ASYMPTOTIC RELATIONS BETWEEN CROSS SECTIONS IN LOCAL FIELD THEORY

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A number of asymptotic relations between the differential and total cross sections for high-energy scattering processes are derived on basis of general principles of relativistic local quantum field theory. All conclusions are based on the assumption that the cross sections do not oscillate at high energies and on the Phragmén–Lindelöf theorem in the theory of analytic functions.

1. INTRODUCTION

IN the present paper, when we speak of local theory, we mean the satisfaction of the main postulates of relativistic quantum field theory [1,2] (see also [3]), without requiring the knowledge of the specific Lagrangian. We recall that these postulates include, along with the requirement of invariance relative to the inhomogeneous Lorentz group, the existence of a complete system of physical states with positive energy, and microcausality, also a certain assumption of a mathematical character, wherein it is required that the elements of the scattering matrix be generalized functions with moderate growth (that is, that they be linear continuous functionals on the space S of infinitely smooth rapidly decreasing functions—see, for example, [4]). The significance of this requirement, which appears to be purely formal at first glance, becomes clear as we study the order of the growth of the analytic continuation of the matrix element in momentum space. It turns out that it ensures, for example, polynomial boundedness of the Fourier transformation $f_r(k)$ of the retarded amplitude in the entire region of analyticity of this function (see [4], Theorem 1). Relaxation of this requirement causes the analytic continuation of the matrix elements in momentum space to have an arbitrary growth (for example, exponential), which would correspond in the Lagrangian formalism to nonrenormalizable theory [1)]. It is not at all essential here that such a growth be exhibited by the cross sections on the real axis, as may occur in the investigation of the n-th term of a perturbation theory series, which increases polynomially in any direction in the complex energy plane. Since the infinity is in this case an essential singularity, the amplitude can increase exponentially along some directions and remain bounded along others.

The inclusion of the assumed moderate growth of the matrix elements among the main postulates of the local theory is justified by the fact that without this assumption it is impossible to obtain dispersion relations, which are in practice the only experimentally verifiable consequences of the general principles of quantum theory. In addition, as noted in [1,2], in order to compensate for the exponential increase in the amplitude in the upper half-plane with respect to the energy E, it is necessary to introduce a factor $\exp(iaE)$, where the positive constant a has the dimension of length and can be interpreted as some measure of nonlocality of the theory ("elementary length").

It turns out that if we add to the general principles of the local theory the physical assumption that the cross sections of the scattering processes do not oscillate but have a definite power-law or logarithmic growth when the energy goes to infinity (at a fixed momentum transfer), then we can obtain several experimentally verifiable equations relating the cross sections of the different processes. The first relation of this type is the equality of the total interaction cross sections of particles and antiparticles at high energies, and was obtained by Pomeranchuk [5]. Different generalizations and refinements of Pomeranchuk's statement are given in several papers [6-9]. A simple and at the same time rigorous proof of this statement under more general assumptions than in [5] was given by Meĭman [10] on the basis of the Phragmén–Lindelöf theorem.

In the present paper we shall show that, on the basis of the Phragmén–Lindelöf theorem, it is

[1)]See the discussion of this question in [3], Sec. 2.

possible to establish several asymptotic relations not only between the total cross sections but also between differential cross sections of different processes. Section 2 is introductory in character. Using the scattering of scalar particles as an example (under certain assumptions concerning the asymptotic behavior of the scattering amplitude), the method of proof is illustrated in detail and, in particular, the Pomeranchuk theorem is derived for this case. The method developed in Sec. 2 is applied in the succeeding sections to study scattering processes that are of greater interest from the physical point of view, namely the scattering of charged particles with spin and form factors. The main physical results are summarized in Sec. 6.

