

ENERGY SPECTRUM OF A CONDUCTION ELECTRON IN A MAGNETIC FIELD

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The shape of the energy spectrum of a charged quasiparticle with a periodic dispersion law in a magnetic field is determined. It is shown that each Landau level splits into sublevels, each of which consists of other sublevels, etc. in accordance with the expansion of the reduced magnetic field into a continued fraction. The distance between levels, sublevels, etc., and the width of each "superstructure" oscillate with variation of the magnetic field H (i.e., they have an essential singularity at $H = 0$), with a universal period which is independent of the form of the quasiparticle dispersion law. In general, the indicated deviations from the usual level scheme are exponentially small; however, they turn out to be of decisive importance and have a relative magnitude of the order of unity near classical orbits which are infinite in any direction in the plane perpendicular to the magnetic field (see Fig. 1b). In this connection the wave functions are generalized functions. The features of the spectrum which have been mentioned lead (for appropriate directions of the magnetic field) to a new type of quantum oscillations with a universal period, such oscillations being possible for both degenerate and nondegenerate Fermi gases; these features also lead to "magnetic breakdown," to an abrupt change in the quantum cyclotron resonance line, and to the appearance of a periodic dependence of its periods on the inverse magnetic field.

1. PHYSICAL AND MATHEMATICAL REASONS FOR OSCILLATIONS OF THE LEVEL STRUCTURE WITH VARIATION OF THE MAGNETIC FIELD

IT is well known that the distance $\Delta\epsilon$ between the levels of a conduction electron in a magnetic field is given by [1]

$$\Delta\epsilon = \hbar\Omega, \tag{1}$$

where Ω is the classical frequency of the electron's revolution in orbit (the so-called cyclotron frequency). It is easy to arrive at formula (1) starting from simple physical considerations. The electron will resonantly absorb energy from an electromagnetic field of frequency ω , if the distance between its energy levels is equal to the energy of a quantum: $\omega = \Delta\epsilon/\hbar$. It is therefore obvious that in the classical case, when $\hbar = 0$, resonance corresponds to equality of ω to the frequency of orbital revolution: $\omega = \Omega$. Hence, $\Omega = \lim (\Delta\epsilon/\hbar)$ as $\hbar \rightarrow 0$, which is equivalent to formula (1).

Thus, regardless of the particle Hamiltonian, for trajectories on which the concept of classical

frequency of revolution has meaning corrections to (1) can arise only in subsequent approximations in (higher powers of) \hbar , and these corrections will be of interest only if they lead to substantially new additions to $\Delta\epsilon$, which are easily distinguishable from the usual corrections. Large deviations from (1) appear near trajectories where the very concept of Ω loses meaning, i.e., near the trajectories shown in Fig. 1, where it is impossible to indicate single-valued motion of the quasiparticle.

The spectrum in the neighborhood of trajectories of the type shown in Fig. 1a was investigated by Zil'berman.^[2] It was found that in the neighborhood of such trajectories the Landau levels corresponding to closed orbits broaden into bands, but gaps appear in the continuous spectrum cor-

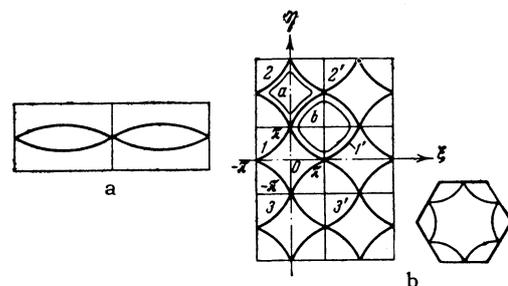


FIG. 1

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responding to open orbits. The broadening of levels into bands must also occur for trajectories of the type illustrated in Fig. 1b. Besides, such trajectories lead in addition to fundamentally new physical results, in particular, to an oscillating dependence of the bandwidths and the distances between them on the inverse magnetic field.²⁾

The physical reason for the oscillations is as follows. It is well known (see, for example, [3]) that the components cp_x/eH and p_y (z is the direction of the magnetic field) of the conduction-electron momentum behave like canonically conjugate coordinate and momentum. This allows one to talk about a de Broglie wave, but with a wavelength which depends on the magnetic field. Naturally, the propagation of such a wave through a periodic lattice will have singularities when the lattice period is equal to an integral number of wavelengths. The effect turns out (in the quasi-classical theory) to be exponentially not small if the interference between different cells of the crystal lattice is not small, i.e., if the classically inaccessible region is small the orbits are similar to those shown in Fig. 1b.

Let us turn to the mathematical reason for the oscillations. Since the probability for a transition between bands is exponentially small, and the probability for intersection is proportional to \hbar^4 (see [2]), we confine our investigation to a single band. In this connection, according to [2-3], the conduction-electron energy operator has the form

$$\hat{\varepsilon} = \sum_{\mathbf{a}} a_{\mathbf{a}} \exp(i\hat{\mathbf{p}}\mathbf{a}), \quad (2)$$

if in the classical case $\varepsilon(\mathbf{p}) = \sum_{\mathbf{a}} a_{\mathbf{a}} \exp(i\mathbf{p} \cdot \mathbf{a})$.

The summation in (2) is carried out over all vectors \mathbf{a} of the crystal lattice.

We assume that the magnetic field \mathbf{H} is directed along one of the principal axes—the z axis. Choosing the vector potential \mathbf{A} in the form $\mathbf{A} = (-Hy, 0, 0)$, recognizing that in the absence of interband transitions $\hat{\mathbf{r}} = i\hbar\partial/\partial\mathbf{P}$ (see [3]), and assuming

$$y = c(\xi - P_x a_1)/eHa_1, \quad \beta = eH\hbar a_1 a_2/c \quad (3)$$

(a_1 and a_2 are the periods of the lattice along the principal axes x , and y), we obtain (for brevity, the spin contribution $\pm\mu_0\mathbf{H}$ is not written out)

$$\begin{aligned} \hat{\varepsilon} &= \sum_{n_1, n_2} a_{n_1 n_2}(p_z) \exp\left(in_1 \xi + \beta n_2 \frac{d}{d\xi}\right) \\ &= \sum_{n_1, n_2} a_{n_1 n_2} \exp\left(\frac{1}{2} in_1 n_2 \beta + in_1 \xi\right) \hat{L}_{\beta n_2} \\ &= \sum_n S_n \left(\xi + \frac{1}{2} \beta n\right) \hat{L}_{n\beta}; \end{aligned} \quad (4)$$

$$S_n(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \varepsilon(\varphi, \eta) e^{-in\eta} d\eta, \quad \hat{L}_x \psi(\xi) \equiv \psi(\xi + \chi); \quad (4a)$$

$$\varepsilon(p_x a_1, p_y a_2, p_z) \equiv \varepsilon(\xi, \eta, p_z)$$

$$= \sum_{n_1, n_2} a_{n_1 n_2}(p_z) \exp(in_1 \xi + in_2 \eta). \quad (5)$$

As is clear from Eq. (4), the system of levels is infinitely degenerate with respect to P_x and is periodic in the parameter β with period 2π , since the Schrödinger equation, $\hat{\varepsilon}\psi = \varepsilon\psi$, is invariant with respect to the change

$$\beta \rightarrow \beta + 2\pi, \quad \psi(\xi) \rightarrow \psi(\pi + \xi) \exp(i\pi^2/\beta + i\pi\xi/\beta). \quad (5a)$$

(This statement was proved in [5] for the general case of arbitrary direction of the magnetic field.)

