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ON THE SYMMETRY OF THE CLEBSCH-GORDAN COEFFICIENTS

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Some consequences of the new symmetry of the Clebsch-Gordan coefficients, discovered by Regge,[1] are considered. All the formulas of the theory of angular momentum are written in invariant form (R- symbols). A whole set of new relations are obtained between Clebsch-Gordan coefficients, Racah coefficients, and transformation matrices. In particular, Racah coefficients depending on the projections are considered. A graphical interpretation of the Regge symmetry is given.

We are all aware of the wide use in contemporary physics of the Clebsch-Gordan and Racah coefficients and of the transformation matrices. They have been the object of very careful study for a long time. It was therefore an unexpected and interesting event when Regge[1] in 1958 exhibited new symmetry laws for the Clebsch-Gordan coefficients. Despite its obvious importance, the Regge symmetry has hardly been used up to now. The present paper discusses questions related to the Regge symmetry and derives some of its consequences for the theory of angular momentum.

According to Regge, the 3jm-symbol (Wigner coefficient)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sqrt{2j_3+1}\,(-1)^{j_3-m_3}\,(j_1 j_2 m_1 m_2 \,|\, j_3 - m_3)$$

can be written in the form:

$$\begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix} = \begin{vmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{vmatrix}. \qquad (1)$$

The elements R_{ik} of the symbol are nonnegative integers. The sums of the elements of all rows and columns are equal to $j_1 + j_2 + j_3 = h$.

The value of the Wigner coefficient is invariant with respect to even permutations of rows and columns, and also to transposing of the matrix (1), and is multiplied by $(-1)^h$ for odd permutations. These rules exhibit the symmetry between definite linear combinations of angular momenta j_1, j_2, j_3, and their projections. There are altogether 72 symmetry types, whereas before Regge's work only 12 were known.

Replacing the 3jm-symbols in the 6j-symbol

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \sum \begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} \begin{pmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & -m \end{pmatrix} \begin{pmatrix} j_3 & j_2 & j_{23} \\ -m_3 & -m_2 & m_{23} \end{pmatrix}$$

$$\times \begin{pmatrix} j_1 & j & j_{23} \\ -m_1 & m & -m_{23} \end{pmatrix} (-1)^{\alpha},$$

$$\alpha = j_1 + j_2 + j_{12} + j_3 + j_{23} + j + m + m_{23} + m_3 \qquad (2)$$

by the expression (1), one can write the 6j-symbol in the form [1)]

$$\begin{vmatrix} R_{11} & R_{21} & R_{31} & R_{41} \\ R_{12} & R_{22} & R_{32} & R_{42} \\ R_{13} & R_{23} & R_{33} & R_{43} \end{vmatrix}$$

$$= \begin{vmatrix} j+j_3-j_{12} & j_1+j-j_{23} & j_2+j_3-j_{23} & j_1+j_2-j_{12} \\ j_3+j_{23}-j_2 & j_1+j_{12}-j_2 & j_3+j_2-j & j_1+j_{23}-j \\ j+j_{23}-j_1 & j+j_{12}-j_3 & j_2+j_{12}-j_1 & j_2+j_{23}-j_3 \end{vmatrix}. \qquad (3)$$

In the 6j-symbol written in this form all 12 elements are nonnegative integers. The differences between corresponding elements of rows and columns are constants, and (3) can be rewritten as

$$\begin{vmatrix} R_{11} & R_{11}+\Delta_1 & R_{11}+\Delta_2 & R_{11}+\Delta_3 \\ R_{11}+\Delta' & R_{11}+\Delta_1+\Delta' & R_{11}+\Delta_2+\Delta' & R_{11}+\Delta_3+\Delta' \\ R_{11}+\Delta'' & R_{11}+\Delta_1+\Delta'' & R_{11}+\Delta_2+\Delta'' & R_{11}+\Delta_3+\Delta'' \end{vmatrix}. \qquad (4)$$

