A RELATIVISTIC EQUATION FOR THE S-MATRIX IN THE *p*-REPRESENTATION

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The covariant equation of motion for the scattering matrix obtained in a previous paper[1] is investigated by perturbation theory. A specific diagram technique is developed.

1. DIAGRAM TECHNIQUE. QUASIPARTICLES.

In the present paper, which is a continuation of an earlier paper[1] 1), we investigate the fundamental equation I, (2.11)

$$R(\lambda\tau) = \tilde{\mathcal{L}}(\lambda\tau) + \frac{1}{2\pi}\int \tilde{\mathcal{L}}(\lambda\tau - \lambda\tau')\frac{d\tau'}{\tau' - i\varepsilon}R(\lambda\tau'),$$

where, as before,

$$\tilde{\mathcal{L}}(\lambda\tau) = \frac{g}{\sqrt{2\pi}}\int \delta(\lambda\tau - k_1 - k_2 - k_3) : \varphi(k_1) \times \varphi(k_2)\varphi(k_3) : dk_1 dk_2 dk_3, \tag{1.1}$$

by perturbation theory.

Let

$$R(\lambda\tau) = \sum_{n=1}^{\infty} R_n(\lambda\tau) \tag{1.2}$$

be the expansion of the operator $R(\lambda\tau)$ in powers of the coupling constant g.

It then follows from I (2.11) that

$$R_n(\lambda\tau) = \frac{1}{(2\pi)^{n-1}}\int \tilde{\mathcal{L}}(\lambda\tau_1)\frac{d\tau_1}{\tau - \tau_1 - i\varepsilon}\tilde{\mathcal{L}}(\lambda\tau_2 - \lambda\tau_1) \times \frac{d\tau_2}{\tau - \tau_2 - i\varepsilon}\cdots\frac{d\tau_{n-1}}{\tau - \tau_{n-1} - i\varepsilon}\tilde{\mathcal{L}}(\lambda\tau - \lambda\tau_{n-1}). \tag{1.3}$$

Obviously we are faced with the problem of reducing (1.3) to normal form. Since this expression is the ordinary product of operators of the type (1.1), then during the process of its N-ordering it is necessary to use only ordinary pairing. We write out the definitions of such pairing:

$$\underbrace{\varphi(p)\,\varphi}(k) = \delta(p + k)\,\theta(k_0)\,\delta(k^2 - m^2) = \delta(p + k)\,D^{(+)}(k), \tag{1.4}$$

or

$$\underbrace{\varphi(p)\,\varphi(k)} = \delta(p + k)\,\theta(-p_0)\,\delta(p^2 - m^2) = \delta(p + k)\,D^{(-)}(p). \tag{1.5}$$

Comparing (1.4) with (1.5) we see that if we take in the pairing the argument of the "right-side" operator $\varphi(k)$, then it is necessary to use the function $D^{(+)}$ and, to the contrary, the use of the argument of the "left-side" operator $\varphi(p)$ calls for the use of $D^{(-)}$.

We shall now assume that the Lagrangians $\tilde{\mathcal{L}}$ in (1.3) are numbered, with the number 1 assigned to $\tilde{\mathcal{L}}(\lambda\tau_1)$, the number 2 assigned to $\tilde{\mathcal{L}}(\lambda\tau_2 - \lambda\tau_1)$, etc., so that the number of the last operator $\tilde{\mathcal{L}}(\lambda\tau - \lambda\tau_{n-1})$ is the number n (in other words, $\tau_n \equiv \tau$). Further, each of the operators $\varphi(k)$ is assigned the number of the Lagrangian $\tilde{\mathcal{L}}$ to which this operator belongs. Then, recognizing that the Lagrangian $\tilde{\mathcal{L}}$ is already specified in normal form, we can state that when $R_n(\lambda\tau)$ is reduced to normal form it is necessary to pair only the operators φ with different numbers. Here, according to (1.4), we insert the function $D^{(+)}$ in the pairing if we use the argument of the operator φ with the larger number.

We assume that we have carried out N-ordering in $R_n(\lambda\tau)$ and represent this operator in the form

$$R_n(\lambda\tau) = \sum_{m=0}^{3n}\int K_m^{(n)}(\lambda\tau, k_1, \ldots, k_m) : \varphi(k_1) \ldots \varphi(k_m) : dk_1 \ldots dk_m. \tag{1.6}$$

It is easy to establish that the coefficient functions $K_m^{(n)}$ should contain as a factor a δ-function, which ensures conservation of the 4-momentum 2):

$$K_m^{(n)} = \delta(\lambda\tau - k_1 - k_2 - \ldots - k_m)\,f_m^{(n)}(\tau;\ k_1, \ldots, k_m). \tag{1.7}$$

1)Henceforth the formulas from[1] will be preceded by a Roman numeral I.

2)The argument of the δ-function in (1.7) is the sum of the argument of the δ-functions contained in all the operators $\tilde{\mathcal{L}}$ in (1.3), and of the arguments of the δ-functions of those pairings which are contained in $K_m^{(n)}$.