The operator $\hat{\varepsilon}$ commutes with the unitary operators

$$\hat{A} = \exp(2\pi i\xi/\beta), \quad \hat{B} = \exp(2\pi d/d\xi), \quad (6)$$

associated with periodicity of the classical ε with respect to p_x and p_y . The operator \hat{A} corresponds to the integral of motion $\exp(2\pi i\xi/\beta)$, whose real or imaginary part can be called the "periodic quasicordinate"; the operator \hat{B} corresponds to the "periodic quasimomentum."

In general, the "period quasicordinate" and the "periodic quasimomentum" do not have simultaneously determined values, so that the eigenfunctions of the Hamiltonian can be written in either the \hat{A} - or in the \hat{B} -representation. However, if

$$\beta = 2\pi/s, \quad (7)$$

where s is an integer, then the operators \hat{A} and \hat{B} commute. It is natural to expect that such values of β will prove to be singular and that, with respect to $2\pi/\beta$, periodicity will occur with period

$$\Delta(2\pi/\beta) = 1. \quad (8)$$

2. NATURE OF THE SPECTRUM IN THE GENERAL CASE

According to Eq. (4), the Schrödinger equation for the problem under consideration has the form

²⁾We emphasize that if the magnetic field is parallel to a three-fold symmetry axis, or to a symmetry axis of high order trajectories of the type shown in Fig. 1b will certainly appear. We also remark that formula (2) is not exact, as asserted in [3], but is valid for arbitrary \mathbf{p} to within an accuracy of order \hbar^2 inclusively.

$$\sum_n S_n (\xi + \beta n/2) \psi (\xi + \beta n) = \varepsilon \psi (\xi). \quad (9)$$

We assume

$$\xi = \xi_0 + \beta m, \quad m - \text{integer}; \quad (10a)$$

$$\psi (\xi_0 + \beta m) \equiv \varphi_m (\xi_0); \quad (10b)$$

$$S_{m'-m} [\xi_0 + \beta (m + m')/2] = \tilde{S}_{mm'} (\xi_0), \quad \tilde{S}_{mm'}^* = \tilde{S}_{m'm}. \quad (11)$$

We obtain

$$\hat{\varepsilon} (\xi_0) \varphi_m (\xi_0) \equiv \sum_{m'} \tilde{S}_{mm'} (\xi_0) \varphi_{m'} (\xi_0) = \varepsilon \varphi_m (\xi_0). \quad (12)$$

(It is curious that Eq. (11) describes Brownian motion in a periodic lattice with period β (a "jump" of several sites at once is allowed, where the probabilities to "jump" and to remain on a given site depend periodically, with period 2π , on the position in the lattice.)

It is obvious that the quantity ξ_0 in Eq. (12) plays the role of a parameter and varies according to Eq. (10a) in an (arbitrary) interval of length β . Let

$$\beta = 2\pi r/n; \quad r, n - \text{integers}. \quad (12a)$$

It is easy to verify that the Hermitian operator $\hat{\varepsilon}$ commutes with the unitary operators $\hat{L}_{2\pi r} \equiv \hat{L}_{\beta n}$, and since both operators have a complete set of eigenfunctions, this means that $\hat{\varepsilon} (\xi_0)$ has common eigenfunctions with $\hat{L}_{\beta n}$, i.e.,

$$\varphi_m (\xi_0) = e^{i\omega m} f_m (\xi_0), \quad f_{m+n} (\xi_0) = f_m (\xi_0). \quad (12b)$$

Relation (12b) enables us, having solved Eq. (12), to find the eigenfunctions and eigenvalues of $\hat{\varepsilon}$ which depend on two continuous parameters: ξ_0 and ω . As will be clear from what follows, it is more convenient to first obtain a general solution $\psi (\xi)$ of Eq. (9), which is not an eigenfunction (i.e., increasing at infinity), for arbitrary values of ξ . Then, according to Eq. (12b), the relationship

$$\psi (\xi + 2\pi r) = e^{i\omega} \psi (\xi), \quad \xi = \xi_0 + j\beta, \quad (12c)$$

written at an arbitrary point ξ_0 , gives the eigenvalues and eigenfunctions of $\hat{\varepsilon}$, which depend on ξ_0 and ω , and ψ , and which are determined, naturally, only at discrete points [Eq. (12) does not apply in general to other points]. The reason for the appearance of ξ_0 as a parameter and its meaning will be explained below.

Let us turn to the determination of the form of the general solution. We shall assume, only for the sake of simplifying the discussion, that the Fourier series (5) contains a finite number $\mu < 2\pi/\beta$ of terms, so that the sum in Eq. (12) contains μ terms. Then the Schrödinger equation

reduces to simply a recursion relation expressing $\psi (\xi_0 + \mu\beta)$ in terms of $\psi \{ \xi_0 + (u-1)\beta \}$, $\psi \{ \xi_0 + (\mu-2)\beta \}$, ..., $\psi (\xi_0)$. The assignment of μ "initial" conditions—the values of the function ψ at the indicated μ points—for a given ξ_0 determines the function at all points displaced relative to ξ_0 by integer multiples of β . Further, the μ independent initial conditions (for example,

$$\varphi_i (\xi_0) = \delta_{is}, \quad 0 \leq i' \leq \mu - 1, \quad 0 \leq s \leq \mu - 1, \quad (13)$$

where each condition corresponds to a prescribed s), determine μ linearly independent solutions $\Phi_m^{(s)} (\xi_0)$, in terms of which the general solution for arbitrary initial conditions $\varphi_i (\xi_0) = C_i (\xi_0)$, $0 \leq i \leq \mu - 1$) can be expressed:

$$\varphi_m (\xi_0) = \sum_s G_s (\xi_0) \Phi_m^{(s)} (\xi_0). \quad (14)$$

(For conditions (13) $G_s = C_s$.)

In order to write down the general form of the function ψ on the entire axis, it is necessary to assign arbitrary "initial conditions" $C_s (\xi_0)$ [which determine $G_s (\xi_0)$] at all points of the interval of length β . It is more convenient to regard C_s and G_s as given on the entire axis, but periodic with period β , so that

$$\psi (\xi) = \sum_s G_s (\xi) \Phi^{(s)} (\xi) = \mathbf{G} (\xi) \mathbf{\Phi} (\xi), \quad \mathbf{F} = (F_1, \dots, F_\mu),$$

$$\mathbf{\Phi} (\xi) = \mathbf{\Phi}_m (\xi_0); \quad (15)$$

$\mathbf{\Phi}_m (\xi_0)$ is defined on an interval of length β by some kind of "standard" conditions, for example, by conditions (13).

Thus, the general solution of Eq. (9) contains μ arbitrary periodic functions (with period β), owing to the infinite order of the operator $\hat{\varepsilon}$ [see Eq. (4)]. (In this connection, nonsmooth functions are also admissible as solutions, because the operator $\hat{\varepsilon}$ of infinite order has a rather special form and reduces to translation operators). The infinite order is associated with periodicity of the lattice, and if in the quasiclassical situation the function $\Phi^{(1)}$ determines the motion along a given branch 1 of the orbit, then the function G_1 ensures the motion along branches 2, 3, and so on (see Fig. 1b).

It is important to emphasize that "from the point of view" of any operator defined on a crystal lattice and corresponding therefore to a periodic physical quantity (energy, velocity, etc.), the function $\mathbf{G} = \sum_n \gamma_n A^n$, which commutes with these operators, is an integral of the motion, i.e., a constant.

Knowledge of the general solution (15) of the Schrödinger equation enables us, in principle, to

determine the energy levels. Since in the \hat{B} -representation [see Eq. (6)] the eigenfunctions have the Bloch form, the following equality must hold on the interval of ξ equal to $(0, 2\pi)$:

$$G(\xi + 2\pi) \Phi(\xi + 2\pi) = e^{i\omega} G(\xi) \Phi(\xi), \tag{16}$$

$$G(\xi) = \sum_{m=-\infty}^{\infty} \gamma_m e^{2\pi i m \xi / \beta}$$

or

$$\int_0^{2\pi} e^{in\xi} G(\xi + 2\pi) \Phi(\xi + 2\pi) d\xi = e^{i\omega} \int_0^{2\pi} e^{in\xi} G(\xi) \Phi(\xi) d\xi, \tag{17}$$

where $-\infty < n < \infty$, n is an integer.