The relation between the angular momenta j and the R_{ik} is given by the equations

$$\begin{aligned}
2j &= R_{11} + R_{23} = R_{21} + R_{13} = 2R_{11} + \Delta_1 + \Delta'', \\
2j_{12} &= R_{23} + R_{32} \\
&= R_{22} + R_{33} = 2R_{11} + \Delta_1 + \Delta_2 + \Delta' + \Delta'', \\
2j_3 &= R_{11} + R_{32} = R_{31} + R_{12} = 2R_{11} + \Delta_2 + \Delta', \\
2j_1 &= R_{21} + R_{42} = R_{22} + R_{41} = 2R_{11} + \Delta_1 + \Delta_3 + \Delta', \\
2j_{23} &= R_{13} + R_{42} = R_{12} + R_{43} = 2R_{11} + \Delta_3 + \Delta' + \Delta'', \\
2j_2 &= R_{31} + R_{43} = R_{33} + R_{41} = 2R_{11} + \Delta_2 + \Delta_3 + \Delta''.
\end{aligned} \qquad (5)$$

There are all together six independent elements in (3): the arbitrary element R_{11} and the five differences $\Delta_i^{(k)}$. The Racah coefficient written in the

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[1)]Cf.[2], formula (56.19).

form (3) is invariant under an arbitrary permutation of rows and columns. There are all together $3! \times 4!$ symmetry types, which are a direct consequence of the symmetry (1) and formula (2). These rules were found by Regge in 1959[3] as relations between 6j-symbols written in the usual form.

We shall refer to (1) and (3) as R-symbols (9R-symbol and 12R-symbol). All the formulas of the theory of angular momentum will be written as relations between R-symbols. This (Regge-invariant) way of writing automatically includes all the consequences of the new symmetry rules.

Carrying out the Regge symmetry transformation and going back to the jm-representation, we obtain a whole series of new relations between Clebsch-Gordan coefficients, Racah coefficients, and transformation matrices. Some examples of these new formulas are orthogonality relations and recursion formulas for the Clebsch-Gordan and Racah coefficients. In terms of R-symbols, the orthogonality relations for the Clebsch-Gordan coefficients have the following form:

$$\sum_{\alpha} \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21}-\alpha & R_{22}+\alpha & R_{23} \\ R_{31}+\alpha & R_{32}-\alpha & R_{33} \end{vmatrix} \begin{vmatrix} R'_{11} & R'_{12} & R'_{13} \\ R_{21}-\alpha & R_{22}+\alpha & R'_{23} \\ R_{31}+\alpha & R_{32}-\alpha & R'_{33} \end{vmatrix} = \frac{1}{R_{11}+R_{12}+1}\,\delta_{R_{ik}R'_{ik}}, \tag{6}$$

$$\sum_{\alpha} \begin{vmatrix} R_{11}+\alpha & R_{12}+\alpha & R_{13}-\alpha \\ R_{21} & R_{22} & R_{23}+\alpha \\ R_{31} & R_{32} & R_{33}+\alpha \end{vmatrix} \begin{vmatrix} R_{11}+\alpha & R_{12}+\alpha & R_{13}-\alpha \\ R'_{21} & R'_{22} & R_{23}+\alpha \\ R'_{31} & R'_{32} & R_{33}+\alpha \end{vmatrix} \times (R_{11}+R_{12}+2\alpha+1) = \delta_{R_{ik}R'_{ik}}. \tag{7}$$

From these one gets, in particular, new orthogonality relations for the Clebsch-Gordan coefficients:

$$\sum_{\gamma} (j_1+r+\gamma \;\; j_2+r-\gamma \;\; m_1+r+\gamma \;\; m_2+r-\gamma \mid j_3 \; m_3+2r)$$
$$\times (j_1+r'+\gamma \;\; j_2+r'-\gamma \;\; m_1+r'+\gamma \;\; m_2+r'-\gamma \mid j_3 \; m_3+2r') = \frac{2j_3+1}{j_1+j_2-m_1-m_2+1}\,\delta_{rr'}, \tag{8}$$

$$\sum_{r} (j_1+r+\gamma \;\; j_2+r-\gamma \;\; m_1+r+\gamma \;\; m_2+r-\gamma \mid j_3 \; m_3+2r)$$
$$\times (j_1+r+\gamma' \;\; j_2+r-\gamma' \;\; m_1+r+\gamma' \;\; m_2+r-\gamma' \mid j_3 \; m_3+2r) \times \frac{j_1+j_2-m_1-m_2+4r+1}{2j_3+1} = \delta_{\gamma\gamma'}. \tag{9}$$