The functions $f_m^{(n)}(\tau; k_1, \ldots, k_m)$, which can be regarded as symmetrical in the arguments $k_1, \ldots, k_m$, represent multiple integrals of the products of the functions $D^{(\pm)}$ and functions of the type $(2\pi)^{-1}(\tau - \tau_s - i\epsilon)^{-1}$. The integration is carried out here over all the variables τ_s $(s = 1, 2, \ldots, n-1)$, and also over the independent 4-momenta q_j, which remain after integration of all the δ-functions [with the exception of the function in (1.7)]. The effective method for the construction of the quantities $f_m^{(n)}(\tau; k_1, \ldots, k_m)$ results from a unique diagram technique, which differs from the customarily used Feynman technique.

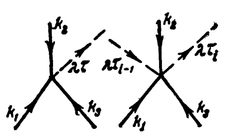
Proceeding to the formulation of the diagram technique, we should, first, find a suitable graphic description for the Lagrangian $\widetilde{\mathscr{L}}(\lambda\tau)$ responsible for the first-order processes. We shall agree to represent this operator by diagram a of Fig. 1, that is, to each field $\varphi(k)$ in (1.1) we set in correspondence a solid line with 4-momentum k, directed towards the vertex, and in order to satisfy the conservation law $\lambda\tau = k_1 + k_2 + k_3$ we connect to this vertex an "outgoing" dashed line with 4-momentum $\lambda\tau$.

It is natural to set in correspondence with Lagrangian $\widetilde{\mathscr{L}}(\lambda\tau_i - \lambda\tau_{i-1})$ the diagram of Fig. 1.

Using diagrams a and b, we can arbitrarily represent[3] the second-order processes, described by the operator

$$R_2(\lambda\tau) = \frac{1}{2\pi}\int \widetilde{\mathscr{L}}(\lambda\tau_1)\,\frac{d\tau_1}{\tau - \tau_1 - i\varepsilon}\,\widetilde{\mathscr{L}}(\lambda\tau - \lambda\tau_1), \quad (1.8)$$

by the scheme shown in Fig. 2, where the "internal" dashed line with 4-momentum $\lambda\tau_1$ is set in correspondence with the factor $(2\pi)^{-1}(\tau - \tau_1 - i\epsilon)^{-1}$. This diagram recalls the graphic description of a second-order process in nonrelativistic perturbation theory.



a b

FIG. 1

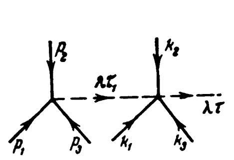


FIG. 2

In N-ordering of (1.8) we should connect pairwise some solid lines which enter respectively the first and second vertices (see Fig. 2), and then set the pairing in correspondence with each of the obtained internal lines. If these lines are assumed directed from the first vertex to the second, then obviously the pairing used should be the functions $D^{(+)}$ (see the rule formulated above).

Figure 3 shows some second-order diagram with allowance for the 4-momentum conservation at each vertex[4]; the internal lines correspond to the functions $D^{(+)}$.

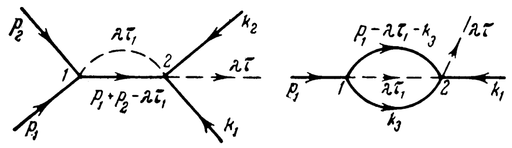


FIG. 3

We now proceed to the general case. Assume that we are given a Feynman diagram with l internal and m external lines (the external momenta $k_1, \ldots, k_m$ will be assumed "incoming"), describing some n-th order process. In order to represent graphically this process in the formalism considered here, and to find the corresponding coefficient function, it is necessary to employ the following procedure.

1. All the vertices of the given Feynman diagram are arbitrarily numbered.

2. The first vertex is connected with the second, the second with the third, the third with the fourth, etc., by dashed lines and, in addition, with a free dashed line is drawn out of vertex n and assigned a 4-momentum $\lambda\tau = \Sigma k_i$ (summation from 1 to m).

3. All the internal lines, including the dashed lines, are so oriented that they leave the vertex with the lower number and enter the vertex with the larger number.

4. The dashed internal lines are assigned 4-momenta $\lambda\tau_s$, where $s = 1, 2, \ldots, n-1$ is the number of the vertex from which this line leaves. On the solid internal lines we mark the 4-momenta p_ν $(\nu = 1, 2, \ldots, l)$, taking account of the orientations of these lines (item 3) and the 4-momentum conservation at each vertex (with the dashed lines included!). If necessary we introduce the required number of independent momenta q_j $(j = 1, 2, \ldots, l - n + 1)$.

5. Each internal dashed line with 4-momentum $\lambda\tau_s$ is set in correspondence with a function $G(\tau_s) = (2\pi)^{-1}(\tau - \tau_s - i\epsilon)^{-1}$, and each solid internal line with 4-momentum p_ν is set in correspondence with a function $D^{(+)}(p_\nu) = \theta(p_\nu^0)\,\delta(p_\nu^2 - m^2)$.

