MEAN FIELD STRENGTH IN AN INHOMOGENEOUS MEDIUM

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The field in an inhomogeneous medium which can be described by a random function $l(x)$ or by a random distribution of inclusions is considered. An expansion is found for the equation which is obeyed by the mean value of the field with respect to moments of $l(x)$ or with respect to the powers of inclusion concentrations. An electromagnetic field, the dielectric constant of a gas of nonpolar molecules, and the dielectric constant of a medium with spherical inclusions are considered as examples.

1. INTRODUCTION

STUDY of the propagation of heat, sound, and electromagnetic waves in a medium with random inhomogeneities, study of the scattering of an electron by impurities in the crystal, or the determination of the static characteristics of an inhomogeneous medium, such as the dielectric constant and the permeability, etc. (for more details see, for example, [1]), all lead to the investigation of the statistical properties of the solution $\Psi(\mathbf{x})$ of the linear equation

$$L\Psi = j, \qquad \Psi = Gj \tag{1}$$

with specified sources $j(\mathbf{x})$, containing a random operator L in the left side. We shall consider an equation that does not contain the time. If the conditions for the propagation of the waves Ψ are not stationary and the separation of variables is impossible, then the time can be included in Eq. (1). No additional changes are produced thereby, except for an increase in the number of variables.

In physical problems it is of interest to calculate $\langle\Psi(\mathbf{x})\rangle$ and $\langle\Psi(\mathbf{x})\Psi(\mathbf{y})\rangle$ (the angle brackets denote statistical averaging). It will be our purpose to investigate the coherent part of $\langle\Psi(\mathbf{x})\rangle = \psi(\mathbf{x})$; we leave out the question of the fluctuations of $\Psi(\mathbf{x}) - \psi(\mathbf{x})$. We shall constantly use the equation

$$\mathcal{L}\psi = j, \qquad \psi = \mathcal{G}j, \tag{2}$$

which is satisfied by $\psi(\mathbf{x})$. As follows from the averaging of the second equation of (1),

$$\mathcal{G} = \langle G\rangle, \qquad \mathcal{L} = \langle L^{-1}\rangle^{-1}.$$

The problem of the mean value of a field in a medium with random inhomogeneities can usually be solved only approximately. In this case the calculation of the operator $\mathcal{L}$ turns out to be more convenient and leads to a more accurate value of $\psi(\mathbf{x})$ than the calculation of the average value of the Green's function $\mathcal{G}$. This circumstance is well known in quantum field theory. It is used also in the problem under consideration[1-6], but to an insufficient degree and is apparently not always recognized, which sometimes leads to incorrect statements[1,4] (this question is discussed in the conclusion).

Thus, the procedure employed consists in an approximate calculation of the operator $\mathcal{L}$ (which in particular cases reduces to finding effective characteristics of the medium such as the dielectric constant, conductivity, thermal conductivity coefficient, etc.) and subsequent determination of $\psi(\mathbf{x})$ from (2).

In order not to interrupt the exposition that follows, we shall list here some of the general properties of the field $\psi(\mathbf{x})$ and of the operator $\mathcal{L}$. First, spatial dispersion always exists for $\psi(\mathbf{x})$. In other words, the operator $\mathcal{L}$ is nonlocal, even if L is local. Thus, if

$$L\Psi(\mathbf{x}) = L_0\Psi(\mathbf{x}) - l(\mathbf{x})\Psi(\mathbf{x}) \tag{3}$$

and only the function $l(\mathbf{x})$ is random, then

$$\mathcal{L}\psi(\mathbf{x}) = L_0\psi(\mathbf{x}) - \int d^3y\, l_{eff}(\mathbf{x}, \mathbf{y})\,\psi(\mathbf{y}). \tag{4}$$

In a statistically homogeneous medium l_{eff} depends only on the difference $\mathbf{x} - \mathbf{y}$. This case is the simplest and will always be assumed in the examples considered below. When the mean value of the field $\psi(\mathbf{x})$ is homogeneous or the scale of its homogeneity is much larger than the scale of nonlocality l_{eff}, then $l_{\text{eff}}(\mathbf{x}, \mathbf{y})$ can be replaced by $\delta(\mathbf{x}-\mathbf{y}) \int l_{\text{eff}}(\mathbf{x}, \mathbf{z})\, d^3z$, so that the nonlocality does not manifest itself. In this precisely lies the meaning of local static dielectric constant of a mixture, the expansion of which was obtained by Brown[6]. Finally, if (1) describes wave propagation, then these waves will unavoidably become damped as a result of the averaging [Eq. (2)].