2. ASYMPTOTIC PROPERTIES OF THE SCATTERING AMPLITUDE OF SCALAR PARTICLES

In this section we consider the related scalar-particle processes

$$a_1 + b_1 \rightarrow a_2 + b_2, \qquad \text{(I)}$$

$$\bar{a}_2 + b_1 \rightarrow \bar{a}_1 + b_2, \qquad \text{(II)}$$

where the bar denotes the transition to the antiparticle. Let the momenta of particles a_i and b_i be equal to q_i and p_i, and let their masses be m_i and M_i, respectively. The differential cross section of process I is expressed in the following manner in terms of the invariant amplitude of this process:

$$\frac{d\sigma_I(s,t)}{dt} = \frac{\pi}{k_1k_2}\frac{d\sigma_I}{d\Omega} = \frac{1}{64\,\pi s k_1^2}\,|T_I(s,t)|^2; \qquad (1)$$

Here $s = (p_1 + q_1)^2$, $t = (p_1 - p_2)^2$, and the k_i are the three-dimensional momenta in the center-of-mass system:

$$k_i^2 = \frac{1}{4s}[s^2 - 2(M_i^2 + m_i^2)s + (M_i^2 - m_i^2)^2], \quad i = 1, 2. \qquad (2)$$

An analogous formula holds also for the cross section of process (II). The amplitude of process (II) is connected, for real s and t, with the amplitude T_I by the crossing symmetry relation:

$$T_{II}(s, t; M_1^2, m_2^2, M_2^2, m_1^2) = T_I^*(u, t; M_1^2, m_1^2, M_2^2, m_2^2),$$
$$u = M_1^2 + M_2^2 + m_1^2 + m_2^2 - s - t \qquad (3)$$

(the asterisk denotes the complex conjugate).

Let us assume that the masses M_i and m_i and the interactions of the particles a_i and b_i are such that the principles of the local theory imply analyticity of the amplitude $T_I(s,t)$ for fixed t in the complex s plane with cuts along the real axis. In addition to the cuts, the amplitude T_I has as a rule a finite number of poles on the real s axis. In the investigation of the asymptotic behavior of the amplitude as $s \rightarrow \infty$ it is convenient to subtract beforehand from the amplitude those poles terms which have a known asymptotic value (c/s), and to investigate the asymptotic behavior of the function which is bounded at finite points of the real axis. We shall henceforth denote by T_I the amplitude with the pole terms subtracted.

In order to encompass the class of amplitudes with more or less arbitrary asymptotic behavior, we introduce an auxiliary concept. We call the function $\varphi(s,t)$ admissible if for fixed t (from some interval) the function $[\varphi(s,t)]^{-1}$ is analytic, polynomially bounded in s in the upper half-plane, continuous on the real axis, and, in addition,

$$\lim_{s\to\infty}[\varphi(s,t)/\varphi(-s,t)] = e^{-i\pi\alpha(t)}, \qquad (4)$$

where $\alpha(t)$ is an arbitrary real function. An example of an admissible function is

$$\varphi(s,t) = (s+i)^{\alpha(t)}[\ln(s+i)]^{\beta(t)}[\ln\ln(s+i)]^{\gamma(t)}\ldots.$$

The following theorem holds true:

Theorem 1. Assume that for some admissible function $\varphi(s,t)$ there exist finite limits

$$V_I(t) = \lim_{s\to+\infty}[T_I(s,t)/\varphi(s,t)],$$
$$V_{II}(t) = \lim_{s\to+\infty}[T_{II}^*(s,t)/\varphi(-s,t)]. \qquad (5)$$

Then in the local theory these limits are equal to each other:

$$V_I(t) = V_{II}(t). \qquad (6)$$

From this follows also the asymptotic equality of the differential cross sections of processes (I) and (II):

$$\lim_{s\to\infty}\left\{\frac{d\sigma_I}{dt}\left(\frac{d\sigma_{II}}{dt}\right)^{-1}\right\} = 1. \qquad (7)$$

Proof. By virtue of the foregoing assumptions with respect to the function $T_I(s,t)$, the function

$$V(s,t) = T_I(s,t)/\varphi(s,t) \qquad (8)$$

is analytic, polynomially bounded in the upper half plane of s, and bounded on the real axis. In addition it follows from (3) and (5) that

$$\lim_{s\to+\infty}V(s,t) = V_I(t), \qquad \lim_{s\to-\infty}V(s,t) = V_{II}(t). \qquad (9)$$