6. Integration between infinite limits is carried

[3]We speak here of an arbitary representation, since the expression for $R_2(\lambda\tau)$ has not yet been reduced to normal form.

[4]The variables k_3 and τ_1 are independent, so that integration must be carried out with respect to them.

out over all the independent variables q_j and τ_s.

7. The operations called for in items 2—6 are repeated for all n! numberings of the vertices of the given diagram, and the resultant coefficient functions added. The summary coefficient function turns out to be partially or fully symmetrized with respect to the variables $k_1, \ldots, k_m$.

8. The total coefficient function is symmetrized over those variables $k_1, \ldots, k_m$, with respect to which this function remained asymmetrical after application of item 7, with allowance for the corresponding factorial multiplier[5)] and the factor $(g/\sqrt{2\pi})^n$.

The performance of operations 1—8 leads us to the sought coefficient function[6)]. Let us illustrate this procedure with concrete examples.

We first return to the second order and consider the self-energy diagram. It is clear that in this case we need take into account diagrams a and b of Fig. 4. The corresponding coefficient function is of the form

$$K_2^{(2)}(\lambda\tau; k_1, k_2) = \delta(\lambda\tau - k_1 - k_2) f_2^{(2)}(\tau; k_1; k_2); \quad (1.9)$$

$$f_2^{(2)} = (3!)^2 \frac{g^2}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\tau_1}{\tau - \tau_1 - i\varepsilon} \left\{ \int dq \, [D^{(+)}(q) \times D^{(+)}(k_1 - \lambda\tau_1 - q) + D^{(+)}(q) D^{(+)}(k_2 - \lambda\tau_1 - q)] \right\}. \quad (1.10)$$

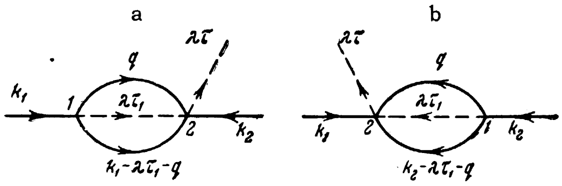


FIG. 4

By way of another example, Fig. 5 shows a fifth-order diagram with three free ends for some arbitrary numbering of the vertices (only the independent momenta q_1 and q_2 are indicated on the solid internal lines).

We see thus that the diagrams in the investigated scheme differ topologically from the Feynman diagrams, in that they contain additional dashed lines interconnecting all the vertices. The

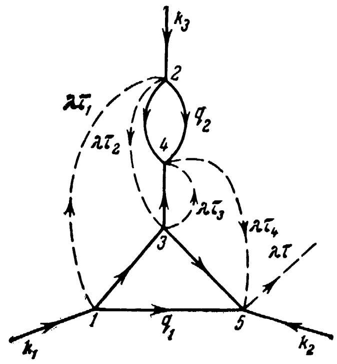


FIG. 5

particles in the intermediate states are in this case real particles, since the relations $p^2 = n^2$ and $p_0 > 0$ are satisfied for them.

Taking into consideration this circumstance, we can attempt to interpret the dashed lines of the diagrams as the representations of some "quasiparticles," which interact with the real physical particles. Then it follows from the first-order diagram (Fig. 1a), corresponding to the Lagrangian

$$\tilde{\mathscr{L}}(\lambda\tau) = \int e^{-i\lambda\tau x} \mathscr{L}(x)\, dx$$

[see I, (2.8)] that the wave function of the quasiparticle is the plane wave[7)] $\psi(x) = \exp(-i\lambda\tau x)$. According to I, (3.1) we have

$$\tilde{\mathscr{L}}(\lambda\tau) = \int e^{-i\tau\sigma} L(\sigma)\, d\sigma, \quad (1.11)$$

and therefore the function ψ can also be written in the α-representation:

$$\psi(\sigma) = e^{-i\tau\sigma}. \quad (1.12)$$

The parameter σ has here the meaning of the proper time of the quasiparticle, since the equality $\sigma = x^0$ is satisfied in the rest system of the quasiparticle ($\lambda = 0$). Consequently, the quantity τ must be regarded as equal to the proper mass of the quasiparticle, all the more since the relation $(\lambda\tau)^2 = \tau^2$ holds. Taking this interpretation into account, we henceforth put $\tau \geq 0$.

The wave function $\psi(\sigma)$ obviously satisfies the equation

$$id\psi/d\sigma = \tau\psi. \quad (1.13)$$

It is easy to verify that the propagation function

$$G(\tau') = \frac{1}{2\pi} \frac{1}{\tau - \tau' - i\varepsilon}, \quad (1.14)$$

5)The form of this factor will be indicated when we consider more realistic interactions than $g\varphi^3$.

6)If we go over from this coefficient function to the matrix element, that is, we break up the momenta k_1 into "incoming" and "outgoing," then at $\tau = 0$ the vector λ can be regarded as directed along the summary 4-momentum of the "incoming" (or "outgoing") particles[1]. As a result, the matrix element acquires a completely invariant form.