Inasmuch as exact calculation of $\mathscr{L}$ is impossible, in the presence of a small parameter it is necessary to seek an expansion of $\mathscr{L}$ in powers of this parameter. We consider below two cases when such an expansion can be written in general form. This is, first, expansion in moments of the random function $l(\mathbf{x})$. Second, if the inhomogeneities represent inclusions of a second phase in a continuous first phase—a matrix—then it is possible to expand $\mathscr{L}$ in powers of the concentration of the inclusions. The random properties of the medium are then described by the distribution functions of one, two, three, etc., inclusions.

In applications it is necessary to confine oneself only to the lowest terms of these expansions. The reason for this, in addition to computational difficulties, is the absence of information on the higher moments of $l(\mathbf{x})$ and the higher distribution functions of the inclusions. By way of examples we shall consider, first, an electromagnetic field, since it has some singularities connected with the singularity of its Green's function, and also the dielectric constant of a gas of nonpolar molecules and the effective dielectric constant of a medium with spherical inclusions. In the last two cases new formulas have been obtained.

2. EXPANSION OF l_{eff} IN POWERS OF THE MOMENTS OF $l(\mathbf{x})$

Let L be given by formula (3), in which the function $l(\mathbf{x})$ will be written for the time being with a factor λ. Inverting the known expansion of $\mathscr{G}$ in powers of λ,

$$\mathscr{G} = G_0 + \lambda \langle G_0 l G_0\rangle + \lambda^2 \langle G_0 l G_0 l G_0\rangle + \ldots,$$

where $G_0 = L_0^{-1}$, we obtain the following expansion for l_{eff}:

$$\begin{aligned} l_{eff} &= \lambda \langle l\rangle + \lambda^2 (\langle l G_0 l\rangle - \langle l\rangle G_0 \langle l\rangle) \\ &+ \lambda^3 (\langle l G_0 l G_0 l\rangle - \langle l G_0 l\rangle G_0 \langle l\rangle - \langle l\rangle G_0 \langle l G_0 l\rangle \\ &+ \langle l\rangle G_0 \langle l\rangle G_0 \langle l\rangle) + \ldots, \end{aligned} \tag{5}$$

The s-th term of which is equal to

$$\begin{aligned} &\lambda^s \int d^3x_1 \ldots d^3x_{s-2} K(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_{s-2}, \mathbf{y}) \\ &\times G_0(\mathbf{x}, \mathbf{x}_1) \ldots G_0(\mathbf{x}_{s-2}, \mathbf{y}), \end{aligned} \tag{6}$$

and K is expressed in terms of the moments of the function $l(\mathbf{x})$

$$\begin{aligned} K(\mathbf{x}_1, \ldots, \mathbf{x}_s) &= \langle l(\mathbf{x}_1) \ldots l(\mathbf{x}_s)\rangle \\ &- \sum_{1 \le i < s} \langle l(\mathbf{x}_1) \ldots l(\mathbf{x}_i)\rangle \langle l(\mathbf{x}_{i+1}) \ldots l(\mathbf{x}_s)\rangle \\ &+ \sum_{1 \le i < j < s} \langle l(\mathbf{x}_1) \ldots l(\mathbf{x}_i)\rangle \langle l(\mathbf{x}_{i+1}) \ldots l(\mathbf{x}_j)\rangle \\ &\times \langle l(\mathbf{x}_{j+1}) \ldots l(\mathbf{x}_s)\rangle \\ &+ \ldots + (-1)^{s+1} \langle l(\mathbf{x}_1)\rangle \langle l(\mathbf{x}_2)\rangle \ldots \langle l(\mathbf{x}_s)\rangle. \end{aligned}$$

