ_0 is so large that the approximate equation (16) is not valid, then for relativistic speeds $v < c\varepsilon^{-1/2}$ the calculation of the integrals $I^{\pm}$ requires a knowledge not only of the pole part, but also of the whole behavior of the dielectric constant $\varepsilon(z)$ near the pole. In the nonrelativistic limit, on the other hand, it is enough to know the residue ε_r at the pole Furthermore, in the formulas (21) we must replace the expression for the integral

$$\int_{\zeta_1}^{\zeta_0} p\, d\zeta = -\pi\lambda,$$

which holds only for Eq. (16), by the general expression

$$\int_{\zeta_1}^{\zeta_0} \sqrt{\varepsilon(\zeta)\frac{\omega^2}{c^2} - k^2}\, d\zeta.$$

In addition, for $\operatorname{Im}\lambda \gg 1$ one can always neglect the last term in I^+, Eq. (21), which appears because of the Stokes phenomenon.

b) As our second example let us consider the case in which the singularity nearest to the real axis is a branch point of the function $\varepsilon(z, \omega)$. At this point $\varepsilon(z, \omega)$ goes to zero according to the law

$$\varepsilon(z, \omega) = \left[\frac{\zeta - \zeta_0}{a(\omega)}\right]^{\alpha}. \quad (22)$$

Near this point ($|\zeta - \zeta_0| \ll a$) this main contribution is that from the term $\varepsilon^{1/2}(\varepsilon^{-1/2})''$, and the wave equation (8) takes the form

$$A'' + \left[\left(\frac{\zeta - \zeta_0}{a}\right)^{\alpha} \frac{\omega^2}{c^2} - k^2 + \frac{1 - (1 + \alpha)^2}{4(\zeta - \zeta_0)^2}\right] A = 0.$$

By neglecting the first term in the square brackets in the neighborhood of ζ_0 and making the change of variables

$$\xi = 2k(\zeta - \zeta_0), \qquad (1 + \alpha)/2 = \mu \quad (23)$$

we reduce this equation to the form (17). Then the solutions we need, which go over into the solutions (10) with the lower limit of integration $\zeta_1 = \zeta_0 - (i/2)[(1 + \alpha)^2 - 1]^{1/2}/k$ (the location of the level lines is given in Fig. 2), can be ex-

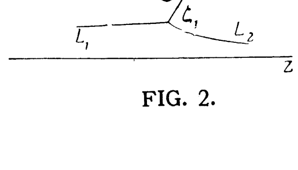

FIG. 2.

pressed in terms of Bessel functions of imaginary argument $K_\mu(\xi/2)$ [6]:

$$\psi^\pm(\zeta) = K^{-1/2}\exp\left(i\frac{\pi}{2}\sqrt{\mu^2-1/4}\right)W_{0,\mu}(\pm\xi)$$
$$= K^{-1/2}\exp\left(\pm i\frac{\pi}{2}\sqrt{\mu^2-1/4}\right)$$
$$\times(\pm\xi/\pi)^{1/2}K_\mu(\pm\xi/2). \qquad (24)$$

The calculation of the integrals $I^\pm$ along the path C can be done easily by using the connection between the cylinder functions on the two sides of the cut leaving the point ζ_0 [6]:

$$K_\mu(e^{m\pi i}z) = e^{-m\pi\mu i}K_\mu(z) - i\pi\frac{\sin m\mu\pi}{\sin\mu\pi}I_\mu(z).$$

We give the result only for the most interesting case, that of nonrelativistic speeds and a simple zero of $\varepsilon(z)$ ($\alpha = 1$):

$$I^\pm = \frac{\sqrt{8\pi a}}{k}\exp\left[i\frac{\omega}{v}\zeta_0 \pm i\frac{\pi(\sqrt{3}-1)}{4} - i\frac{\pi}{4}\right]. \qquad (25)$$

For arbitrary v and α the result can be expressed in terms of hypergeometric functions [see Eq. (20)].

c) If the singularity nearest to the real axis is a branch point ζ_0 of the function $\varepsilon(\zeta)$ and

$$\varepsilon(\zeta) \approx \varepsilon_0 + \left(\frac{\zeta-\zeta_0}{a(\omega)}\right)^\alpha,$$

then we cannot make the calculation of $I^\pm$ in general form by the method described above, since near ζ_0 the presence of the term $\varepsilon^{1/2}(\varepsilon^{-1/2})''$ in the "potential" leads to singularities of the type

$$\left(\frac{\zeta-\zeta_0}{a}\right)^\alpha\frac{1}{(\zeta-\zeta_0)^2},$$

and it is difficult to solve the wave equation.