7)That is, we regard the operator $\tilde{\mathscr{L}}(\lambda\tau)$ as the integral $\int \mathscr{L}'(x)dx$, where $\mathscr{L}(x)$ is the Lagrangian of the "quaternary" interaction (three fields φ and a plane wave ψ).

which in the diagram technique is set in correspondence with the internal dashed lines, is the Green's function of (1.13) in the τ' representation. Indeed, putting

$$G(\sigma - \sigma') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\tau - \tau' - i\varepsilon} e^{-i(\sigma - \sigma')\tau'} d\tau', \quad (1.15)$$

we get

$$(id/d\sigma - \tau) G(\sigma - \sigma') = -\delta(\sigma - \sigma'). \quad (1.16)$$

Consequently, the internal dashed line can be regarded as the graphical representation of a quasiparticle (with mass τ) which is in the virtual state.

Calculating the integral (1.15), we get

$$G(\sigma - \sigma') = i\theta(\sigma - \sigma') e^{-i\tau(\sigma - \sigma')}. \quad (1.17)$$

Thus, $G(\sigma - \sigma')$ is the "retarded" Green's function of (1.13), this being a manifestation of the causality principle in this formalism.

Summarizing the foregoing, we arrive at the following conclusions. The mechanism whereby real physical particles interact can be represented as multiple exchange of similar real particles and, in addition, certain quasiparticles. If the quasiparticles have mass $\tau > 0$ (we shall arbitrarily designate these quasiparticles as "heavy"), then any process in which real particles participate proceeds with nonconservation of the 4-momentum, for in this case a free quasiparticle with 4-momentum $\lambda\tau$ must be radiated. On the other hand, if the real particles interact with the "light" quasiparticles ($\tau = 0$)[8], then the energy-momentum is conserved. In the space-time picture (see Sec. 3 of [1]) the former case corresponds to examination of physical processes and instants of time belonging to a finite spacelike plane $\lambda x = \sigma$, while the latter case corresponds to an analysis with $\sigma = \infty$. Using intuitive notions, we can state that the use of "heavy" quasiparticles for exchange with real particles corresponds as it were to a study of the "short-range action" of the latter, while the use of "light" quasiparticles corresponds to a study of "long-range action."

We thus have grounds for assuming that the quasiparticles constitute in a definite sense a dynamic equivalent of space-time or, in other words, the statement that the physical particles propagate in space-time is equivalent to stating that these particles interact with quasiparticles [9].

[8]"Light" quasiparticles in the virtual state are obviously described by a propagation function $G(\tau') = -(2\pi)^{-1} \times (\tau' + i\epsilon)^{-1}$.

2. ANALYSIS OF CONVERGENCE OF INTEGRALS. REPRESENTATION OF THE COEFFICIENT FUNCTIONS IN DISPERSION FORM

The formalism which we consider is based on the same principles as field theory in the ordinary approach, and therefore should lead to the same results. In particular, in the calculation of the diagrams which diverge in the usual formalism, divergences should appear in this scheme, too. However, it is typical that in this case the divergences are contained only in the one-dimensional integrals with respect to the parameters τ_s, while the integrals with respect to the momenta q_j converge (at fixed τ_s and external momenta[10] k_i). We shall prove the latter statement for an arbitrary connected diagram of order n, having l internal and m external lines.

Using the notation adopted above, we can represent the coefficient function $f_m^{(n)}$ of the given diagram in the form

$$f_m^{(n)}(\tau; k_1, \ldots, k_m) = \frac{1}{(2\pi)^{n-1}} \left(\frac{g}{\sqrt{2\pi}}\right)^n \int \prod_{s=1}^{n-1} \frac{d\tau_s}{\tau - \tau_s - i\varepsilon}$$
$$\times \int \prod_{\nu=1}^{l} \theta(p_\nu^0) \delta(p_\nu^2 - m^2) \prod_{j=1}^{l-n+1} dq_j. \quad (2.1)$$

The 4-momentum conservation law in each vertex i of the diagram is satisfied, and this is conveniently written in the form of the incidence matrix [11] $\epsilon_{i\nu}$:

$$k_i + \lambda\tau_{i-1} - \lambda\tau_i = \sum_{\nu=1}^{l} \varepsilon_{i\nu} p_\nu; \quad (2.2)$$

where $\tau_0 \equiv 0$ and $\tau_n \equiv \tau$.

The compatibility of (2.2) is ensured by the conservation law

$$\sum_{i=1}^{n} k_i = \lambda\tau.$$

We note that in our method of orienting the lines

[9]We emphasize in particular that the very structure of space-time is closely related with the properties of the quasiparticle. This is seen already from the fact that the parameter τ is canonically conjugate to the σ-metric of 4-space along the motion of the quasiparticle.

[10]It will be convenient here to put i = 1, 2, ..., n, signifying identical vanishing of some k_i.

[11]We use for this matrix the definition and designation used in[2]: $\varepsilon_{i\nu} = 1$ if the line ν goes out of the vertex i; $\epsilon_{i\nu} = -1$ if the line ν enters the vertex i; $\epsilon_{i\nu} = 0$ if the vertex i does not belong to the line ν.