The condition for the solvability of Eqs. (17) gives the eigenvalues ε and γ_m . However, for a concrete calculation and even for an analysis of the structure of the spectrum, expression (17) is not very suitable, since it does not take the specific form of the Schrödinger equation (9) into account. It is more convenient to use condition (12c).

Upon transition from the point ξ in cell 1 (Fig. 1) to the point $\xi + 2\pi$, we again obtain one of the solutions at the point ξ , as a consequence of the invariance of the operator \hat{E} with respect to a displacement of 2π [see (4)], so that $\Phi^{(i)}(\xi + 2\pi) = g_{ik}(\xi) \Phi^{(k)}(\xi)$, i.e.,

$$\Phi(\xi + 2\pi) = \hat{g}(\xi) \Phi(\xi), \quad \hat{g}(\xi + \beta) = \hat{g}(\xi), \tag{18}$$

and $G(\xi) \cdot \Phi(\xi)$ goes over into $G(\xi + 2\pi) \hat{g}(\xi) \cdot \Phi(\xi)$ (since G is conserved and plays the role of a constant with respect to \hat{E}). Physically the operator \hat{g} corresponds to a quasiclassical transition from one orbit to another one displaced by a lattice period, for example, from orbit 1 to orbit 2' (see Fig. 1b).

Translating r times in succession by 2π , we obtain ($2\pi r = n\beta$), using (18) and (12c):

$$\begin{aligned} G(\xi + 2\pi r) \Phi(\xi + 2\pi r) &\equiv G(\xi) \Phi(\xi + 2\pi r) \\ &= G(\xi) \hat{g}\{\xi + 2\pi(r-1)\} \hat{g}\{\xi + 2\pi(r-2)\} \\ &\dots \hat{g}(\xi) \Phi(\xi) = G(\xi) \hat{T}(\xi) \Phi(\xi) = e^{i\omega} G(\xi) \Phi(\xi), \\ \xi &= \xi_0 + j\beta, \quad 0 \leq \beta \leq \mu - 1. \end{aligned} \tag{19}$$

Since here $G \neq 0$, so

$$\hat{T}(\xi) \Phi(\xi) = e^{i\omega} \Phi(\xi), \tag{20}$$

$$\hat{T}(\xi) = \prod_{\sigma=1}^r \hat{g}(\xi + 2\pi\sigma - 2\pi) = \prod_{\sigma=1}^r \hat{g}(\xi - 2\pi\sigma), \tag{21}$$

and the equation for the eigenvalues (the condition for the solvability of Eq. (20) with $\Phi \neq 0$) has the form

$$|\hat{T}(\xi) - e^{i\omega}| = 0. \tag{22}$$

The order of this determinant as well as the rank of the matrices \hat{T} and \hat{g} is obviously equal to μ . It is possible to determine the form of the matrix \hat{T} by using the recurrence relation

$$\hat{T} = \hat{\tau}_r, \quad \hat{\tau}_{k+1} = \hat{\tau}_k \hat{g}(\xi - 2\pi k), \tag{23a}$$

$$\hat{\tau}_0 = 1. \tag{23b}$$

The matrices \hat{T} , $\hat{\tau}_k$ and \hat{g} are periodic with period β ; therefore, marking with a tilde the same operators but with dependence on the argument multiplied by $2\pi/\beta$ and with period 2π , we obtain a recursion relation analogous to (9), where the following quantity will play the role of β :

$$\begin{aligned} \beta_1 &= (2\pi/r)(n - r[n/r]) = 2\pi n_1/r \\ &= 2\pi v = 2\pi(2\pi/\beta - [2\pi/\beta]), \end{aligned} \tag{24}$$

where β_1 is a periodic function of $2\pi/\beta$ with period equal to unity ($[x]$ is the integer closest to x and smaller than x).

The solution of this problem is quite analogous to that already carried out, but this time we need not eigenfunctions, but a solution satisfying the given "initial condition" (23b). Since it is necessary to translate r times to go from $\hat{\tau}_0$ to $\hat{\tau}_r$, i.e., to translate by $2\pi n_1$ in the new variables ($\xi_1 = 2\pi\xi_0/\beta - \beta_1 k$), is necessary but the parameter is $\beta_1 = 2\pi n_1/r$, we get once again a matrix \hat{T}_1 playing the role of \hat{T} , a parameter $\beta_2 = 2\pi(2\pi/\beta_1 - [2\pi/\beta_1])$, and so forth.

The nature of the spectrum is determined by the continued-fraction expansion of β :

$$\frac{\beta}{2\pi} = \frac{1}{s + \frac{1}{s_1 + \frac{1}{s_2 + \dots + \frac{1}{s_l}}}}, \tag{25}$$

and the calculation terminates with the continued fraction cut off ($\beta < 2\pi$ since—see Sec. 1—the levels are periodic in β with period 2π).

Since each "superstructure" is determined by its "own" β_i , which is periodic in $2\pi/\beta_{i-1}$, it is clear that structure of the spectrum has oscillating dependences on $2\pi/\beta$ (which corresponds to a change of s by unity), on $2\pi/\beta_1$ (which corresponds to a change of s_1 by unity), etc. each with a period equal to unity. In this connection, instability of the spectrum with respect to β occurs.

Let $\beta = 2\pi/s$. Then the eigenvalues of Eq. (12) are determined by a matrix of rank s and there are s energy bands [since ε has a discrete index: $\varepsilon = \varepsilon_\nu(\xi_0, \omega)$]. A small change in β :

$$\frac{\beta}{2\pi} = \frac{1}{s + 1/s_1}, \quad s_1 \gg 1,$$

splits each band into a series of adjacent narrow subbands. Whenever s_1 decreases by an integer the subbands become wider, merging together and decreasing in number. For $s_1 = 2$ the spectrum consists of $2s + 1$ bands, for $s_1 = 1$ the spectrum is similar to the initial spectrum (oscillation in $2\pi/\beta$), but consists of $s + 1$ bands.

We can investigate analogously the complete fraction (25). For $s, s_1, s_2, \dots, s_l \gg 1$ there are s levels, each of which splits into s_1 sublevels, each sublevel consists of s_2 sublevels, etc.; bands appear where the fraction is cut-off. Each of these "systems" oscillates with its own period. This case will be investigated in more detail below. The periodicity in $2\pi/\beta \sim 1/H$ means, obviously, the presence of an essential singularity in the spectrum at $H = 0$.

Let us turn to the determination of the role of the parameters ξ_0 and ω . As noted above, the assignment of ψ at the points $\xi_0, \xi_0 + \beta, \dots, \xi_0 + (\mu - 1)\beta$ (points c_1, c_2, \dots, c_μ) enables us, after a finite number of steps, to obtain the values of ψ at the points c'_1, \dots, c'_μ , displaced relative to the initial points by $(M + 1)\beta$, $M = [2\pi/\beta]$.