The convergence of the integrals (6) is ensured by the decrease of the function K, which follows from the weakening of the correlation of $l(\mathbf{x})$ at different points as they move farther apart. For normally distributed $l(\mathbf{x})$ the integrals (6) can be represented by diagrams [1,7]. We note that if l_{eff} ends with the term $\sim\lambda^{N+1}$, then $L_0 - l_{\text{eff}}$ coincides with the operator L_N which is determined in the paper of Tatarskiĭ and Gertsenshteĭn [4].

The separation of L into L_0 and a random part $l(\mathbf{x})$ [or $l(\mathbf{x}, \mathbf{y})$] remains arbitrary. It is usually assumed that $\langle l\rangle = 0$, but it is also possible to proceed in a different way. Confining ourselves to the second moment of $l(\mathbf{x})$ we can, for example, seek a self-consistent solution by carrying out the expansion in $L - \mathscr{L} = l_{\text{eff}} - l$:

$$\begin{aligned} &\langle l - l_{eff}\rangle + \langle (l - l_{eff})\, \mathscr{G}\, (l - l_{eff})\rangle \\ &\quad - \langle l - l_{eff}\rangle\, \mathscr{G}\, \langle l - l_{eff}\rangle \\ &\quad \equiv \langle l\rangle - l_{eff} + (\langle l \mathscr{G} l\rangle - \langle l\rangle\, \mathscr{G}\, \langle l\rangle) = 0, \\ &\mathscr{G} = (L_0 - l_{eff})^{-1}. \end{aligned} \tag{7}$$

3. ELECTROMAGNETIC FIELD

So far no actual choice of the field $\Psi(\mathbf{x})$ was made. It could be both scalar or vector. In the latter case the operators should be tensors. Let us stop to discuss in detail one peculiarity of the electromagnetic field, assuming for simplicity a permeability $\mu = 1$.

In accordance with (1)—(4), the quantity ϵ_{eff} is determined by the relation

$$\langle \mathbf{D}\rangle = \varepsilon_{eff} \langle \mathbf{E}\rangle, \tag{8}$$

where $\mathbf{E}$ is the electric field, $\mathbf{D}(\mathbf{x}) = \epsilon(\mathbf{x})\mathbf{E}(\mathbf{x})$ the induction, and $\epsilon(\mathbf{x})$ the dielectric constant, assumed scalar. The equation for* $\mathbf{E}$

$$\text{rot rot}\, \mathbf{E} - (\omega^2/c^2)\, \varepsilon \mathbf{E} = 4\pi i \omega \mathbf{j}/c^2$$

*rot = curl.

(where ω—frequency, c—velocity of light, $i^2 = -1$, j—current density) can be rewritten in terms of the field $\mathbf{E}_0$ corresponding to a medium with dielectric constants ϵ_0 in lieu of the random $\epsilon(\mathbf{x})$:

$$\text{rot rot } \mathbf{E} - \omega^2 \varepsilon \mathbf{E}/c^2 = \text{rot rot } \mathbf{E}_0 - \omega^2 \varepsilon_0 \mathbf{E}_0/c^2$$

or

$$\text{rot rot } (\mathbf{E} - \mathbf{E}_0) - (\omega^2/c^2)\,\varepsilon_0\,(\mathbf{E} - \mathbf{E}_0) = \omega^2 (\varepsilon - \varepsilon_0)\,\mathbf{E}/c^2 \qquad (9)$$

(compare with (3); here $L = \text{curl curl} - \omega^2\epsilon/c^2$ and $L_0 = \text{curl curl} - \omega^2\epsilon_0/c^2$).