ν of the diagram, the upper of the two nonvanishing elements in each column of the matrix $\epsilon_{i\nu}$ must be unity. In particular, in the first line there are consequently no negative elements, nor positive elements in the last line.

We now multiply both halves of each of the equations in (2.2) by the vector λ:

$$(k_i\lambda) + \tau_{i-1} - \tau_i = \sum_{\nu=1}^{l} \varepsilon_{i\nu}(p_\nu\lambda). \quad (2.3)$$

Since, by virtue of (2.1),

$$p_\nu^0 > 0, \qquad p_\nu^2 = m^2, \quad (2.4)$$

all the quantities $(p_\nu\lambda)$ are positive. Thus, the minus signs in front of the individual terms in the sums (2.3) are due only to the matrix $\epsilon_{i\nu}$. Using the definition of $\epsilon_{i\nu}$, we can write these negative terms in explicit form in each equation of (2.3):

$$(k_1\lambda) - \tau_1 = \sum_{1<j}(p_{1j}\lambda),$$

$$(k_2\lambda) + \tau_1 - \tau_2 = \sum_{2<j}(p_{2j}\lambda) - \sum(p_{12}\lambda), \ldots,$$

$$(k_n\lambda) + \tau_{n-1} - \tau = -\sum(p_{1n}\lambda) - \sum(p_{2n}\lambda) - \ldots - \sum(p_{n-1,n}\lambda), \quad (2.5)$$

where p_{ij} is used for convenience to designate the momentum on the line joining the vertices i and j, and the sums in the right halves extend over all lines ν with corresponding indices (for example, in $\Sigma(p_{12}\lambda)$ the summation is over all the lines which join the first and second vertices).

We then repeat verbatim the procedure from [3] (see Sec. 16). Namely, transferring the negative terms of (2.5) to the left half and taking the obvious upper bounds, we have

$$k_1\lambda - \tau_1 = \sum_{1<j}(p_{1j}\lambda), \qquad k_2\lambda + k_1\lambda - \tau_2 \geqslant \sum_{2<j}(p_{2j}\lambda), \ldots,$$

$$k_s\lambda + k_{s-1}\lambda + 2k_{s-2}\lambda + \ldots + 2^{s-2}k_1\lambda - \tau_s - \tau_{s-2} - 2\tau_{s-3} - \ldots - 2^{s-3}\tau_1 \geqslant \sum_{s<j}(p_{sj}\lambda), \ldots . \quad (2.6)$$

It follows from (2.6) that in any case for arbitrary $(p_\nu\lambda)$ the following inequality holds

$$(p_\nu\lambda) \leqslant 2^{n-3}(|k_1\lambda| + \ldots + |k_n\lambda| + |\tau_1| + \ldots + |\tau_{n-1}|), \quad (2.7)$$

so that, allowing for (2.4), all four components of the vector p_ν are bounded. Since the independent vectors q_j are contained among the vectors p_ν, the region of integration with respect to q_j in (2.1) turns out to be finite and the corresponding integrals converge.

Consequently, the divergences in (2.1) can be encountered only in the integrals with respect to τ_S. It is easy to verify that these divergences have, as before, an "ultraviolet" character, they arise for large $|\tau_S|$, and the index of the diagram ω over the variables τ_S coincides with the arbitrary growth exponent [3] with respect to the external momenta (in our case $\omega = -n - m + 4$).

Recalling the interpretation of the dashed lines of the diagrams, we can draw the following conclusion: the divergences in quantum field theory appear because the quasiparticles which interact with the real particles are allowed to carry too large a 4-momentum.

Let us illustrate the foregoing results using as an example a second-order self-energy diagram (Fig. 4), which diverges logarithmically in the usual formalism. According to (1.10), for the given case the integral over the independent momentum q is of the form

$$A = \int [D^{(+)}(q)\, D^{+}(k_1 - \lambda\tau_1 - q) + D^{(+)}(q)\, D^{(+)}(k_2 - \lambda\tau_1 - q)]\, dq, \quad (2.8)$$

that is, it is actually the sum of the imaginary parts of the corresponding Feynman diagrams with external momenta $k_1 - \lambda\tau_1$ and $k_2 - \lambda\tau_1$. From this we obtain

$$A = \frac{\pi}{2}\Big\{\theta(k_1^0 - \lambda^0\tau_1)\,\theta((k_1 - \lambda\tau_1)^2 - 4m^2) \times \left[\frac{(k_1-\lambda\tau_1)^2 - 4m^2}{(k_1-\lambda\tau_1)^2}\right]^{1/2} + \theta(k_2^0 - \lambda^0\tau_1)\,\theta((k_2 - \lambda\tau_1)^2 - 4m^2)\left[\frac{(k_2-\lambda\tau_1)^2 - 4m^2}{(k_2-\lambda\tau_1)^2}\right]^{1/2}. \quad (2.9)$$