Requiring fulfillment of the Bloch condition $\psi(\xi) = e^{-i\omega} \psi(\xi + 2\pi)$, one can obtain values of ψ at the points $c'_i - 2\pi = \tilde{c}_i$ on the initial interval $(0, 2\pi)$ and start the recurrence relations all over. If β and 2π are commensurable we return after a finite number of "revolutions" to the starting points, where the initial values of ψ must be obtained. (By the same token, obviously, the finiteness of ψ at infinity and its Bloch form are guaranteed.) This condition also gives an equation for the eigenvalues and eigenfunctions of $\hat{\mathcal{E}}$, which depend (in addition to the parameters P_X and p_Z) on two continuous parameters: ω and the "initial" value of ξ_0 . If β and 2π are not commensurable, we approach only arbitrarily close to the initial c_i , and it is necessary (in order to guarantee that ψ is not an increasing function at infinity) to require coincidence of the corresponding values of ψ at the limit. Of course, here ψ turns out to be defined only on a denumerable set of points, and we need not count on the smoothness of ψ .

Mathematically such a periodicity "every 2π " is associated with the fact that the Hermitian operator $\hat{\mathcal{E}}$ and the unitary operators \hat{A} and \hat{B}^r commute with each other. Commutation of unitary operators possessing complete sets of eigenfunctions and an Hermitian operator means that all

these operators have common eigenfunctions

$$\psi = D_\beta(\xi - \xi_0) e^{i\omega\xi} U(\xi);$$

$$U(\xi + 2\pi r) = U(\xi), \quad D_\beta(x) = \sum_{m=-\infty}^{\infty} \delta(x - m\beta). \quad (26)$$

(We note that D_β is a special case of the function G , but any periodic function with period β can be expanded in D_β .) Functions corresponding to different ξ_0 are defined, in general, at different points. (The exception is the denumerable set of points ξ_0 for which $\xi'_0 + q_1\beta = \xi''_0 + q_2\beta + 2\pi q_3$, where q_1, q_2, q_3 , are integers. Such values of ξ'_0 and ξ''_0 correspond, in the main, to the same function.) Therefore, if the same eigenvalue ε corresponds to them, it is possible to regard these function as values of the same eigenfunction ψ of the Schrödinger equation, given at different points.

It is most convenient to write this function, by taking a linear combination of the functions (26) corresponding to all possible ξ_0 [in the fundamental interval $(0, \beta)$] for a given value of ε :

$$\Psi_\varepsilon(\xi) = \int_0^\beta G(\xi_0) D_\beta(\xi - \xi_0) \times \exp\{i\xi\omega(\varepsilon, \xi_0)\} U\{\xi; \xi_0, \omega(\varepsilon, \xi_0)\} d\xi_0; \quad (27)$$

$$\varepsilon(\xi_0, \omega) = \varepsilon, \quad \text{Im } \omega(\varepsilon, \xi_0) = 0, \quad (28)$$

where $G(\xi_0)$ is an arbitrary function; since it is defined in the interval $(0, \beta)$, we can periodically continue it along the entire axis. The integration in (27) is carried out only with respect to such values of ξ_0 (and only such values of ε are eigenvalues) for which (28) has a solution. From (27) we obtain the eigenfunction in the form (15). Similarly, it would be possible to obtain the Bloch solution by determining from $\varepsilon = \varepsilon(\xi_0, \omega)$ not $\omega(\xi_0, \varepsilon)$ but $\xi_0 = \xi_0(\omega, \varepsilon)$, i.e., by choosing all values of ω leading to a given value of ε .

It is essential that, in accordance with (27) and (28), whole intervals of ξ exist on which $\Psi_\varepsilon(\xi) \equiv 0$. By virtue of the periodicity in ξ_0 with period β , the number of such intervals in the segment $(0, 2\pi)$ is of order $ss_1 \dots s_l$. This indicates the singular nature of the eigenfunctions. The $\Psi_\varepsilon(\xi)$ are generalized functions, because only such functions vanish over an entire region without being identically equal to zero. This can be seen even from the form (4) of the operator $\hat{\mathcal{E}}$, which has in the Fourier representation a form similar to (4) and coinciding with (4) if $\varepsilon(p_X, p_Y) = \varepsilon(p_Y, p_X)$. Therefore, the Fourier components of the eigenfunction, as well as the function itself, have the

Bloch form and are not damped at infinity, as for a smooth function.

The presence of only one parameter ε defining the state is associated with conservation of the number of states. The number of states is determined only by the number of basis vectors of the given function space, and different $\hat{\varepsilon}$ only lead to different choices of basis vectors, the different choices corresponding to different complete sets of eigenfunctions of the different operators $\hat{\varepsilon}$. For a Fermi gas, conservation of the number of states is physically due to conservation of the number of particles, since a decrease of the number of states in some case, when all states are occupied, would lead to the necessity for a reduction in the number of particles.

It is convenient to trace the noted regularities to the case of strong coupling, $\hat{\varepsilon} = A_1 \cosh(\beta d/d\xi) + A_2 \cos \xi$ for $\beta = \pi, \pi/2$.

Out of all the eigenvalues found, only those which correspond to $\xi_0 = 0$, i.e., wave functions which are periodic in quasimomentum space (this corresponds to the definition of p-space itself), have physical significance.

For arbitrary boundary conditions on the interval $0 \leq \xi < 2\beta$, these levels in general do not correspond to stable solutions (for these levels only the class of functions having physical significance—periodic in p-space—are stable, i.e., do not increase at infinity). The latter (always stable solutions) are obtained when no finite intervals of ξ are present on which $\psi \equiv 0$, and these solutions correspond to a discrete spectrum (for given P_x and p_z).

It is curious that two different definitions of the eigenvalues ε are thus possible for the type of equations considered: 1) eigenvalues for which some kind of bounded-at-infinity solution exists, and 2) eigenvalues for which any solution is non-increasing (usually, for example, these two definitions are the same for differential equations of finite order).

3. QUASICLASSICAL QUANTIZATION

It was shown in Sec. 2 that the determination of the energy levels reduces to a determination of the operator \hat{g} , i.e., to the continuation of the general solution of the Schrödinger equation from the point ξ to the point $\xi + 2\pi$ with account of the possibility of quantum transition of the electron from one cell of the periodic lattice to another. In the case of trajectories that are far from open, such a transition is associated with penetration due to the tunnel effect through the region which

is forbidden for classical motion of particles; this effect is exponentially small—its probability is of order $e^{-a/\beta}$, where $a \sim 1$ and, of course, cannot be observed in actually attainable magnetic fields (in a field $H \sim 10^5$ Oe the transition probability $\sim \exp(-10^4)$ to $\exp(-10^5)$).

If the trajectories are close to trajectories which are open in one direction (trajectories of the type shown in Fig. 1a), the electron can go from one cell to another but stays always on the same branch $\eta = \eta(\xi)$, $\varepsilon(\xi, \eta) = \varepsilon$ (with exponential accuracy). Thus the level system is stable with regard to the parameter β , the wave functions are regular functions, and there are no oscillations of the levels with variation of the magnetic field. The spectrum in this case was investigated by Zil'berman (see [2,3]).

We are interested in the case when the trajectories are similar to trajectories passing through the entire periodic lattice (Fig. 1b). Solely for the sake of concreteness, we assume that the unit cell is square in the plane $p_z = \text{const}$ (i.e., the unit cell of the crystal lattice is a rectangular parallelepiped), that the energy corresponds to a classical orbit of the type a [Fig. 1b; the calculation for the energies corresponding to orbits of type b is quite similar, if one selects as the "basic" cell the cell with vertices at the points $(0, 0), (0, 2\pi), (2\pi, 0), (2\pi, 2\pi)$, and with center at the point (π, π)], and that in the given cell the equation $\varepsilon(\xi, \eta) = \varepsilon$ has relative to η only two real solutions ($\pm \eta$).