Therefore it is possible to apply to the function V(s,t) the following theorem of Phragmén and Lindelöf[10-12]:

Theorem 2. A. Let f(z) be an analytic function of $z = r\exp(i\theta)$, regular in the domain D enclosed between two half-lines Γ_1 and Γ_2 (which make an angle π/σ with vertex at the origin) and bounded on these half-lines: $|f(z)| < C$. Then we have the following alternative: either $|f(z)| < C$ at all points of the domain D, or there exists a sequence $z_n \to \infty$ such that

$$M(r_n) = \max_{\substack{|z|=r_n \\ z\in D}} |f(z)| \geqslant \exp(\nu r_{n}^{\sigma}), \qquad \nu > 0.$$

In particular, if the function f(z) is polynomially bounded in the angle D, then the first possibility should be realized, that is, f(z) is bounded by a constant in the entire domain D.

B. Let the function $w = f(z)$ be regular and bounded in the angle D. We denote by ϵ_i $(i = 1, 2)$ the set of limiting values of these functions at $z \to \infty$ along the half-line Γ_i. Then either the sets ϵ_1 and ϵ_2 have a common point, or else one surrounds the other, separating it from the circle $|w| = C$.

In particular, if finite limits a_1 and a_2 exist when $z \to \infty$ along Γ_1 and Γ_2, then $a_1 = a_2 = a$, and $f(z) \to a$ as $z \to \infty$ uniformly in the domain D.

The function V(s, t) (8) satisfies all the conditions of Theorem 2 (in our case D is the upper half-plane, $\sigma = 1$). Since this function is polynomially bounded, its limiting values as $s \to \pm\infty$ should coincide. This proves the equality (6).

If the discarded pole terms decrease as $s \to \infty$ more rapidly than the function $T_I(s, t)$ itself, then as $s \to \infty$ the formula (1) remains valid for this part of the amplitude, too, and we obtain the asymptotic equality (7) of the differential cross sections. In the opposite case, if the total amplitude behaves at infinity like 1/s, we arrive at the same result by choosing $\varphi(s, t) = 1/s$. This proves Theorem 1.

In the particular case of elastic scattering (when $M_1 = M_2 = M$, $m_1 = m_2 = m$, and $k_1 = k_2 = k$), if we make the additional assumption that the function (4) satisfies the condition

$$\alpha(0) = 1 \qquad (10)$$

(this is the case, for example, if the forward scattering amplitude behaves like $s(\ln s)^{\beta}(\ln\ln s)^{\gamma}$ ... as $s \to \infty$), then we get from Theorem 1 the Pomeranchuk theorem on the asymptotic equality of the total cross sections of the interaction of particles and antiparticles. To this end it is sufficient to note that by virtue of the optical theorem the total cross sections $\sigma_{J\,tot}(f)$ corresponding to processes (I) and (II) are expressed in terms of the imaginary parts of the amplitudes of these processes by means of the formula

$$\sigma_{J\,tot}(s) = \frac{1}{2k\sqrt{s}}\,\mathrm{Im}\,T_J(s, 0), \qquad J = \mathrm{I}, \mathrm{II}. \qquad (11)$$

If a is a neutral scalar (or pseudoscalar) particle that coincides with its own antiparticle, then the amplitudes of processes (I) and (II) coincide:

$$T_{\mathrm{I}}(s, t) = T_{\mathrm{II}}(s, t) \equiv T(s, t), \qquad (12)$$

whereas in the case of forward scattering [when $\alpha(0) = 1$] Theorem 1 yields by virtue of (4)

$$\lim_{s\to\infty}[T_{\mathrm{I}}(s, 0)/T_{\mathrm{II}}^{*}(s, 0)] = -1. \qquad (13)$$

From (12) and (13) it follows that

$$\lim_{s\to\infty}[\mathrm{Re}\,T(s, 0)/\mathrm{Im}\,T(s, 0)] = 0, \qquad (14)$$

that is, the amplitude is asymptotically (as $s \to +\infty$) pure imaginary [2)].