The first and second recursion relations[2] have the following form in terms of R-symbols:

$$\begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23}+1 \\ R_{31} & R_{32} & R_{33}-1 \end{vmatrix} \sqrt{R_{23}(R_{33}+1)} + \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21}+1 & R_{22} & R_{23} \\ R_{31}-1 & R_{32} & R_{33} \end{vmatrix} \sqrt{R_{21}(R_{31}+1)}$$
$$+ \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22}+1 & R_{23} \\ R_{31} & R_{32}-1 & R_{33} \end{vmatrix} \sqrt{R_{22}(R_{32}+1)} = 0. \tag{10}$$

One can write the third and fourth recursion relations similarly.

An example of one of the new recursion formulas is

$$(j_1 j_2\, m_1 m_2 \mid j_3 - m_3) = \sqrt{\frac{(2j_3+1)(j_1+m_1+1)(j_1-m_1)}{2(j_3+1)(j_3-m_3+1)(j_3+m_3)}}$$
$$\times \left(j_1-\tfrac{1}{2}\, j_2\, m_1-\tfrac{1}{2}\, m_2 \,\middle|\, j_3+\tfrac{1}{2}-m_3-\tfrac{1}{2}\right)$$
$$+\sqrt{\frac{(2j_3+1)(j_2+m_2+1)(j_2-m_2)}{2(j_3+1)(j_3-m_3+1)(j_3+m_3)}}$$
$$\times \left(j_1 j_2-\tfrac{1}{2}\, m_1 m_2-\tfrac{1}{2} \,\middle|\, j_3+\tfrac{1}{2}-m_3-\tfrac{1}{2}\right). \tag{11}$$

For the Racah coefficient, the orthogonality formula expressed in terms of R-symbols has the form

$$\sum_{\beta} \begin{vmatrix} R_{11}-\alpha & R_{21}-\beta & R_{31}-\beta & R_{41}-\alpha \\ R_{12}+\beta & R_{22}+\alpha & R_{32}+\alpha & R_{42}+\beta \\ R_{13}+\beta & R_{23}+\alpha & R_{33}+\alpha & R_{43}+\beta \end{vmatrix}$$
$$\times \begin{vmatrix} R_{11}-\alpha' & R_{21}-\beta & R_{31}-\beta & R_{41}-\alpha' \\ R_{12}+\beta & R_{22}+\alpha' & R_{32}+\alpha' & R_{42}+\beta \\ R_{13}+\beta & R_{23}+\alpha' & R_{33}+\alpha' & R_{43}+\beta \end{vmatrix}$$
$$\times (R_{13}+R_{42}+2\beta+1)(R_{23}+R_{32}+2\alpha+1) = \delta_{\alpha\alpha'}. \tag{12}$$

From this one can get a new orthogonality formula

$$\sum_{\alpha} \begin{Bmatrix} j_1-\alpha+\beta' & j_2 & j_{12}+\alpha+\beta' \\ j_3-\beta'+\alpha & j & j_{23}+\alpha+\beta' \end{Bmatrix} \begin{Bmatrix} j_1-\alpha+\beta & j_2 & j_{12}+\alpha+\beta \\ j_3-\beta+\alpha & j & j_{23}+\alpha+\beta \end{Bmatrix}$$
$$\times (j_{12}+j_{23}+j_1-j_3+4\beta+1)$$
$$\times (j_{12}+j_{23}+j_3-j_1+4\alpha+1) = \delta_{\beta\beta'}. \tag{13}$$

It is not difficult to write a recursion formula in R-symbols for the Racah coefficient, which gives us new recursion relations.