The motion of the particle is quasiclassical everywhere, with the exception of a small neighborhood of points near the self-intersection ($\xi = k\pi$, k is an integer). In order to determine the solution in the quasiclassical region it is convenient to use the fact that, as was shown in [6], an Hermitian operator, accurate to terms of order β^2 , does not depend on the method of symmetrization (one can demonstrate this for the operator (4) by direct calculation), so that if $\varepsilon = \sum d_{mn} \xi^m \eta^n$ classically, then

$$\hat{\varepsilon} = \sum \frac{1}{2} d_{mn} \left\{ \xi^m \left(i^{-1} \beta \frac{d}{d\xi} \right)^n + \left(i^{-1} \beta \frac{d}{d\xi} \right)^n \xi^m \right\}. \quad (29)$$

In the quasiclassical region Eq. (29) gives³⁾

³⁾It is convenient, operating with the operator (29) on the function Ψ in the form given by the Wentzel-Kramers-Brillouin method, for calculation of the quantity $\hat{K}^n \Psi = (i^{-1} \beta d/d\xi)^n \Psi$ by using the identity

$$\sum_{n=0}^{\infty} (n!)^{-1} x^n \hat{K}^n \equiv \hat{L}_{-ix}.$$

$$\psi = \frac{1}{\sqrt{|v|}} \exp \left\{ \frac{i}{\beta} \int_{\xi}^{\xi'} \eta(\xi') d\xi' \right\}, \quad v = \frac{\partial \varepsilon(\xi, \eta)}{\partial \eta}; \quad (30)$$

$$\varepsilon(\xi, \eta) = \varepsilon. \quad (30a)$$

Equation (30a) in the case considered by us has the solution:

$$\eta = \pm \tilde{\eta} + 2\pi n, \quad n - \text{integer} \quad (31)$$

(since, by assumption, $\varepsilon(\xi, \eta) = \varepsilon(\xi, -\eta) = \varepsilon(-\xi, \eta) = \varepsilon(\xi, \eta + 2\pi) = \varepsilon(\xi + 2\pi, \eta)$, and in a single cell there are only two real solutions). In addition to the real solutions (31), still other complex and purely-quantum solutions exist in general. However, we can prove that these give, owing to the Bloch condition, an exponentially small contribution to the levels, and we shall therefore not consider them.

Writing down the general solution and noting that arbitrary values of n ensure the appearance of the functions G_+ and G_- , we find

$$\psi = \frac{G_+(\xi)}{\sqrt{|v|}} \exp \left\{ \frac{i}{\beta} \int_0^{\xi} \tilde{\eta}(\xi') d\xi' \right\} + \frac{G_-(\xi)}{\sqrt{|v|}} \exp \left\{ -\frac{i}{\beta} \int_0^{\xi} \tilde{\eta}(\xi') d\xi' \right\} \\ \equiv G_+(\xi) \Phi_+(\xi) + G_-(\xi) \Phi_-(\xi), \quad (32)$$

where Φ_+ corresponds to the branch α_{11} , and Φ_- corresponds to the branch β_{11} (see Fig. 2).

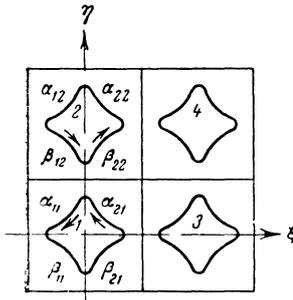


FIG. 2

Now let us investigate how the transition from $\xi (-\pi < \xi < 0)$ to $\xi + 2\pi$ takes place. Near the point $\xi = 0$ there is an appreciable probability that the particle shifts from branch α_{21} (Fig. 2) to the nearby branch β_{22} (the probability of a direct transition to, say, α_{22} is exponentially small). In this connection, a "joining together" of the quasiclassical solutions corresponding to the branches $\alpha_{11}, \beta_{12}, \alpha_{21}$, and β_{22} takes place. In order to find the "transition rule" from $\xi < 0$ to $\xi > 0$, it is necessary to write down the exact solution of the Schrödinger equation, which assumes an extremely simple form in the neighborhood of the point $\xi = 0$. As expected (and as can be shown by starting from (4), since the point

$\xi = 0, \eta = \pi$ corresponds classically to self-intersection of the orbits and $\partial \varepsilon / \partial \xi = \partial \varepsilon / \partial \eta = 0$, so that

$$\varepsilon(\xi, \eta) \approx -\frac{1}{2} \alpha \xi^2 + \frac{1}{2} \alpha (\eta - \pi)^2, \quad (33)$$

we obtain in the quantum case

$$\left\{ \left(i^{-1} \beta \frac{d}{d\xi} - \pi \right)^2 - \xi^2 \right\} \Phi = \frac{2\varepsilon}{\alpha} \Phi. \quad (34)$$

(Since it makes no difference where the origin of energy measurement is, we measure from the energy $\varepsilon_0(p_z)$ at which self-intersection occurs for a given value of p_z . Everywhere in what follows, $\varepsilon - \varepsilon_0(p_z)$ is denoted by ε and is assumed to be equal to $\beta\mu, \mu > 0$, since we are interested in orbits close to self-intersection and located inside the chosen cell.)

Introducing

$$\Phi = e^{i\pi\xi/\beta} \chi_1, \quad \xi = x \sqrt{\frac{1}{2}\beta}, \quad \rho = \frac{\mu}{\alpha} > 0, \quad (35)$$

$$\chi_1(\xi) = \tilde{\chi}(x),$$

we obtain for $\tilde{\chi}(x)$ the parabolic-cylinder equation

$$\tilde{\chi}'' + \left(\frac{1}{4} x^2 + \rho \right) \tilde{\chi} = 0, \quad (36)$$

whose asymptotic solutions for $|x| \gg 1$ are well-known both for $x > 0$ and for $-x > 0$ (see, for example, [7]).

As always in the quasiclassical theory, a region of values of ξ exists where Eq. (34) is already valid and the asymptotic solution of (36) holds, yet the quasiclassical treatment is still valid with the branches α_{11}, α_{21} corresponding to $\eta(\xi) = \pi - \sqrt{\xi^2 + 2\beta\rho}$ (the arithmetical root) and to the functions $C_1 \Phi_+$ (for $\xi < 0$) and $C_2 \Phi_+$ (for $\xi > 0$), and the branches β_{12}, β_{22} corresponding to $\eta(\xi) = \pi + \sqrt{\xi^2 + 2\beta\rho}$ and to the functions $D_1 A \Phi_-$ (for $\xi < 0$) and $D_2 A \Phi_-$ (for $\xi > 0$); $A = \exp(2\pi i \xi / \beta)$. "Joining together" the value in this region (for $1 \gg |\xi| \gg \beta^{1/2}$), we obtain a connection between D_1, D_2 and C_1, C_2 , and ascertain the contributions of the functions Φ_+ and $A \Phi_-$ (and hence Φ_+ and Φ_-) on going from $\xi > 0$ to $\xi < 0$.