Inverting L_0 we get

$$E^\mu(\mathbf{x}) - E_0^\mu(\mathbf{x}) = \frac{1}{4\pi} \mathrm{P} \int \Lambda^{\mu\nu}(\mathbf{x}-\mathbf{y}) \frac{\varepsilon(\mathbf{y}) - \varepsilon_0}{\varepsilon_0} E^\nu(\mathbf{y})\, d^3y - \frac{1}{3} \frac{\varepsilon(\mathbf{x}) - \varepsilon_0}{\varepsilon_0} E^\mu(\mathbf{x}), \qquad (10)$$

where the Greek indices, which assume values 1, 2, and 3, denote the Cartesian components of the vectors and tensors, repeated indices imply summation,

$$\Lambda^{\mu\nu}(\mathbf{x}) = \left(k_0^2 \delta^{\mu\nu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu}\right) \frac{\exp(ik_0|\mathbf{x}|)}{|\mathbf{x}|}, \qquad (11)$$

and $k_0^2 = \omega^2\epsilon_0/c^2$; the symbol P denotes that the integral is taken in the sense of principal value at the point $\mathbf{x} = \mathbf{y}$.

Equation (10), rewritten in the form

$$F^\mu(\mathbf{x}) = E_0^\mu(\mathbf{x}) + \mathrm{P} \int \Lambda^{\mu\nu}(\mathbf{x}-\mathbf{y})\, \varkappa(\mathbf{y})\, F^\nu(\mathbf{y})\, d^3y,$$

where

$$\mathbf{F} = \frac{\varepsilon + 2\varepsilon_0}{3\varepsilon_0}\mathbf{E}, \qquad \varkappa = \frac{3}{4\pi}\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0},$$

shows that it is more convenient to calculate not ϵ_{eff} but κ_{eff}, which is determined by means of

$$\langle \varkappa \mathbf{F} \rangle = \varkappa_{eff} \langle \mathbf{F} \rangle; \qquad (12)$$

As follows from (8) and (12)

$$3(\varepsilon_{eff} - \varepsilon_0)/4\pi = \varkappa_{eff}(\varepsilon_{eff} + 2\varepsilon_0).$$

The expansion of κ_{eff} in powers of the moments of the random function $\kappa(\mathbf{x})$ is given by (6) with Λ taken for G_0.

It is thus clear that the frequent use of the quantity κ in place of ϵ in problems involving dielectric constants of gases (the Lorenz-Lorentz formula)[5,6,8] and mixtures etc., is connected with the singularity of the Green's function of Eq. (9), which includes the term $\delta(\mathbf{x})$. It is useful to bear this remark in mind also in other problems with singular Green's functions.

4. GROUP EXPANSION OF l_{eff}

If the homogeneities of the medium are represented by random inclusions, then l_{eff} can be expanded in powers of the concentration of the inclusions, which is similar to the group expansion in statistical physics: the s-th approximation is expressed in terms of the solution of the propagation of the field in the presence of s inclusions and their distribution function. In order to obtain the series it is sufficient to use the scheme of an earlier paper by the author[9], after first correcting an error contained there, namely failure to take account of the nonlocality of l_{eff}.

We shall first assume that the number of inclusions N is finite and that they are contained in a volume V. The concentration of the inclusions is $n = N/V$. Let the coordinates ξ_i describe the position and the orientation of the i-th inclusion. The operator L_0 [see (3)] and the field corresponding to it will be assigned to the medium without inclusions. Ψ and $l\Psi$ are linearly connected with Ψ_0:

$$\langle l\Psi(\mathbf{x})\rangle = \int \langle A(\mathbf{x}, \mathbf{y})\rangle\, \Psi_0(\mathbf{y})\, d^3y,$$

$$\langle \Psi(\mathbf{x})\rangle = \int \langle B(\mathbf{x}, \mathbf{y})\rangle\, \Psi_0(\mathbf{y})\, d^3y.$$

The averaging is carried out here over different configurations of the inclusions.