From (1), (11), and (14) we obtain the following asymptotic connection between the differential and total cross section of the process under consideration.

$$\lim_{s\to\infty}\frac{16\pi}{\sigma_{tot}^{2}(s)}\frac{d\sigma(s, t=0)}{dt} = 1. \qquad (15)$$

3. ASYMPTOTIC RELATION BETWEEN THE DIFFERENTIAL CROSS SECTIONS FOR PARTICLES WITH SPIN

We shall show that the results of the preceding section can be generalized to include scattering amplitudes of particles with spin.

Let us consider first the reactions (I) and (II) under the assumption that the particles b_i have spin $1/2$ and the particles a_i have spin 0. Then the amplitudes of these processes can be written in the form [3)]

$$T_J = \bar{u}(p_2)\Big[A_J(s, t) + B_J(s, t)\,\frac{i}{2}\,(\hat{q}_1 + \hat{q}_2)\Big]u(p_1). \qquad (16)$$

For real s and t the crossing-symmetry conditions take the form

$$A_{\mathrm{II}}(s, t) = A_{\mathrm{I}}^{*}(u, t), \qquad B_{\mathrm{II}}(s, t) = -B_{\mathrm{I}}^{*}(u, t). \qquad (17)$$

The differential cross section of the process (I) is equal to

$$\frac{d\sigma_{\mathrm{I}}(s, t)}{dt} = \frac{1}{64\pi k_1^2 s}\Big\{[(M_1 + M_2)^2 - t]\,|A_{\mathrm{I}}|^2$$
$$+ \frac{1}{4}[(u - s)^2 - (m_1^2 - m_2^2)^2 - (t - 2(m_1^2 + m_2^2))$$

2) Relations (7) and (17) are derived also in a recent preprint by Van Hove [13] through generalization of the Pomeranchuk method [5] under the assumption that $\varphi(s, t) \equiv s$. Only the scattering of scalar particles is considered in [13].

3) We assume here that the relative parities $a_1 - b_1$ and $a_2 - b_2$ coincide. This assumption exerts no influence on the final results.

$$\times (t-(M_1-M_2)^2)]|B_{\mathrm{I}}|^2 + [M_1(u-s+m_2^2-m_1^2) + M_2(u-s+m_1^2-m_2^2)]\,\mathrm{Re}\,A_{\mathrm{I}}B_{\mathrm{I}}^*\Big\}, \qquad (18)$$

and the cross section of process (II) is obtained from (18) by replacing A_{I} and B_{I} with A_{II} and B_{II}, and m_1 and m_2 with m_2 and m_1.

Let us assume that the functions A_{I} and sB_{I} have the same asymptotic behavior as $s \to \pm\infty$, that is, that under some choice of the admissible function $\varphi(s,t)$ there exist finite limits

$$U_{\pm}(t) = \lim_{s\to\pm\infty} [A_{\mathrm{I}}(s,t)/\varphi(s,t)],$$

$$V_{\pm}(t) = \lim_{s\to\pm\infty} [sB_{\mathrm{I}}(s,t)/\varphi(s,t)]. \qquad (19)$$

(In the opposite case, if the functions A_{I} and sB_{I} have different asymptotic values as $s \to \infty$, then one of them can be neglected and the equality of the cross sections is ensured by Theorem 1.) By virtue of Theorem 2, the limiting values (19) are equal to each other:

$$U_+(t) = U_-(t), \qquad V_+(t) = V_-(t), \qquad (20)$$

and consequently [taking (20) into account]

$$\lim_{s\to+\infty} [A_{\mathrm{I}}(s,t)/A_{\mathrm{II}}^*(s,t)] = e^{-i\pi\alpha(t)} = \lim_{s\to+\infty} [B_{\mathrm{I}}(s,t)/B_{\mathrm{II}}^*(s,t)]. \qquad (21)$$