Denoting (Φ_+, Φ_-) by Φ , and the functions for $\xi > 0$ or $\xi < 0$ by plus or minus superscripts, respectively, we have

$$\Phi^+ \rightarrow \hat{R} \Phi^-, \quad \hat{R} = \begin{pmatrix} s & i e^{-\pi\rho} A^{-1} \\ -i e^{-\pi\rho} A & s^* \end{pmatrix}, \quad (37)$$

$$s = \sqrt{1 + e^{-2\pi\rho}} e^{i\chi}, \quad \chi = \rho \ln(\rho e^{-1}) - \arg \Gamma(1/2 + i\rho). \quad (37a)$$

Since, as already mentioned earlier, the functions G and A play the roles of constants with respect to the energy operator, (37) implies that

$$G \Phi^+ \rightarrow G \hat{R} \Phi^- \quad (38)$$

and that for the subsequent motion with respect to ξ it is once again sufficient to consider the functions Φ_+ and Φ_- on the branches α_{21} and β_{21} . It is obvious that

$$\Phi_{\pm}^+ = \exp\left(\pm \frac{i}{\beta} \int_0^{\pi-\delta} \tilde{\eta} d\xi'\right) \tilde{\Phi}_{\pm}^-, \quad \varepsilon(\pi - \delta, 0) = \varepsilon, \quad \delta > 0; \quad (39)$$

$$\tilde{\Phi}_{\alpha}^{\gamma} = \frac{1}{V|\nu|} \exp\left(\alpha \frac{i}{\beta} \int_{\pi+\gamma\delta}^{\xi} \tilde{\eta} d\xi'\right), \quad \alpha, \gamma \sim \pm. \quad (40)$$

In order to find the rule for the transformation from $\tilde{\Phi}^+$ to $\tilde{\Phi}^-$, it is necessary to carry out near the point $\xi = \pi$, where

$$\varepsilon(\xi, \eta) = \varepsilon(\pi + \xi', \eta) \approx \frac{1}{2} \alpha \xi'^2 - \frac{1}{2} \alpha \eta^2 = \varepsilon = \beta \mu, \quad (41)$$

$$\eta(\xi') = \sqrt{\xi'^2 - 4\rho}, \quad \delta = 2\sqrt{\rho} \quad (42)$$

[the notation is the same as in (34)], the "joining" of the quasiclassical solution with the asymptotic solution of the Schrödinger equation, $\hat{E}\psi = \varepsilon\psi$, corresponding to (41):

$$\begin{aligned} \Phi(\xi) &= \Phi(\pi + x\sqrt{\beta/2}) = \tilde{\chi}(x), \\ \tilde{\chi}'' + (x^2/2 - \rho)\tilde{\chi} &= 0. \end{aligned} \quad (43)$$

The calculation shows that

$$\tilde{\Phi}^+ \rightarrow \hat{U} \tilde{\Phi}^-, \quad \hat{U} = \begin{pmatrix} se^{\pi\rho} & -ie^{\pi\rho} \\ ie^{\pi\rho} & s^*e^{\pi\rho} \end{pmatrix}. \quad (44)$$

Finally, noting that $\tilde{\Phi}_{\pm}^{\pm}(\xi + 2\pi) = \exp\left(\pm \frac{i}{\beta} \int_{\pi+\delta}^{2\pi} \tilde{\eta} d\xi'\right) \tilde{\Phi}_{\pm}^{\pm}(\xi)$, one can write down, taking into account the "constancy" of G and A , how the transformation from a general solution at the point $\xi + 2\pi$ to a general solution at the point ξ occurs, and find that

$$\begin{aligned} G(\xi + 2\pi) \Phi(\xi + 2\pi) &\rightarrow G(\xi) \hat{g}(\xi) \Phi(\xi); \\ \hat{g}(\xi) &= \begin{pmatrix} e^{\pi\rho} s^2 e^{-i2\pi} - A^{-1} & -e^{\pi\rho} i s^* e^{i2\pi} + i s A \\ e^{\pi\rho} i s e^{-i2\pi} - i s^* A^{-1} & e^{\pi\rho} s^{*2} e^{i2\pi} - A \end{pmatrix}, \end{aligned} \quad (45)$$

$$\hat{g} = \frac{1}{\beta} \int_{-\pi+\delta}^{\pi-\delta} \tilde{\eta}(\xi') d\xi'. \quad (45a)$$

Before going on to the following concrete calculations, we note that in the quasiclassical theory the form of the matrices \hat{R} and \hat{U} (uniquely determining the matrix \hat{g}) can be found, apart from inessential details from simple physical considerations. This is useful in practice, since it enables us to establish general rules (valid with exponential precision), to determine the order of the terms discarded during the above calculation, and to justify the neglect of these terms; is much more complicated to do all

this starting from the equations themselves.

Let us return to Fig. 2. A particle moving along the segment β_{12} is either reflected from the classically inaccessible region and goes along the path β_{22} , or else penetrates this region and goes onto the path segment α_{11} . Here

$$A\Phi_-^- \rightarrow s_{21}\Phi_+^- + s_{22}A\Phi_-^+, \quad (46)$$

and since the sum of the reflection and transmission coefficients is equal to unity, we get

$$|s_{21}|^2 + |s_{22}|^2 = 1. \quad (47)$$

We can obtain a second relation either by considering the fate of a particle moving along α_{21} or, what is simpler, by taking the complex conjugate of (46) and noting that $\Phi_-^* = \Phi_+$. We have

$$A^{-1}\Phi_+^- \rightarrow s_{21}^*\Phi_-^- + s_{22}^*A^{-1}\Phi_+^+. \quad (48)$$

The form of \hat{R} follows from (46) and (48).

If the "barrier" is only slightly transparent (it is easy to verify that in this case the transmission coefficient is of order $\Delta = e^{-\pi\rho}$), then in the basic approximation $s_{21} = \gamma_{21}\Delta$ ($\gamma_{21} \sim 1$) and $s_{22} = 1$. The subsequent terms in s_{21}/Δ and in s_{22} are expansions in powers of Δ^2 , since they are associated with return to that same region, each time with double penetration of the barrier. This in particular, governs the main correction to s_{22} , viz., $s_{22} = 1 - \gamma_{22}\Delta^2$, and according to Eq. (47), $2 \operatorname{Re} \gamma_{22} = |\gamma_{21}|^2$.

Similar arguments can also be made near the point $\xi = \pi$, and for our case of a square cell we can relate the transition coefficients at these points. As a result, if the requirement $|\hat{g}| = 1$ is taken into consideration, it turns out that formula (45a) ensures the accuracy needed for what follows.

In the case under investigation $|\hat{T}| = \Pi |\hat{g}| = 1$, with $T_{11} = T_{22}^*$ and $T_{12} = T_{21}^*$. (It is easy to check by induction that the product of operators possessing the latter property also has this property. Both relations have a common nature: the first, $|\hat{T}| = 1$, is associated with conservation of the number of particles; the second is associated with the reversibility of wave propagation—see the transition from (46) to (48). Therefore Eq. (22) for the determination of the levels takes the form

$$\frac{1}{2} \operatorname{Sp} \hat{T} = \operatorname{Re} T_{11} = \cos \omega. \quad (49)$$

Since we are only interested in the $\operatorname{Sp} \hat{T}$, cyclic permutation is permissible in the product (21), and we can take

$$\hat{\tau}_k = \begin{pmatrix} s_k & \sigma_k \\ \sigma_k^* & s_k^* \end{pmatrix} (\hat{T} = \hat{\tau}_r, \quad \hat{\tau}_{k+1} = \hat{\tau}_k \hat{g}(\xi_0 - 2\pi k); \quad \hat{\tau}_0 = 1). \quad (50)$$

From (50) and (49) we find

$$\begin{aligned} \operatorname{Re} s_r &= \cos \omega; \\ \beta_{k-1}^* s_{k+1} + \beta_k^* s_{k-1} &= (\alpha_{k-1}^* \beta_k^* + \alpha_k \beta_{k-1}^*) s_k; \quad s_0 = 1, \quad s_1 = \alpha_0; \\ \alpha_k &= e^{\pi \rho} s^2 e^{-i \Sigma} - \psi_k^{-1}, \quad \beta_k = e^{-\pi \rho} i s^* e^{i \Sigma} + i s \psi_k; \\ \psi_k &= e^{i \varphi_k}, \quad \varphi_k = 2\pi \beta^{-1} (\xi_0 - 2\pi k), \end{aligned} \quad (51)$$

where s is defined by formula (37a). As expected, the connection at the two points ($\xi = 0, \eta = \pi$; $\xi = \pi, \eta = 0$) leads to the equation corresponding to strong coupling.