It is interesting to analyze the following formula for the Racah coefficients:

$$(j_1 j_2 m_1 m_2 \mid j_{12} m_{12})\,(j_{12} j_3 m_{12} m_3 \mid jm)$$
$$= \sum_{j_{23}} ((j_1 j_2)\, j_{12} j_3 j \mid j_1\, (j_2 j_3)\, j_{23} j)\,(j_1 j_{23} m_1 m_{23} \mid jm)$$
$$\times (j_2 j_3 m_2 m_3 \mid j_{23} m_{23}). \tag{14}$$

In R-symbols it has the form

$$(-1)^{A+\alpha} \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A-\alpha \\ A_{31} & A_{32} & A+\alpha \end{vmatrix} \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ A+\alpha & B_{22} & B_{23} \\ A-\alpha & B_{32} & B_{33} \end{vmatrix}$$
$$= (-1)^{B_{23}+B_{32}} \sum_{C} (-1)^{C+\beta}(2C+1)$$
$$\times \begin{vmatrix} D_{11} & A_{11} & B_{12} & C_{11} \\ C_{12} & B_{13} & A_{12} & D_{12} \\ B_{11} & C_{13} & D_{13} & A_{13} \end{vmatrix} \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ B_{22} & A_{22} & C-\beta \\ B_{32} & A_{32} & C+\beta \end{vmatrix} \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ A_{21} & B_{23} & C+\beta \\ A_{31} & B_{33} & C-\beta \end{vmatrix}. \tag{15}$$

A whole set of new formulas follow from (15). An example is

$$(-1)^{j_{12}+j_2-j_1}\begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{pmatrix}\begin{pmatrix} j_{12} & j_3 & j \\ -m'_{12} & m_3 & m \end{pmatrix}$$

$$= (-1)^{2j}\sum_C (-1)^{C+\beta}(2C+1)$$

$$\times \begin{Bmatrix} \frac{1}{2}(j_1+j_2-m_{12}) & \frac{1}{2}(j_1+j_2+m_{12}) & j_{12} \\ j_3 & j & C \end{Bmatrix}$$

$$\times \begin{pmatrix} j_3 & \frac{1}{2}(j_1+j_2+m_{12}) & C \\ m_3 & \frac{1}{2}(j_1-j_2-m_1+m_2) & \beta \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{2}(j_1+j_2-m_{12}) & j & C \\ \frac{1}{2}(j_1-j_2+m_1-m_2) & m & -\beta \end{pmatrix}, \quad (16)$$

where $m'_{12} = j_2 - j_1$. Here the Racah coefficient depends not only on the angular momenta but also on their projections.

One can exhibit three fundamental forms for the Racah coefficients (the others are combinations of these):

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix},$$

$$\begin{Bmatrix} \frac{1}{2}(j_2+j_{12}+m_2+m_{12}) & \frac{1}{2}(j_1+j_{12}+m_1+m_{12}) \\ j_3 & j \end{Bmatrix.$$

$$\left.\begin{matrix} \times\frac{1}{2}(j_1+j_2+m_1+m_2) \\ C \end{matrix}\right\},$$

$$\begin{Bmatrix} \frac{1}{2}(j_1+j_2-m_{12}) & \frac{1}{2}(j_1+j_2+m_{12}) & j_{12} \\ j_3 & j & C \end{Bmatrix}. \quad (17)$$

Thus m can appear among the arguments of the Racah coefficient.

In this sense there is no difference in principle between the Clebsch-Gordan and Racah coefficients, more precisely between the jm- and j-symbols. In this treatment they are all included in the category of R-symbols. The way is thus opened for obtaining entirely new relations for the theory of angular momentum. A paper of the author [4] treated the general properties of the transformation matrices by means of which one makes the transition from one coupling scheme to another. Here too, using the Regge symmetry gives new relations which were not considered in [4]. For example, the transformation matrix

$$(((j_1 j_2) j_{12} j_3) j_{123} j_4 j \mid j_1 ((j_2 j_3) j_{23} j_4) j_{234} j) \quad (18)$$

is symmetric under the interchanges

$$j_1 \to \tfrac{1}{2}(j'_1 + j'_{123} - j_3 + j_2), \qquad j_{123} \to \tfrac{1}{2}(j'_1 + j'_{123} + j_3 - j_2),$$

$$j_2 \to \tfrac{1}{2}(j'_1 - j'_{123} + j_3 + j_2), \qquad j_3 \to \tfrac{1}{2}(j'_{123} - j'_1 + j_3 + j_2)$$

(the other angular momenta are unchanged). For the matrix (18) and more complicated ones we must consider 18R-symbols and symbols containing more elements R_{ik}. As was the case for the Racah coefficients, one can write transformation matrices depending on the projections m.