From (18) and (21) follows the asymptotic equality of the cross sections of processes (I) and (II) in the case when the particles b_i have spin ½. In particular, this proves, for example, the equality of the following differential cross sections:

$$\frac{d\sigma(\pi^+ + p \to \pi^+ + p)}{dt} \sim \frac{d\sigma(\pi^- + p \to \pi^- + p)}{dt},$$

$$\frac{d\sigma(K + p \to K + p)}{dt} \sim \frac{d\sigma(\bar{K} + p \to \bar{K} + p)}{dt},$$

$$\frac{d\sigma(\pi^+ + p \to K^+ + \Sigma^+)}{dt} \sim \frac{d\sigma(K^- + p \to \pi^- + \Sigma^+)}{dt}. \qquad (22)$$

We now proceed to investigate processes (I) and (II) under the assumption that the processes are elastic and that the particles a and b have spin ½. The amplitudes of processes (I) and (II) can be written in this case in the form

$$T_J = \sum_{i=1}^{6} \bar{u}_a(p_2)\Gamma_i^{(a)}u_a(p_1)\,\bar{u}_b(q_2)\Gamma_i^{(b)}u_b(q_1)\,F_{Ji}(s,t), \quad J = \mathrm{I}, \mathrm{II}, \qquad (23)$$

where

$$\Gamma^{(a)} = \Big\{1, \gamma_\mu, \gamma_5, \frac{i}{2}(\hat{q}_1+\hat{q}_2), \frac{i}{2}(\hat{q}_1+\hat{q}_2), 1\Big\},$$

$$\Gamma^{(b)} = \Big\{1, \gamma_\mu, \gamma_5, 1, \frac{i}{2}(\hat{p}_1+\hat{p}_2), \frac{i}{2}(\hat{p}_1+\hat{p}_2)\Big\}. \qquad (24)$$

In the case of scattering of particles of the same isotopic multiplet (for example, in the case of nucleon-nucleon scattering) $F_{J4}(s,t) = F_{J6}(s,t)$. The differential cross section $d\sigma_J/dt$ (after averaging over the spins) is given by the following formula in terms of the invariant amplitudes F_{Ji}:

$$\frac{d\sigma_J(s,t)}{dt} = \frac{1}{16\pi s k^2}$$
$$\times\Big\{4(4M^2-t)(4m^2-t)|F_{J1}|^2 + 8[(s-M^2-m^2)^2 + (u-M^2-m^2)^2 + 2(M^2+m^2)t]$$
$$\times |F_{J2}|^2 + 4t^2|F_{J3}|^2 + (4m^2-t)[(s-u)^2 - t(t-4m^2)]|F_{J4}|^2 + \frac{1}{4}[(s-u)^2 - t(t-4m^2)]$$
$$\times[(s-u)^2 - t(t-4M^2)]|F_{J5}|^2 + (4M^2-t)[(s-u)^2 - t(t-4M^2)]|F_{J6}|^2 + 2\mathrm{Re}\,[16Mm(s-u)F_{J1}F_{J2}^*$$
$$+ 4M(u-s)(4m^2-t)F_{J1}F_{J4}^* + 4Mm(u-s)F_{J1}F_{J5}^* + 4m(u-s)(4M^2-t)F_{J1}F_{J6}^*$$
$$- 4m[(s-u)^2 - t(t-4M^2)]F_{J2}F_{J4}^* + (s-u)[(s-u)^2 - t^2 + 4(M^2+m^2)t]F_{J2}F_{J5}^*$$
$$- 4M[(s-u)^2 - t(t-4m^2)]F_{J2}F_{J6}^* + m(u-s)\times[(s-u)^2 - t(t-4m^2)]F_{J4}F_{J5}^*$$
$$+ 4Mm(u-s)^2F_{J4}F_{J6}^* + M(u-s)\times[(s-u)^2 - t(t-4m^2)]F_{J5}F_{J6}^*\Big\}. \qquad (25)$$

The invariant amplitudes $F_{\mathrm{I}j}$ and $F_{\mathrm{II}j}$ are related for real s and t by the crossing relation

$$F_{\mathrm{II}j}(s,t) = (-1)^{j+1}F_{\mathrm{I}j}^*(u,t), \quad j = 1, \ldots, 6. \qquad (26)$$