Substituting (2.9) in (1.10) we have

$$f_2^{(2)} = \frac{g^2}{\pi}\frac{(3!)^2}{2^4}\left\{\int_{-\infty}^{a(k_1)}\frac{d\tau_1}{\tau - \tau_1 - i\varepsilon}\left[\frac{(k_1-\lambda\tau_1)^2 - 4m^2}{(k_1-\lambda\tau_1)^2}\right]^{1/2} + \int_{-\infty}^{a(k_2)}\frac{d\tau_1}{\tau - \tau_1 - i\varepsilon}\left[\frac{(k_2-\lambda\tau_1)^2 - 4m^2}{(k_2-\lambda\tau_1)^2}\right]^{1/2}\right\}, \quad (2.10)$$

where [12]

$$a(k) = (k\lambda) - \sqrt{(k\lambda)^2 + 4m^2 - k^2}. \quad (2.11)$$

Thus, the function $f_2^{(2)}(\tau; k_1, k_2)$ is represented in the form of a dispersion integral over the variable τ, and has in the τ plane two cuts along the real

[12] From (2.11) it follows that if $k_1^2 < 4m^2$ and $k_2^2 < 4m^2$, then the imaginary part of the function $f_2^{(2)}$ vanishes when $\tau > 0$. This circumstance justifies the choice of the sign of τ, made in Sec. 1.

axis. Since the quantities a(k) in (2.11) are determined from the conditions $(k - \lambda\tau_1)^2 - 4m^2 = 0$ and $k_0 - \lambda_0\tau_1 > 0$, the first of the integrals (2.10) can be conditionally assumed connected with the "channel" $(k_1, \lambda\tau)$ and the second with the "channel" $(k_2, \lambda\tau)$ (see Fig. 4).

Of course, all these arguments are only symbolic in meaning, since both integrals (2.10) diverge logarithmically at large τ_1 ($\omega = 0$). It is therefore necessary to carry out subtractions in (2.10).

For $\tau = 0$ we have $k_1 = -k_2 \equiv k$. In this case expression (2.10) reduces, after making simple changes of variable ($z = (k - \lambda\tau_1)^2$ for the first integral and $z = (k + \lambda\tau_1)^2$ for the second), to the usual Kallen-Lehmann integral:

$$f_2^{(2)}(0; k) = \frac{g^2}{\pi}\frac{(3!)^2}{2^4}\int_{4m^2}^{\infty}\frac{dz}{z - k^2 - i\varepsilon}\sqrt{\frac{z - 4m^2}{z}}. \quad (2.12)$$

The dispersion representation with respect to the variable τ holds for a coefficient function of any connected diagram. To verify this, we carry out the N-ordering of the operator $R_n(\lambda\tau)$, starting not from (1.3) but from the equivalent expression

$$R_n(\lambda\tau) = \frac{1}{(2\pi)^{n-1}}\int \mathscr{L}(\lambda\tau_1)\frac{d\tau_1}{\tau - \tau_1 - i\varepsilon}\mathscr{L}(\lambda\tau - \lambda\tau_1 + \lambda\tau_2) \times \frac{d\tau_2}{-\tau_2 - i\varepsilon}\mathscr{L}(\lambda\tau_3 - \lambda\tau_2)\ldots\frac{d\tau_{n-1}}{-\tau_{n-1} - i\varepsilon}\mathscr{L}(-\lambda\tau_{n-1}). \quad (2.13)$$

Then the arbitrary coefficient function $f_m^{(n)}$ will, obviously, have the form

$$f_m^{(n)} = \left(\frac{g}{\sqrt{2\pi}}\right)^n\frac{1}{(2\pi)^{n-1}}\int_{-\infty}^{a}\frac{d\tau_1}{\tau - \tau_1 - i\varepsilon}\rho(\tau_1; k_1, \ldots, k_m), \quad (2.14)$$

where

$$\rho(\tau_1; k_1, \ldots, k_m) = \int\prod_{s=2}^{n-1}\frac{d\tau_s}{-\tau_s - i\varepsilon}\int\prod_{\nu=1}^{l}\theta(p_\nu^0)\,\delta(p_\nu^2 - m^2)\prod_{j=1}^{i-n+1}dq_j, \quad (2.15)$$

and the vectors p_ν are linear combinations of the vectors k_i, q_j, and $\lambda\tau_s$ ($s = 1, 2, \ldots, n-1$). The upper limit a in the integral (2.14) must be a finite quantity, as can be directly seen from the first equation in (2.5). Indeed, since $p_{1j}\lambda \geq m$ always, we have in any case $|k_1\lambda| + \kappa m > \tau_1$, where κ is the number of solid internal lines leaving the first vertex.