At a specified inclusion configuration $A(\mathbf{x}, \mathbf{y})$ can be represented in the form of a sum over groups of 1, 2, ..., N inclusions:

$$A(\mathbf{x}, \mathbf{y} \mid \xi_1, \ldots, \xi_N) = \sum_{1 \leqslant i \leqslant N} a(\mathbf{x}, \mathbf{y} \mid \xi_i) + \sum_{1 \leqslant i < j \leqslant N} a(\mathbf{x}, \mathbf{y} \mid \xi_i, \xi_j) + \ldots, \qquad (13)$$

where

$$a(\mathbf{x}, \mathbf{y} \mid \xi_1) = A(\mathbf{x}, \mathbf{y} \mid \xi_1),$$

$$a(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_2) = A(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_2) - A(\mathbf{x}, \mathbf{y} \mid \xi_1) - A(\mathbf{x}, \mathbf{y} \mid \xi_2),$$

$$a(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_2, \xi_3) = A(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_2, \xi_3) - A(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_2) - A(\mathbf{x}, \mathbf{y} \mid \xi_2, \xi_3) - A(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_3) + A(\mathbf{x}, \mathbf{y} \mid \xi_1) + A(\mathbf{x}, \mathbf{y} \mid \xi_2) + A(\mathbf{x}, \mathbf{y} \mid \xi_3)$$

etc. Analogously

$$B(\mathbf{x}, \mathbf{y} \mid \xi_1, \ldots, \xi_N) = \delta(\mathbf{x} - \mathbf{y}) + \sum_{1 \leqslant i \leqslant N} b(\mathbf{x}, \mathbf{y} \mid \xi_i) + \sum_{1 \leqslant i < j \leqslant N} b(\mathbf{x}, \mathbf{y} \mid \xi_i, \xi_j) + \ldots \qquad (14)$$

To average it is necessary to integrate (13) and (14) with respect to ξ with distribution functions $F(\xi_1, \ldots, \xi_s)$, which are connected with the probability dw of observing inclusions at the points $\xi_1, \xi_2, \ldots, \xi_s$ by the relation

$$dw = n^s F(\xi_1, \ldots, \xi_s) \frac{d\xi_1 \ldots d\xi_s}{N(N-1)\ldots(N-s+1)}.$$

This yields

$$\langle A(\mathbf{x}, \mathbf{y})\rangle = \sum_{s=1}^{N} \frac{n^s}{s!} \int a_F(\mathbf{x}, \mathbf{y} \mid \xi_1, \ldots, \xi_s)\, d\xi_1 \ldots d\xi_s,$$

where

$$a_F(\mathbf{x}, \mathbf{y} \mid \xi_1, \ldots, \xi_s) = a(\mathbf{x}, \mathbf{y} \mid \xi_1, \ldots, \xi_s) F(\xi_1, \ldots, \xi_s),$$

and an analogous expression for $\langle B(\mathbf{x}, \mathbf{y})\rangle$.

According to the relation

$$l_{eff}\langle\Psi\rangle = \langle l\Psi\rangle$$

we find

$$l_{eff} = \langle A\rangle \langle B\rangle^{-1}.$$

Combining in this formula the terms with identical powers of n, we arrive at a final result

$$\begin{aligned} l_{eff}(\mathbf{x}, \mathbf{y}) &= n\int a_F(\mathbf{x}, \mathbf{y} \mid \xi_1)\, d\xi_1 \\ &+ \frac{n^2}{2!}\int d\xi_1 d\xi_2 \Big[b_F(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_2) \\ &- 2\int d^3z\, b_F(\mathbf{x}, \mathbf{z} \mid \xi_1)\, a_F(\mathbf{z}, \mathbf{y} \mid \xi_2)\Big] \\ &+ \frac{n^3}{3!}\int d\xi_1 d\xi_2 d\xi_3 \Big[b_F(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_2, \xi_3) \\ &- 3\int d^3z\, b_F(\mathbf{x}, \mathbf{z} \mid \xi_1, \xi_2)\, a_F(\mathbf{z}, \mathbf{y} \mid \xi_3) \\ &- 3\int d^3z\, b_F(\mathbf{x}, \mathbf{z} \mid \xi_1)\, a_F(\mathbf{z}, \mathbf{y} \mid \xi_2, \xi_3) \\ &+ 6\int d^3z\, d^3t\, b_F(\mathbf{x}, \mathbf{z} \mid \xi_1)\, a_F(\mathbf{z}, \mathbf{t} \mid \xi_2) \\ &\times a_F(\mathbf{t}, \mathbf{y} \mid \xi_3)\Big] + \ldots \end{aligned} \quad (15)$$