All six functions F_{Jj} will contribute to the asymptotic value of the differential cross section (28), if the functions

$$F_{J1},\ sF_{J2},\ F_{J3},\ sF_{J4},\ s^2F_{J5},\ sF_{J6} \quad (J = \mathrm{I}, \mathrm{II}) \qquad (27)$$

have identical behavior as $s \to \infty$. We see that in the case under consideration, as in the case of scattering of a particle with spin ½ by a particle with spin zero, the amplitudes, which reverse sign under the crossing transformation (26), are contained in the cross sections with a factor s, whereas the remaining amplitudes are contained in (27) with an even power of s. From this, as before, we obtain the asymptotic equality of the differential cross sections of (I) and (II). In particular, as $s \to +\infty$, the following relations hold between the differential cross sections of the scattering of nucleons and hyperons by nucleons:

$$\frac{d\sigma(N+p\to N+p)}{dt}\sim\frac{d\sigma(\bar N+p\to\bar N+p)}{dt},\quad N=p,n;$$

$$\frac{d\sigma(Y+p\to Y+p)}{dt}\sim\frac{d\sigma(\bar Y+p\to\bar Y+p)}{dt},\quad Y=\Lambda,\Sigma.\quad(28)$$

4. ASYMPTOTIC RELATIONS BETWEEN FORWARD ELASTIC SCATTERING AMPLITUDES

We have seen, with the scalar case as an example (Sec. 2), that in the case of elastic forward scattering, under the supplementary assumption (10) [$\alpha(0) = 1$], we can obtain equality of the total cross sections, and also a few other asymptotic relations of the type (15). We shall now show that even in the case of elastic scattering of particles with spin at zero angle it is possible [making the assumption (10)] to obtain some new relations (in addition to the equality of the differential cross sections). The number of experimentally verifiable relations increases if account is taken of isotopic invariance of the strong interactions.

We begin with an examination of the elastic scattering of a pion by a nucleon. The amplitudes A and B of this process [see (16)] have the following isotopic structure:

$$A_{\beta\alpha}=A^{(+)}\delta_{\beta\alpha}+A^{(-)}\frac{1}{2}[\tau_\beta,\tau_\alpha],$$

$$B_{\beta\alpha}=B^{(+)}\delta_{\beta\alpha}+B^{(-)}\frac{1}{2}[\tau_\beta,\tau_\alpha].\quad(29)$$

The amplitudes of the physical processes

a) $\pi^++p\to\pi^++p$, a′) $\pi^-+p\to\pi^-+p$,
b) $\pi^-+p\to\pi^0+n$, b′) $\pi^0+p\to\pi^++n$,
c) $\pi^0+p\to\pi^0+p$, (30)

and of the processes obtained from (30) by making the substitutions $p\rightleftarrows n$, $\pi^+\rightleftarrows\pi^-$, and $\pi^0\rightleftarrows\pi^0$, are related with the amplitudes (29) by the equations

$$A_a=A^{(+)}-A^{(-)},\quad A_{a'}=A^{(+)}+A^{(-)},$$
$$-A_b=A_{b'}=\sqrt{2}A^{(-)},\quad A_c=A^{(+)};\quad(31)$$

Similar equations hold also for the amplitudes B.

We note that in the case of forward scattering the differential and total cross sections are expressed in terms of the same function:

$$\frac{d\sigma_i(s,0)}{dt}=\frac{1}{64\pi sk^2}|F_i(s)|^2,\quad i=a,a',b,b',c;$$

$$\sigma_{j\,tot}(s)=\frac{1}{2k\sqrt{s}}\operatorname{Im}F_j(s),\quad j=a,a',c,\quad(32)$$

where

$$F_i(s)=2MA_i(s,0)-(s-M^2-m^2)B_i(s,0).\quad(33)$$

It is possible to apply to the functions $F_i(s)$ Theorem 1, which leads, with allowance for (10), to the asymptotic relations

$$F_a\sim-F^*_{a'},\quad F_b\sim-F^*_{b'},\quad F_c\sim-F^*_c.\quad(34)$$

It follows from (31) and (34) that

$$\lim_{s\to\infty}\frac{\operatorname{Im}A^{(-)}}{\operatorname{Re}A^{(-)}}=\lim_{s\to\infty}\frac{\operatorname{Im}B^{(-)}}{\operatorname{Re}B^{(-)}}=\lim_{s\to\infty}\frac{\operatorname{Re}A^{(+)}}{\operatorname{Im}A^{(+)}}$$