In general the application of the Regge symmetry to the theory of angular momentum means not only that we get new formulas having practical value, but also is a significant simplification of the theory. Regge-invariant relations contain sets of formulas in the jm-representation. Making computations or applying recursion relations and other formulas is more convenient in a large number of cases if one uses R-symbols.

For a graphical interpretation of the Regge symbol we use trilinear coordinates in the plane. We construct an equilateral triangle whose sides are the coordinate axes. The positive direction is toward the interior of the triangle. For any point the sum of its trilinear coordinates is the same and equal to h. We shall take for the three rows of a symbol the coordinates of three points. Then according to (1) the sum of the coordinates of each point is h. In addition, because the R_{ik} are non-negative and integral, one should consider only points in the interior of the triangle and lying on the intersections of the coordinate grid, (Fig. 1).

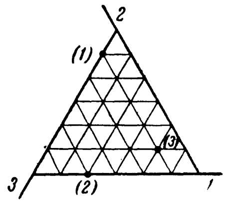

FIG. 1

The sum of the distances of the points from each of the axes is also equal to h. Giving two of the points in this coordinate system automatically determines the third. We number the axes 1, 2, 3, and the points (1), (2) and (3). The 72 symmetry relations permit us to permute the axes and the points, and to interchange their positions in a definite way.

This may be interpreted as follows: The Regge symmetry consists of the permutation symmetry (where the positions of any of the three points can be interchanged) and the symmetry of the coordinate system. The latter consists of the rotations of the triangle, C_3, reflection in the plane of the triangle, σ_h, and reflection with respect to the altitude of the triangle, σ_v, i.e., we have the space group $C_{3v} \times C_s$. This symmetry corresponds to the case of three identical particles located at the vertices of an equilateral triangle. This interpretation may be important for generalizations to the complex domain.

As an example, Fig. 1 shows the graphical description of the symbol

$$\begin{vmatrix} 5 & 1 & 0 \\ 0 & 4 & 2 \\ 1 & 1 & 4 \end{vmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{5}{2} & 3 \\ +\frac{1}{2} & -\frac{3}{2} & +1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & -2 \end{pmatrix} = \dots$$

From the graphical method of writing the Racah coefficient, one gets a simple representation of the symbol (3):

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \begin{vmatrix} \eta_1 - (j_1 + j_3) & \eta_2 - (j_1 + j_3) \\ \eta_1 - (j_2 + j) & \eta_2 - (j_2 + j) \\ \eta_1 - (j_{12} + j_{23}) & \eta_2 - (j_{12} + j_{23}) \end{vmatrix}$$

$$\times \begin{vmatrix} \eta_3 - (j_1 + j_3) & \eta_4 - (j_1 + j_3) \\ \eta_3 - (j_2 + j) & \eta_4 - (j_2 + j) \\ \eta_3 - (j_{12} + j_{23}) & \eta_4 - (j_{12} + j_{23}) \end{vmatrix}. \qquad (19)$$

Here η_1, η_2, η_3, and η_4 are the sums of the angular momenta in the bottom row of the 6j-symbol, $(j_i + j_k)$ are the sums of angular momenta having no common triad. A graphical illustration of (19) is given in Fig. 2. The 12 points characterizing the coefficient $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 5 & 4 \\ 4 & 3 & 5 & 6 \end{vmatrix}$ are at the intersections of the corresponding lines. The Racah coefficient is different only when all the points are in the sector AOB.

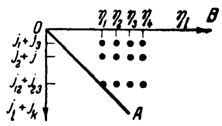

FIG. 2

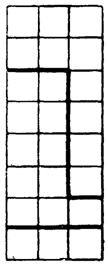

FIG. 3

We also mention the connection of the Regge symmetry with permutation symmetry. The elements of each row are a partition of the number h into nonnegative integers. To each partition there corresponds a definite Young pattern. To different symbols there correspond different partitions of the rectangle $3 \times h$ into Young patterns (Fig. 3). The Regge symbol is related to the permutations of the basis spinors.

There is no doubt that Regge's work[1] is of great importance not only for getting specific new formulas, but primarily from the point of view of general theoretical questions. Its consequences have as yet been by no means exhausted.

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