Here it is possible to go in the limit $N \to \infty$, $V \to \infty$, n = const both in the integrals and in the functions F. The integrals, extending over all of space, are convergent. The proof can be obtained in the same manner as in [9]. The expansion of (15) is not a power series in n, since the functions F are themselves dependent on n.

5. DIELECTRIC CONSTANT OF A SYSTEM OF POINTLIKE ELECTRIC DIPOLES

By way of the first example we consider the dielectric constant of a classical gas, which we assume to have pointlike molecules that have a polarizability α but do not have dipole moments. In this problem it is possible to expand both in α [10] and in n [11]. We are interested in the latter. The role of Ψ is played here by the electric field intensity L, that of $l\Psi$ by the polarization **P**, and that of l by the local polarizability, equal to $\alpha \sum_i \delta(\mathbf{x} - \xi_i)$, where ξ_i are the coordinates of the molecules. The tensors

$$a(\mathbf{x}, \mathbf{y} \mid \xi_1) = \alpha\delta(\mathbf{x} - \xi_1)\,\delta(\mathbf{y} - \xi_1)\,\delta^{\mu\nu},$$

$$b(\mathbf{x}, \mathbf{y} \mid \xi_1) = \alpha\Lambda^{\mu\nu}(\mathbf{x} - \xi_1)\,\delta(\mathbf{y} - \xi_1)$$

and $a(\mathbf{x}, \mathbf{y} \mid \xi_1, \xi_2)$, which are needed for the calculation of $\alpha_{eff} = (\epsilon_{eff} - 1)/4\pi$ accurate to n^2, can be obtained by solving the problem of two molecules in an external field:

$$\begin{aligned} a(\mathbf{x}, \mathbf{y} \mid \xi_1) &= f^{\mu\nu}(\xi_1 - \xi_2)\,[\delta(\mathbf{x} - \xi_1)\,\delta(\mathbf{y} - \xi_1) \\ &+ \delta(\mathbf{x} - \xi_2)\,\delta(\mathbf{y} - \xi_2)] \\ &+ g^{\mu\nu}(\xi_1 - \xi_2)\,[\delta(\mathbf{x} - \xi_1)\,\delta(\mathbf{y} - \xi_2) \\ &+ \delta(\mathbf{x} - \xi_2)\,\delta(\mathbf{y} - \xi_1)], \end{aligned}$$

where

$$\Lambda^{\mu\nu}(\mathbf{x}) = \lambda_1(|\mathbf{x}|)\,\delta^{\mu\nu} + \lambda_2(|\mathbf{x}|)\,x^\mu x^\nu / |\mathbf{x}|^2$$

is given by formula (11) with $k_0 = \omega/c$,

$$f^{\mu\nu}(\mathbf{x}) = \tfrac{1}{2}\alpha\,[h^{\mu\nu}(\mathbf{x}, \alpha) + h^{\mu\nu}(\mathbf{x}, -\alpha)] - \alpha\delta^{\mu\nu},$$

$$g^{\mu\nu}(\mathbf{x}) = \tfrac{1}{2}\alpha\,[h^{\mu\nu}(\mathbf{x}, \alpha) - h^{\mu\nu}(\mathbf{x}, -\alpha)],$$

$$h^{\mu\nu}(\mathbf{x}, \alpha) = \frac{1}{1 - \alpha\lambda_1(|\mathbf{x}|)} \times \left[\delta^{\mu\nu} + \frac{x^\mu x^\nu}{|\mathbf{x}|^2}\,\frac{\alpha\lambda_2(|\mathbf{x}|)}{1 - \alpha\lambda_1(|\mathbf{x}|) - \alpha\lambda_2(|\mathbf{x}|)}\right].$$