$$=\lim_{\to\infty}\frac{\operatorname{Re}B^{(+)}}{\operatorname{Im}F^{(+)}}=0,\quad(35)$$

meaning that

$$\operatorname{Im}F_a\sim\operatorname{Im}F_{a'}\sim\operatorname{Im}F_c,$$

$$\lim_{s\to\infty}(\operatorname{Im}F_b/\operatorname{Re}F_b)=\lim_{s\to\infty}(\operatorname{Im}F_{b'}/\operatorname{Re}F_{b'})=0,$$

$$\operatorname{Re}F_a\sim-\operatorname{Re}F_{a'}\sim\frac{1}{\sqrt{2}}\operatorname{Re}F_b\sim-\frac{1}{\sqrt{2}}\operatorname{Re}F_{b'},$$

$$\lim_{s\to\infty}(\operatorname{Re}F_c/\operatorname{Im}F_c)\sim0.\quad(36)$$

All the relations in (36) can be verified experimentally. Indeed, the equality of the imaginary parts of F for the processes a, a′, and c of (30) leads to asymptotic equality of the total cross sections

$$\sigma_{tot}(\pi^+p)\sim\sigma_{tot}(\pi^-p)\sim\sigma_{tot}(\pi^0p).\quad(37)$$

The first of these equations is the Pomeranchuk theorem[5], while the second was proposed in [14] on the basis of an analysis of the experimental data. Further, noting that the amplitudes F_b and $F_{b'}$ for scattering with charge exchange are real when s is large, and using (36), we obtain the following interesting relation between the differential and total cross sections:

$$\left[\frac{d\sigma_a(s,t)}{dt}-\frac{1}{2}\frac{d\sigma_b(s,t)}{dt}\right]_{t=0}\sim\frac{1}{16\pi}\sigma^2_{a\,tot}(s).\quad(38)$$

Relation (38) is a generalization of (15) to include scattering of charged pions. It is seen from (36) that (15) remains in force without modification for the cross sections of scattering of a π^0 meson by a proton (process c). We can thus prove the correctness of (15) for the cross sections for the scattering of K_1^0 and K_2^0 by mesons on a nucleon (if we neglect weak interactions, then the amplitudes of these two processes are equal). We note also that the amplitude of the process $K_2^0+N\to K_1^0+N$ is real for $t=0$, $s\to\infty$, if (10) holds.

The situation is somewhat more complicated in the case of scattering of particles with spin ½, when we deal with six independent invariant functions (or with five functions in the case of nucleon-nucleon scattering[4]). The matrices Γ_i (24) are

[4]We note that whereas the dispersion relations for the scattering of a pion by a nucleon have been proved on the basis of the general principles of local theory, the analytic properties of the nucleon-nucleon scattering amplitude, which are needed for the application of Theorem 2, have been proved only in arbitrary order of perturbation theory[15,16].

so chosen as to make the hermitian and antihermitian parts of the amplitude (22) obey the following equations ($J = I, II$)

$$D_J = \frac{1}{2}(T_J + T_J^+)$$
$$= \sum_{i=1}^{6} \bar{u}_a(p_2)\Gamma_i^{(a)}u_a(p_1)\bar{u}_b(q_2)\Gamma_i^{(b)}u_b(q_1)\operatorname{Re}F_{Ji}(s,t),$$
$$A_J = \frac{1}{2i}(T_J + T_J^+)$$
$$= \sum_{i=1}^{6} \bar{u}_a(p_2)\Gamma_i^{(a)}u_a(p_1)\bar{u}_b(q_2)\Gamma_i^{(b)}u_b(q_1)\operatorname{Im}F_{Ji}(s,t). \quad (39)$$

In the case of forward scattering (assuming (10)) Theorem 2 enables us to prove the following asymptotic relations:

$$D_I(s,0) \sim -D_{II}(s,0), \qquad A_I(s,0) \sim A_{II}(s,0). \quad (40)$$

The second equality leads to the Pomeranchuk theorem concerning the identical asymptotic behavior of the total cross sections:

$$\sigma_{tot}(pp) \sim \sigma_{tot}(\bar{p}p), \qquad \sigma_{tot}(np) \sim \sigma_{tot}(\bar{n}p),$$
$$\sigma_{tot}(\Sigma N) \sim \sigma_{tot}(\bar{\Sigma}N). \quad (41)$$

The first equation in (40) is also amenable to experimental verification.