Let us return to the determination of the levels. If $\Delta \sim 1$ and $\beta_1 \sim 1$, formula (51) for a specific case demonstrates the reasoning of Sec. 2. We shall illustrate the general calculation for the simpler case $\Delta \ll 1$. We set

$$\begin{aligned} \Sigma &= (m + 1/2) \pi + 2\chi + \Delta \sigma; \\ (-1)^{m-1} \psi_k &= e^{i \varphi_k} = f_k; \quad s_k = (-1)^{k(m-1)} V_k. \end{aligned} \quad (52)$$

To the precision we are interested in, we obtain

$$\begin{aligned} \frac{1}{2} \{1 - i \Delta (\sigma - f_{k-1}^{-1})\} V_{k+1} + \frac{1}{2} \{1 - i \Delta (\sigma - f_k^{-1})\} V_{k-1} \\ + (\cos \rho_{k-1} - \sigma) \{1 - i \Delta (\sigma - f_k^{-1})\} V_k = 0; \end{aligned} \quad (53)$$

$$V_0 = 1, \quad V_1 = i \Delta^{-1} + \sigma - f_0^{-1}. \quad (54)$$

Equation (53) is a recurrence relation for V_k . We shall be interested in the first two terms of the expansion of V_k in powers of Δ , corresponding to the initial conditions (54):

$$V_k = i \Delta^{-1} u_k + w_k; \quad (55)$$

$$\frac{1}{2} (u_{k+1} + u_{k-1}) + \cos \rho_{k-1} \cdot u_k = \sigma u_k; \quad (56)$$

$$u_0 = 0, \quad u_1 = 1; \quad (57)$$

$$\frac{1}{2} (w_{k+1} + w_{k-1}) + \cos \rho_{k-1} \cdot w_k = \sigma w_k + \frac{1}{2} (e^{-i \beta_1} - 1) f_k^{-1}. \quad (58)$$

$$w_0 = 1, \quad w_1 = \sigma - f_0^{-1}, \quad \beta_1 = 2\pi (2\pi / \beta - [2\pi / \beta]). \quad (59)$$

Since the initial conditions for u_k and the coefficients in (56) are real, u_k is real, and the levels are determined by the equation

$$\operatorname{Re} w_r = (-1)^{r(m-1)} \cos \omega = \cos \omega_1. \quad (60)$$

(Equations (58)–(60) are easily obtained from (49) and (53)–(54) both for $\Delta \ll 1$ and for $\beta_1 \ll 1$, when $f_k \approx f_{k-1}$.)

Even now we can draw two important conclusions. First, the oscillating correction to the levels, as is evident from (52) and (56)–(59), is of order Δ . This is natural since the “initial” cell 1 (Fig. 2) induces both in cell 2 and in cell 3

states related by the Bloch condition. The probability of a transition to these cells is of order Δ ; there is no need for a direct transition from 1 to 4 associated with a probability Δ^2 . Second, the splitting of the Landau levels is determined by universal equations, independent of the form of the dispersion law and the corresponding strong coupling in the square lattice. This splitting is the same for all levels without exception; only the “width” $\Delta = \exp(-\pi \rho_m)$ of the splitting varies.

Let us return to the solution of the problem. In order to determine w_k it is sufficient to know two linearly independent solutions of Eq. (56): u_k satisfying conditions (57) and v_k satisfying the conditions

$$v_0 = 1, \quad v_1 = 0. \quad (61)$$

Then

$$\begin{aligned} w_k &= (\sigma - e^{i \beta_1} f_0^{-1}) u_k + v_k \\ &+ (e^{-i \beta_1} - 1) u_k \sum_{l=0}^k \sum_{m=0}^l f_{m-1}^{-1} \frac{u_m u_{m-1}}{u_l u_{l-1}}. \end{aligned} \quad (62)$$

Assuming

$$\xi_1 = 2\pi \xi_0 / \beta + (k - 1) (\pi - \beta_1);$$

$$u_k(\xi_0) = u^{(1)}(\xi), \quad v_k(\xi_0) = u^{(2)}(\xi), \quad (63)$$

we find the differential equation and boundary conditions, equivalent to (56), (57), (61):

$$\{\operatorname{ch}(\beta_1 d / d \xi_1) + \cos \xi_1\} u^{(1,2)}(\xi_1) = \sigma u^{(1,2)}(\xi_1); \quad (64)^*$$

$$u^{(2)}(2\pi \xi_0 / \beta) = u^{(1)}(2\pi \xi_0 / \beta + \beta_1) = 0,$$

$$u^{(1)}(2\pi \xi_0 / \beta) = u^{(2)}(2\pi \xi_0 / \beta + \beta_1) = 1, \quad (65)$$

where it is necessary to translate along ξ_1 from the point $(2\pi \xi_0 / \beta) + \beta_1$ by $2\pi n_1$ [compare with (19)]—the problem solved earlier for the general equation. Now, however, the equation and the boundary conditions are standard, only the position of the “boundaries” changes.

Formula (62) gives the exact solution of the problem for arbitrary β_1 . It is advisable, however, to develop a method which is more convenient for a constructive calculation with $\beta_1 \ll 1$. We shall start from

$$\begin{aligned} \{\operatorname{ch}(\beta_1 \frac{d}{d \xi_1}) + \cos \xi_1\} w(\xi_1) \\ = \sigma w(\xi_1) + \frac{1}{2} (1 - e^{i \beta_1}) e^{-i \xi_1} w(\xi_1 - \beta_1), \end{aligned} \quad (66)$$

$$w(2\pi \xi_0 \beta^{-1} + \beta_1) = 1,$$

$$w(2\pi \xi_0 \beta^{-1}) = \sigma + (-1)^{m-1} \exp(-2\pi i \xi_0 / \beta + i \beta_1), \quad (67)$$

*ch = cosh.

which are equivalent to (58) and (59). We obtain a particular solution of the inhomogeneous equation (66) for the case when $\Phi(\xi_1 - \beta_1)$ (the meaning of Φ is the same as above; the Φ_i correspond to specific branches) stands on the right side in the region near $2\pi\xi_0/\beta$. This, evidently, is sufficient for solution of the problem. In the region separated from $2\pi\xi_0/\beta$ by $2\pi j$, the quantity $\hat{\tau}_j(\xi_1)\Phi(\xi_1 - \beta_1)$ appears on the right side, where $\hat{\tau}_j(\xi_1 + \beta_1) = \hat{\tau}_j(\xi_1)$, $\hat{\tau}_0(\xi_1) = 1$. Supposing in the j -th region that

$$w(\xi_1) = \frac{1}{i\beta_1} \hat{\tau}_j(\xi_1) \hat{C}(\xi_1) \Phi(\xi_1), \quad \hat{C}(\xi_1) = \begin{pmatrix} C_1(\xi_1) & 0 \\ 0 & C_2(\xi_1) \end{pmatrix}, \quad (68)$$

we obtain the equation for $C_\alpha(\xi_1)$:

$$\frac{1}{i\beta_1} \{C_\alpha(\xi_1 + \beta_1) - C_\alpha(\xi_1)\} \Phi_\alpha(\xi_1 + \beta_1) - \frac{1}{i\beta_1} \{C_\alpha(\xi_1) - C_\alpha(\xi_1 - \beta_1)\} \Phi_\alpha(\xi_1 - \beta_1) = 2e^{-i\xi_1} \Phi_\alpha(\xi_1 - \beta_1). \quad (69)$$

Since

$$\Phi_\alpha(\xi_1) = \frac{1}{V|v|} \exp \left\{ \frac{i}{\beta_1} \int_0^{\xi_1} \tilde{\eta}_\alpha(\xi'_1) d\xi'_1 \right\}, \quad \cos \eta_\alpha + \cos \xi_1 = \sigma,$$

then Eq. (69) for $\beta_1 \ll 1$ gives

$$C_\alpha(\xi_1) = \int_0^{\xi_1} \exp \{ -i\xi'_1 - i\tilde{\eta}_\alpha(\xi'_1) \} d\xi'_1 / \sin \tilde{\eta}_\alpha(\xi'_1), \quad (70)$$

and in the region which is sufficient for us [since it is possible to investigate Eq. (65) for $j = 0$, $C_\alpha(\xi_1 + 2\pi r) = C_\alpha(\xi_1)$] we have $\tilde{\eta}_\alpha \neq 0$.

