RESTRICTIONS IMPOSED ON THE MAGNITUDE OF THE CROSS SECTION FOR THE REACTION $e^+ + e^- \to \pi^+ + \pi^-$ *BY ANALYTICITY REQUIREMENTS*

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Inequalities which set a lower limit on the $e^+ + e^- \to \pi^+ + \pi^-$ reaction cross section averaged over energy of the pair are obtained on the basis of the analytic properties of the form factor $F(k^2)$ for the transition of a γ quantum into a $\pi^+ + \pi^-$ pair.

THE cross section for the transformation of an electron-positron pair into a $\pi^+\pi^-$ pair is given by (see, e.g. [1]):

$$\sigma(x) = \tfrac{1}{16}\pi\alpha^2 m^{-2} x^{-5/2}(x-1)^{3/2}|F(x)|^2. \qquad (1)$$

Here m is the mass of the pion, $x = \epsilon^2/4m^2$, ϵ is the energy of the pair in their center-of-mass system, $\alpha = 1/137$ ($\hbar = c = 1$). We ignore terms of order m_e^2/m^2 and confine ourselves to production of pions via a single γ quantum; hence the accuracy of our considerations is of order α.

Let F(x) be the electromagnetic form factor of the $\pi^\pm$ meson due to strong interactions. As a function of x F(x) has the following analyticity properties:

1) F(x) is an analytic function of x in the complex x plane cut along the real axis from x = 1 to infinity.

2) On the real axis to the left of x = 1 the function F(x) is real and consequently assumes complex conjugate values on the upper and lower edges of the cut.

3) In the complex x plane as $|x| \to \infty$ the function F(x) increases no faster than a finite power.

4) The function F(x) is normalized according to F(0) = 1.

The purpose of this note (assuming that the form factor satisfies the analyticity properties 1 — 4) is to obtain certain inequalities which must be satisfied by integrals over the cross section σ for the transition $e^+ + e^- \to \pi^+ + \pi^-$. If experiment should show that these inequalities are not satisfied then one will have to conclude that the form factor fails to satisfy at least one of the analyticity properties 1 – 3.

Let us define the average cross section for the reaction $e^+ + e^- \to \pi^+ + \pi^-$ as follows:

$$\bar{\sigma}_f = \int_1^\infty \sigma(x)\, q(x)\, dx \equiv \int_1^\infty f(x)\,|F(x)|^2\, dx, \qquad (2)$$

where q(x) or f(x) is an arbitrary positive weight function so chosen that the integrals

$$\int_1^\infty f(x)\frac{dx}{x\sqrt{x-1}}, \qquad \int_1^\infty \ln f(x)\frac{dx}{x\sqrt{x-1}}.$$

exist. The minimum of $\bar{\sigma}_f$ over the class of functions F(x), satisfying conditions 1—4, exists, is different from zero and is equal to (see [2-4])

$$\bar{\sigma}_{f\,min} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln \bar{f}(\theta)\, d\theta\right\},$$

$$\bar{f}(\theta) = \frac{1}{2} f\left(\cos^{-2}\frac{\theta}{2}\right)\frac{\sin(\theta/2)}{\cos^3(\theta/2)}. \qquad (3)$$

It follows from formula (3) that regardless of the nature of the strong interactions the cross section for the process $e^+ + e^- \to \pi^+ + \pi^-$ must satisfy the inequality

$$\bar{\sigma}_f \geqslant \bar{\sigma}_{f\,min}. \qquad (4)$$

Let us consider some examples.

1) We set q(x) = 1/x. With this weight function $\bar{\sigma}_{min} = \pi^2\alpha^2/2^9 m^2$. If $F(x) \equiv 1$ then with this weight function $\bar{\sigma} = \bar{\sigma}_0 = \pi\alpha^2/40\ m^2$ and $\bar{\sigma}/\bar{\sigma}_0 \geq 0.25$.

2) With the weight function $q(x) = x^{1/2}/(x-1)^2$ we get $\bar{\sigma}/\bar{\sigma}_0 \geq 1/2$. With the weight function $q(x) = x^{3/2}/(x-1)^2$ we get $\bar{\sigma}/\bar{\sigma}_0 \geq 1$.

It is seen from these examples that the restriction depends on the choice of the weight function q(x). The question arises: can a weight function be found that will give rise to the strongest restriction. Such a weight function may be found if one minimizes the quantity $\bar{\sigma}/\bar{\sigma}_{min}$ over the weight functions, assuming that in $\bar{\sigma}$ the quantity $|F(x)|^2$ is given by experiment. In that case $\bar{f}(\theta) = |F[\cos^{-2}(\theta/2)]|^{-2}$ and the restriction is of the form

$$\int_1^\infty \ln|F(x)|^2 \frac{dx}{x\sqrt{x-1}} \geqslant 0. \qquad (5)$$

Formula (5) may also be obtained in a different way. Let $x_1, x_2, \ldots x_n$ be the zeros of the function $F(x)$. We define the function $F_1(x) = \prod_{k=1}^{n} x_k(x_k - 1)^{-1}$. The function $F_1(x)$ possesses the same analyticity properties 1—4 as the function $F(x)$ and, in addition, has no zeros in the cut plane. Therefore $\ln F_1(x)$ is also an analytic function of x and

$$a = \int_1^\infty \ln |F_1(x)|^2 \frac{dx}{x\sqrt{x-1}} = \int_C \ln F_1(x) \frac{dx}{x\sqrt{x-1}}. \quad (6)$$

The contour C includes the lower and upper edges of the cut and the large circle; since according to assumption 3 the integral over the large circle vanishes we see from the residue theorem and the condition $F_1(0) = 1$ that (6) equals zero. From formula (6) one obtains easily

$$\int_1^\infty \ln |F(x)|^2 \frac{dx}{x\sqrt{x-1}} = -\pi \sum_{k=1}^{n} \ln |z_k|^2,$$

$$z_k = \frac{i - \sqrt{x_k - 1}}{i + \sqrt{x_k - 1}}, \quad (7)$$

from which (5) follows when one takes into account that $|z_k|^2 \leq 1$.

It is seen from formula (7) that if the form factor $F(x)$ has no zeros in the complex x plane then the inequality (5) becomes an equality.[1)]

If it is supposed that for attraction between the pions $F(x) > 1$, and for repulsion $F(x) < 1$, then the pions cannot repel each other at all energies. If the pions attract each other at all energies, i.e. $F(x) > 1$ for all x then the form factor must have complex zeros.

In conclusion we thank L. B. Okun' for useful remarks.

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1)We note that specifying $|F_1(x)|^2$ on the real axis for $x > 1$ uniquely (under the assumption 3) defines the function $F_1(x)$ in the entire complex plane (see [4]), and its value $F_1(0)$ is given by the equation $F_1(0) = e^a$ while equation (7) [or the inequality (5)] follows from the condition $F_1(0) = 1$. At that all possible information about $F_1(x)$ has been used and in that sense the inequality (5) is the strongest possible.