5. ASYMPTOTIC BEHAVIOR OF THE VERTEX PART

On the basis of perturbation theory we can state[17] that the vertex part $\Gamma(M^2, M^2, t)$ is an analytic function of the variable $t = q^2$ with a cut along the positive real t axis. The assumption that the form factor increases not faster than a polynomial in the complex t plane enables us to draw certain conclusions concerning the asymptotics of the processes which are described in the e^2 approximation in terms of electromagnetic form factors.

We consider, for example, the electron-proton scattering and the transformation of a proton and an antiproton into an electron and a positron. The cross section of these processes is expressed in this approximation in terms of the electromagnetic form factors F_1 and F_2 or through their linear combinations[18]:

$$G_E(t) = F_1(t) + \frac{t}{4M^2}\mu F_2(t), \quad G_\mu(t) = F_1(t) + \mu F_3(t), \quad (42)$$

where in the first process we take the values of these functions for negative t, and in the second for positive t. The latest experiments on electron-proton scattering have shown that in a wide range of large negative t the function $G_E(t)$ is approximately constant and differs from zero, while $G_\mu(t)$ tends to zero. If we assume that the form factors tend to definite limits also for positive values $t \to +\infty$, then these limits should coincide with the values of the form factors for $t \to -\infty$.

We note also that since the functions $F_i(t)$ and $G_j(t)$ are real for $t < 0$, by virtue of Theorem 2 these functions should be real for $t \to +\infty$, and $G_j(t) \to G_j(\infty)$ as $t \to \infty$ along any complex path. Here (as in the case of the two-particle Green's function) the vertex functions satisfy the dispersion relation

$$G_j(t) = G_j(\infty) + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\operatorname{Im}G_j(t')}{t'-t}dt', \qquad j = E, \mu,$$
$$\operatorname{Im}G_j(\infty) = 0. \quad (43)$$

6. CONCLUSIONS

On the basis of the principles of local field theory and on the assumption that the amplitudes of the scattering processes do not oscillate for a fixed momentum transfer t when the square of the energy $s \to \pm\infty$, and have a definite growth (for example a power-law or logarithmic growth), we deduce the following physical consequences:

1. The differential cross sections of processes of the type (I) and (II) are asymptotically equal (for example, equations (22) and (28) are asymptotically equal).

2. If the forward elastic scattering amplitude behaves, for example, like $s(\ln s)^\beta$, then the total cross sections for the interactions of particles and antiparticles, (37) and (41), are equal.

3. From the assumptions of item 2 above follows the proportionality of the differential cross section forward to the square of the total cross section (15) in the case of scattering of π^0, K_1^0, or K_2^0 mesons by protons. If we take additional account of the isotopic invariance of strong interactions, we obtain the relations

$$\sigma_{tot}(\pi^+p) \sim \sigma_{tot}(\pi^0 p),$$
$$\left[\frac{d\sigma(\pi^+ + p \to \pi^+ + p)}{dt} - \frac{1}{2}\frac{d\sigma(\pi^- + p \to \pi^0 + n)}{dt}\right]_{t=0}$$
$$\sim \frac{1}{16\pi}\sigma_{ot}^2(\pi^+p),$$

and also a few other relations between the imaginary and real parts of the different amplitudes (36) and (40).

4. As $t \to \pm\infty$ the limiting values of the form factors are equal. This enables us to connect the amplitude for the scattering of an electron by a proton at large momentum transfers with the amplitude of the $p + p \to e^+ + e^-$ reaction at high energies.

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