If, however, the quasi-classical approximation can be used in almost the entire region of integration $\Delta\zeta \sim v/\omega$, as is justified for

$$\left|\varepsilon_0 - \frac{k^2c^2}{\omega^2}\right| \gg \left(\frac{c}{v}\right)^2\left(\frac{v}{\omega a}\right)^\alpha, \qquad (26)$$

then the integral over the path C going around the cut from the point ζ_0 can be calculated easily for nonrelativistic speeds by the method of Laplace [7]:

$$I^\pm = \left(\frac{v}{\omega}\right)^{1+\alpha}a^{-\alpha}\frac{\Gamma(1+\alpha)(2\varepsilon_0 - k^2c^2/\omega^2)}{\sqrt{\omega\varepsilon_0^3/c}\,(\varepsilon_0 - k^2c^2/\omega^2)^{5/4}}e^{\frac{i\pi\alpha}{2}}(1-e^{-2\pi i\alpha})$$
$$\times\exp\left(i\frac{\omega\zeta_0}{v} \pm i\int_{\zeta_1}^{\zeta_0} p\,d\zeta\right). \qquad (27)$$

Thus the three cases we have considered differ in the coefficients of the exponential in the expressions for the vector potential [see Eqs. (23), (25), (27)]:

$$A = A(\omega,k)\frac{e^{i\omega\bar\zeta_0/v}}{\sqrt{\varepsilon(\zeta,\omega)\frac{\omega^2}{c^2}-k^2}}$$
$$\times\exp\left\{\pm i\int_{\bar\zeta}^{\zeta}\sqrt{\varepsilon(\zeta,\omega)\frac{\omega^2}{c^2}-k^2-\sqrt{\varepsilon}\left(\frac{1}{\sqrt{\varepsilon}}\right)''}\,d\zeta\right\}\sqrt{\varepsilon},$$
$$z \to \pm\infty, \qquad (28)$$

where

$$A(\omega,k) = \frac{\pi e}{c}\left(\frac{v}{\omega}\right)^2\frac{\Gamma(7/2)}{\sqrt{-2i\nu\varepsilon_r}}e^{\nu^2/4}$$
$$\times[D_{-7/2}(\mp i\nu) + iD_{-7/2}(\pm i\nu)], \qquad \bar\zeta = \zeta_0,$$

if ζ_0 is a pole of $\varepsilon(\zeta)$ [cf. Eq. (21)];

$$A(\omega,k) = \frac{e}{c}\frac{\sqrt{32\pi^3 a}}{k}\exp\left[i\frac{\omega}{v}\zeta_0 \pm \frac{i\pi(\sqrt{3}-1)}{4} - i\frac{\pi}{4}\right],$$
$$\bar\zeta = \zeta_1,$$

if ζ_0 is a simple zero of $\varepsilon(\zeta)$ [cf. Eq. (25)];

$$A(\omega,k) = \frac{2\pi e}{c}\left(\frac{v}{\omega}\right)^{1+\alpha}\alpha^{-\alpha}\frac{\Gamma(1+\alpha)(2\varepsilon_0 - k^2c^2/\omega^2)e^{i\pi\alpha/2}}{\sqrt{\omega\varepsilon_0^3/c}\,(\varepsilon_0 - k^2c^2/\omega^2)^{5/4}}$$
$$\times(1-e^{-2\pi i\alpha}), \qquad \bar\zeta = \zeta_0,$$

if ζ_0 is a branch point of $\varepsilon(\zeta)$ [cf. Eq. (27)].[3)]

In cases in which $\varepsilon(\zeta)$ has several singularities of the indicated type, which give the same kind of contributions and are at distances much larger than the wavelength, we must sum all of the contributions in the expression (28). The dependence of the effect on the distance from the real axis to the nearest singularity of $\varepsilon(z)$ is always the same and is of exponential nature. Estimating this distance to be $\sim a$, we find that the entire effect is of the order $e^{-\omega a/v}$.

From Eq. (28) we can easily get expressions for the Fourier transforms of the electric and magnetic fields. When, however, we ask for the total spatial dependence of the fields and the angular distribution of the radiation, we find them as integrals of the Fourier transform $A(\omega, k, z)$, and it is impossible to calculate them explicitly without fixing the explicit form of the function $\varepsilon(z,\omega)$.

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3)Each of these expressions is the first term of an asymptotic expansion in powers of v/c.

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