The existence of the representation (2.14) indicates that apparently the coefficient functions $f(\tau; k)$ have definite analytic properties with respect to the variable τ. Since this variable has a universal character and is by no means connected with the concrete form of the considered processes, it is attractive to attempt to formulate a quantum field theory starting only from the analyticity in the τ plane and the conditions of unitarity in τ [see I, (2.23)] [13]. Concluding this section, we point out that equation I, (2.11) for $R(\lambda\tau)$ is equivalent to an infinite chain of integral equations [14] for the amplitudes $f_m(\tau; k_1, \ldots, k_m)$, where $k_1 + \ldots + k_m = \lambda\tau$ and $k_1^2 = \ldots = k_m^2 = m^2$. This chain can be solved approximately by using perturbation theory.

3. CONSTRUCTION OF THE OPERATOR $R(\lambda\tau)$ FROM THE CAUSALITY AND UNITARITY CONDITIONS BY PERTURBATION THEORY

As shown in [7], the causality condition for the operator $R(\lambda\tau)$ is written in the form [see I, (4.10)]

$$R^{(-)}(\lambda\tau) - (R^{(+)}(-\lambda\tau))^+ = \frac{1}{2\pi}\int\frac{d\tau'}{\tau' - i\varepsilon}[(R^{(+)}(\lambda\tau' - \lambda\tau))^+R^{(-)}(\lambda\tau') - (R^{(+)}(\lambda\tau'))^+R^{(-)}(\lambda\tau' + \lambda\tau)], \quad (3.1)$$

where $R^{(-)}(\lambda\tau) \equiv R(\lambda\tau)$ is a solution of Eq. I, (2.11) (the imaginary addition in the denominator is negative), and $R^{(+)}(\lambda\tau)$ is a solution of I, (4.1) (the imaginary addition in the denominator is positive).

Taking into consideration the expansion (1.2) and formulas (1.3), (1.6), and (1.7), we get

$$R^{(\mp)}(\lambda\tau) = \sum_{m=0}^{\infty}\int\delta(\lambda\tau - k_1 - \ldots - k_m) \times f_m(\tau \mp i\varepsilon; k_1, \ldots, k_m) : \varphi(k_1), \ldots, \varphi(k_m) : dk_1 \ldots dk_m. \quad (3.2)$$

Consequently, the coefficient functions of the operators $R^{(-)}$ and $R^{(+)}$ are boundary values of the same function $f_m(\tau; k_1, \ldots, k_m)$ for two different methods of letting Im τ approach zero.

For what follows we shall also need the unitarity condition of the operator $R^{(-)}(\lambda\tau)$. According to I, (2.23) it takes the form

$$R^{(-)}(\lambda\tau) - (R^{(-)}(-\lambda\tau))^+ = \frac{1}{2\pi}\int\frac{d\tau'}{\tau' - i\varepsilon}[(R^{(-)}(-\lambda\tau'))^+R^{(-)}(\lambda\tau - \lambda\tau') + (R^{(-)}(\lambda\tau' - \lambda\tau))^+R^{(-)}(\lambda\tau')]. \quad (3.3)$$

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[13] Of course, before we proceed to a realization of such a program, we must clarify in greater detail the analytic properties of the functions $f(\tau; k_1, \ldots, k_m)$ in the τ plane, without making use of perturbation theory.

[14] An analogous infinite system of equations was investigated by Grigor'ev and Vavilov[4].

We now introduce an operator $\hat{P}$, defined by the fact that when acting on (3.2) it reverses the sign of τ and of ϵ:

$$R^{(-)}(\lambda\tau) = \hat{P}R^{(+)}(-\lambda\tau), \qquad \hat{P}R^{(-)}(\lambda\tau) = R^{(+)}(-\lambda\tau). \quad (3.4)$$

Taking (3.4) into account, the causality condition (3.1) can be rewritten as

$$R^{(-)}(\lambda\tau) - \hat{P}(R^{(-)}(\lambda\tau))^{+} = \frac{1}{2\pi}\int\frac{d\tau'}{\tau'-i\varepsilon}[(R^{(+)}(\lambda\tau'-\lambda\tau))^{+}R^{(-)}(\lambda\tau')] - \hat{P}\left\{\frac{1}{2\pi}\int\frac{d\tau'}{\tau'-i\varepsilon}[(R^{(+)}(\lambda\tau'-\lambda\tau))^{+}R^{(-)}(\lambda\tau')]\right\}^{+}. \quad (3.5)$$

Our problem is to construct an operator $R^{(-)}(\lambda\tau)$ from relations (3.3) and (3.5) by perturbation theory. To this end it is necessary to substitute in (3.3) and (3.5) the expansion (1.2) and then, gathering terms of the same order of smallness, to determine in succession $R_1(\lambda\tau)$, $R_2(\lambda\tau)$, $R_3(\lambda\tau)$, ...

In first order, obviously, we shall have

$$(R_1^{(-)}(\lambda\tau))^{+} = R_1^{(-)}(-\lambda\tau), \quad (3.6)$$

$$R_1^{(-)}(\lambda\tau) = R_1^{(+)}(\lambda\tau) \equiv R_1(\lambda\tau). \quad (3.7)$$

From the last relation it follows that the coefficient functions of the operator $R_1(\lambda\tau)$ should be real [see (3.2)].