Thus, the problem reduces to the expression of u with boundary conditions (65) in terms of the function Φ and the subsequent calculation of (67). The following results are obtained for $\beta_1 \ll 1$, $|\sigma/\beta_1| \gg 1$. In the basic approximation there exists a system of quasi-equidistant Landau levels, which are determined by formula (52) for $\Delta = 0$. The distance between levels is of order β , their total number equals $[2\pi/\beta]$. In the next approximation each of the levels splits into quasi-equidistant sublevels, the distance between which is of order $\beta_1 \Delta = \beta_1 \exp(-\pi\rho_m)$ and is determined by a formula corresponding to (52) for (64), but the distance oscillates in $2\pi/\beta$ with a period equal to unity (because β_1 oscillates in the same way).

Having the rules for the general case, it is now easy to write down the subsequent corrections leading to structures and oscillations of higher order. The bands correspond to the last term in the continued fraction (25). We notice that knowledge of the eigenvalues and of the general solution on the entire axis enables us, by using formulas (19), (27), (28), and (30)–(32), to find

the eigenfunctions of the problem. As can be seen from these formulas, the presence of a large number ($\sim ss_1 \dots s_l \gg 1$) of narrow sections on each of which the eigenfunction is identically equal to zero (the number of these sections changes very little or goes away from the singular trajectory, whereas their width decreases exponentially) is characteristic of the quasi-classical theory. Their appearance is related to the "fine structure." The latter, in relative units (referred to Δ_m), has a universal character for all levels.

4. DISCUSSION OF THE RESULTS

It was shown in Secs. 2 and 3 that the system of levels near a particular type of singular trajectories is unstable with respect to the magnetic field. When $\beta = 2\pi/s$ levels appear, determined by the equation $\text{Re } \alpha_0 = \cos \omega$, $\xi_0 = 0$ (see p. 639), tending to quasi-equidistant Landau levels with increase of the number of levels (the zeroth number corresponds most closely to the level of the singular trajectory). The distances between levels oscillate with respect to the inverse magnetic field, and the amplitude of these oscillations also decreases exponentially with the number. Following an infinitesimal change of the magnetic field, $\beta = 2\pi/(s + s_1^{-1})$, ($s_1 \rightarrow \infty$), each of the levels splits into s_1 infinitesimally close sublevels, which change exponentially rapidly into quasi-equidistant infinitesimally close sublevels upon moving away from the singular (with respect to energy) point. The distance between the latter has in turn an oscillating component (with a new period with respect to the magnetic field). The wave functions corresponding to energy eigenvalues are generalized functions.

Of course, the finite mean free path (due to any cause, in particular, due to the Fermi-liquid interaction) leads to a natural line width of eigenvalues and annihilates both the instability with regard to the magnetic field and the singularity of the wave functions, because the calculation of a discrete distance between energy levels, smaller than or of the order of the natural line width, has no physical meaning. As a result, for a given mean free path l there remains a finite number of "superstructure" Landau levels and a finite number of periods of oscillations with respect to the magnetic field. The most "stable" are the oscillations with period $\Delta(2\pi/\beta) = 1$; for their existence it is only necessary that $l > 2\pi r \sim hc/eHa$, where r is of the order of the Larmor radius. This is clear both physically (it is suffi-

cient for the electron to be able to move into cells 2 and 3 of Fig. 2, and by the same token this also guarantees $\hbar v/l < \hbar\Omega$) and mathematically; if y which appears in the formula for ε through the vector potential, changes by an amount l , the change of the argument must be of the order of or larger than β . (The fact that the distance between levels changes by $\sim\beta\hbar\Omega$ when the field changes by one period has no significance; it is important that the total number of levels changes by a quantity of order unity, and a similar situation occurs also for "ordinary" quantum oscillations.)

Oscillations of the levels lead to quantum oscillations of thermodynamic and kinetic quantities, with universal period and amplitude (relative to the classical quantity) of order $\beta\delta\varepsilon/kT$, where $\delta\varepsilon$ is the energy interval in which the concept of quasiparticle has meaning in spite of the Fermi-liquid interaction, and $l > 2\pi r$. Usually $\delta\varepsilon \gg kT$. Since the "new" quantum oscillations are caused by oscillations of the energy spectrum itself and are not associated, like the "usual" oscillations, with discrete passage of levels through the boundary Fermi, they may also occur for degenerate as well as for nondegenerate Fermi gases, and, moreover, at sufficiently high temperatures corresponding to $l > 2\pi r$. The latter circumstance is a consequence of the universality of the period of oscillations, excluding "interference" for different electrons. Of course, special directions of the magnetic field, for which trajectories of the type under investigation exist, are necessary for observation of the "new" oscillations.

If $l \gg 2\pi r$, observation of the "fine structure" of these oscillations is possible—oscillations having a smaller period which depends on the mean path length. In fact, upon change of β from $2\pi/s$ to $2\pi/(s + \delta) = 2\pi/(s + N^{-1})$ ($N \gg l/2\pi r$) the level structure does not change, since the distance between sublevels will be less than the natural linewidth, and the amplitude of the oscillations is exponentially small in comparison with Nr/l . But even for $\delta \sim 2\pi r/l$, oscillations of the level structure begin upon transition from $N \sim l/r$ to $N + 1$ with the appropriate period. As long as $N \sim l/r$, only these oscillations will be observed. For $1 \ll N \ll l/r$ the following superstructure appears with a new period, etc. The number of superstructures is determined by the requirement [see Eq. (25)] that $s_1 s_2 \dots s_i \lesssim l/2\pi r$, $s_1, \dots, s_i \gg 1$ (if $s_j \sim 1$, the structure of the corresponding "superstructures" is extremely

complicated, they may overlap, giving a single "superstructure"⁴⁾).

Upon going ultimately from s to $s + 1$ the entire picture begins to be repeated all over again. Of course, in order to construct a systematic theory of oscillations of the "fine structure" (having an extremely small period) it is necessary to solve the problem from the very beginning with account of the mean free path.

"Magnetic breakdown"^[8] due to orbits exactly like those shown in Fig. 1b is another effect, which requires knowledge of the spectrum in the case under investigation for its theoretical interpretation.

Third, for directions of the magnetic field corresponding to the appearance of the orbits shown in Fig. 1b, the specific structure of the spectrum essentially manifests itself in quantum cyclotron resonance.^[9] The periodic dependence of the distance between levels on the inverse magnetic field leads to a periodic dependence (with universal period in the inverse magnetic field) of the distance between the resonance frequencies; the presence of "superstructures" sharply changes the shape of the resonance line. Apparently not sharp minima, but "kinks" turn out to be characteristic of the line shape for the corresponding fields.

Finally, since the condition $l > r$ is very "mild" and is realized even in weak magnetic fields, perhaps singularities in the behavior of the various thermodynamic and kinetic quantities appear in such fields.

Until now a comparison of the experimental results with theory would enable one in all cases (also including those cases related to quantization in "ordinary" trajectories—see the beginning of Sec. 1) to examine only the validity of introducing the conduction electron as a quasiparticle with a "classical" dispersion law. The present work for the first time gives the possibility to establish the validity of introducing the operator $\hat{\varepsilon}$ for the electron as a quasiparticle and to establish the validity of writing it in the form (2).

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