Using (15), we get

$$\begin{aligned} \alpha_{eff} &= n\alpha\delta^{\mu\nu}\,\delta(\mathbf{x} - \mathbf{y}) \\ &+ n^2\Big\{\delta(\mathbf{x} - \mathbf{y})\int d^3\xi\, F(\mathbf{x}, \xi)\, f^{\mu\nu}(\mathbf{x} - \xi) + \\ &+ F(\mathbf{x}, \mathbf{y})\, g^{\mu\nu}(\mathbf{x} - \mathbf{y}) - \tfrac{1}{2}\alpha^2\Lambda^{\mu\nu}(\mathbf{x} - \mathbf{y})\Big\} + \ldots \end{aligned} \quad (16)$$

6. ϵ_{eff} FOR A MEDIUM WITH SPHERICAL INCLUSIONS

We consider here a problem, solved by Lewin[5] in first order of n, concerning the effective dielectric constant of a medium with spherical inclusions of radius a. To this end it is sufficient to find the tensor $b(\mathbf{x}, \mathbf{y} \mid \xi_1)$, when ξ_1—coordinate of the center of the sphere. For $l(\mathbf{x})$ we take $\epsilon(\mathbf{x})$

$-\epsilon_1$, where $\epsilon(\mathbf{x}) = \epsilon_2$ and ϵ_2 inside and outside the spheres respectively.

The tensor $b(\mathbf{x}, \mathbf{y} | \xi) = b(\mathbf{x} - \xi, \mathbf{y} - \xi | 0)$ can be expressed as a series in spherical vector harmonics. We do not present this expression, since it is too cumbersome. We merely note that the tensor $b(\mathbf{x}, \mathbf{y} | 0) = 0$ when $|\mathbf{x}| > a$ or $|\mathbf{y}| > a$. In first order in n, therefore, $\epsilon_{\text{eff}}(\mathbf{x} - \mathbf{y}) = 0$ when $|\mathbf{x} - \mathbf{y}| > 2a$.

In the case of the field considered by Lewin[5], with a wavelength large compared with a, the nonlocality of ϵ does not appear (see the introduction). Calculations for a homogeneous field leads to

$$\varepsilon_{eff} - \varepsilon_1 = \frac{(\varepsilon_2 - \varepsilon_1)\,\varepsilon_1}{\varepsilon_2}\left[\frac{2(\varepsilon_2 - \varepsilon_1)\,F(\theta_2)}{\varepsilon_2 F(\theta_2)\,G(\theta_1) + 2\varepsilon_1} + 1\right], \quad (17)$$

where

$$\theta_1 = \omega\sqrt{\varepsilon_1\mu_1}\,a/c, \qquad \theta_2 = \omega\sqrt{\varepsilon_2\mu_2}\,a/c,$$

$$F(\theta) = \frac{2(\sin\theta - \theta\cos\theta)}{(\theta^2 - 1)\sin\theta + \theta\cos\theta}, \qquad G(\theta) = \frac{i + \theta - i\theta^2}{i + \theta};$$

μ_2 and μ_1 are the permeabilities of the material inside and outside the sphere.

By virtue of the condition $\omega\sqrt{\epsilon_{\text{eff}}\mu_{\text{eff}}}\,a/c \ll 1$, we should also have $\theta_1 \ll 1$ so that $G(\theta_1) \approx 1$. Nonetheless, our result differs from that of Lewin[5].

7. CONCLUSION

Bourret states[1] that the approximation he has used is not connected with perturbation theory in $l(\mathbf{x})$. This is not true, however, since he actually obtained[1] the lower terms of the expansion in powers of $l(\mathbf{x})$ not only for $\mathcal{G}$, but also for l_{eff}. On the other hand, in papers devoted to media with random inclusions, the results of which are based on solution of the problem of one body in an external field[5,8,12], the question of the character of the approximation is usually not made more precise. It must be borne in mind that in these cases one always deals with a first approximation in the concentration n of the inclusions, although the region of applicability of the result may not be limited to very small n.

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