In second order, condition (3.5) is written, with allowance for (3.6) and (3.7),

$$R_2^{(-)}(\lambda\tau) - \hat{P}(R_2^{(-)}(\lambda\tau))^{+} = \frac{1}{2\pi}\int\frac{d\tau'}{\tau'-i\varepsilon}R_1(\lambda\tau-\lambda\tau')R_1(\lambda\tau') - \hat{P}\left\{\frac{1}{2\pi}\int\frac{d\tau'}{\tau'-i\varepsilon}R_1(\lambda\tau-\lambda\tau')R_1(\lambda\tau')\right\}^{+}. \quad (3.8)$$

It follows therefore that $R_2^{(-)}(\lambda\tau)$ can be represented in the form

$$R_2^{(-)}(\lambda\tau) = \frac{1}{2\pi}\int\frac{d\tau'}{\tau'-i\varepsilon}R_1(\lambda\tau-\lambda\tau')R_1(\lambda\tau') + \Lambda_2^{(-)}(\lambda\tau), \quad (3.9)$$

where $\Lambda_2^{(-)}(\lambda\tau)$ is some second-order operator, which satisfies the condition

$$\Lambda_2^{(-)}(\lambda\tau) = \hat{P}(\Lambda_2^{(-)}(\lambda\tau))^{+}. \quad (3.10)$$

Substituting (3.9) in (3.3), which is written for second-order quantities, we get

$$[\Lambda_2^{(-)}(\lambda\tau)]^{+} = \Lambda_2^{(-)}(-\lambda\tau), \quad (3.11)$$

from which with allowance for (3.10) it follows that

$$\Lambda_2^{(-)}(\lambda\tau) = \Lambda_2^{(+)}(\lambda\tau) \equiv \Lambda_2(\lambda\tau). \quad (3.12)$$

Thus, the operator $\Lambda_2(\lambda\tau)$ behaves like the operator $R_1(\lambda\tau)$ under Hermitian conjugation, and its coefficient functions are also real.

Reasoning in perfect analogy with the foregoing, we can obtain expressions for $R_3^{(-)}(\lambda\tau)$, $R_4^{(-)}(\lambda\tau)$, etc. For example, the operator $R_3^{(-)}(\lambda\tau)$ takes the form

$$R_3^{(-)}(\lambda\tau) = \frac{1}{(2\pi)^2}\int R_1(\lambda\tau-\lambda\tau_1)\frac{d\tau_1}{\tau_1-i\varepsilon}R_1(\lambda\tau_1-\lambda\tau_2)\frac{d\tau_2}{\tau_2-i\varepsilon} \times R_1(\lambda\tau_2) + \frac{1}{2\pi}\int\frac{d\tau'}{\tau'-i\varepsilon}[R_1(\lambda\tau-\lambda\tau')\Lambda_2(\lambda\tau') + \Lambda_2(\lambda\tau-\lambda\tau')R_1(\lambda\tau')] + \Lambda_3(\lambda\tau),$$

$$\Lambda_3^{+}(\lambda\tau) = \Lambda_3(-\lambda\tau), \quad \Lambda_3(\lambda\tau) \equiv \Lambda_3^{(+)}(\lambda\tau) = \Lambda_3^{(-)}(\lambda\tau). \quad (3.13)$$

If we identify $R_1(\lambda\tau)$ with the Lagrangian $\mathscr{L}(\lambda\tau)$ (this is dictated by correspondence considerations[3]) and put $\Lambda_2 = \Lambda_3 = 0$ in (3.9) and (3.13), we obtain the usual expansion of the operator $R(\lambda\tau)$ up to third order inclusive, resulting when I, (2.11) is solved by perturbation theory[15)] (compare with (1.2) for n = 1, 2, 3). It is therefore clear that the operators Λ_2, Λ_3, ... play the role of "counterterms"[3], i.e., they can be included from the very beginning in the interaction Lagrangian and thus introduced into I, (2.11):

$$R(\lambda\tau) = \mathscr{L}(\lambda\tau) + \sum_{n=2}^{\infty}\Lambda_n(\lambda\tau) + \frac{1}{2\pi}\int\frac{d\tau'}{\tau'-i\varepsilon} \times\left\{\mathscr{L}(\lambda\tau-\lambda\tau') + \sum_{n=2}^{\infty}\Lambda_n(\lambda\tau-\lambda\tau')\right\}R(\lambda\tau'). \quad (3.14)$$

It is easy to verify that (3.9) and (3.13) are iterations of (3.14). The explicit form of the operators $\Lambda_n(\lambda\tau)$ for the model under consideration is obtained by taking the Fourier transforms of the corresponding counter terms, given in [3].

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[15)] We emphasize in particular that for relativistic invariants of R(0) it is necessary to satisfy the condition of "locality" of the operator $\mathscr{L}$ in the x-representation[1] $[\mathscr{L}(x), \mathscr{L}(y)] = 0$ for $(x-y)^2 < 0$ (see Sec. 2 of [1]). The same obviously pertains also to the operators Λ_2, Λ_3 ...

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128