ON THE THEORY OF ELECTROMAGNETIC EFFECTS IN THE PRESENCE OF THE HALL EFFECT

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Specific features of stationary and quasistationary electromagnetic processes in homogeneous rigid media in the presence of the Hall effect are considered. It is found that: (1) Only equilibrium current and field configurations can exist a considerable time in media with a pronounced Hall effect and high conductivity. (2) If the geometry of a conductor with a pronounced Hall effect and high conductivity is such that equilibrium configurations cannot be produced, a stationary current in the conductor will involve a strongly oscillating current pattern over the cross section. However the possibility remains that the current flow is in reality non-stationary. (3) The electromagnetic damping length may differ appreciably from the thickness of the skin effect layer; this length grows with increase of the conductivity. In this case the wave velocity in the medium is of the order of the velocity (or of the "effective" velocity) of the current carriers.

1. INTRODUCTION

THE important role of the Hall effect in plasma phenomena and particularly its role in quasistationary processes, is becoming increasingly clear at present. Since the manifestations of the Hall effect in a plasma are made complicated by many other factors, it is meaningful to analyze the specific role of the Hall effect using as an example a homogeneous solid medium [1)] for which Ohm's law can be written in the form

$$\mathbf{u} = k\,(\mathbf{E} + c^{-1}\,[\mathbf{uH}]). \qquad (1.1)^*$$

Here u—carrier velocity, **E** and H—intensities of electric and magnetic fields respectively, and k—mobility connected with the carrier concentration n and with the conductivity σ by the relation

$$k = \sigma/en, \qquad (1.2)$$

where e—charge of carrier.

The conductivity and the concentration of the carriers will henceforth be assumed constant.

For the case of quasistationary processes equation (1.1) must be supplemented by the two Maxwell equations

$$\mathrm{rot}\,\mathbf{E} = -\,c^{-1}\partial\mathbf{H}/\partial t, \quad \mathrm{rot}\,\mathbf{H} = \gamma\mathbf{u} \quad (\gamma = 4\pi c^{-1}en). \qquad (1.3)$$

For specific calculations it is convenient to transform the system (1.1) and (1.3) by eliminating the electric field **E**. The resultant system takes the form

$$\partial\mathbf{H}/\partial t = \nu_m\Delta\mathbf{H} + \mathrm{rot}\,[\mathbf{uH}], \qquad \mathrm{rot}\,\mathbf{H} = \gamma\mathbf{u}, \qquad (1.4)$$

where $\nu_m = c^2/4\pi\sigma$—magnetic viscosity of the medium.

The difficulty of investigating the system (1.4) lies in the fact that the latter is nonlinear. In many cases the interpretation of the results of the investigation of system (1.4) is facilitated if it is written in integral form. To this end we take a liquid contour (Fig. 1) formed by moving carriers, then

$$\oint_\Gamma \mathbf{H}\,d\mathbf{l} = \frac{4\pi}{c}J, \qquad -\frac{c}{k}\oint_\Gamma \mathbf{u}\,d\mathbf{l} = \frac{d\Phi}{dt}. \qquad (1.5)$$

Here Φ—magnetic flux and J—electric current through contour Γ.

$$\Phi = \iint_{(\Gamma)} \mathbf{H}\,d\mathbf{s}, \qquad J = en\iint_{(\Gamma)} \mathbf{u}\,d\mathbf{s}.$$

It is necessary to add to (1.5) still another condition for the incompressibility of the flow

$$\mathrm{div}\,\mathbf{u} = 0. \qquad (1.6)$$

It follows from the first equation of (1.4) that as $|k| \to \infty$ the field becomes frozen in the carrier stream. Consequently, the main difference between an ideally conducting medium with strongly pronounced Hall effect from a medium where this effect is small is that in the former case the magnetic flux through the moving con-

1)By a solid medium we mean a medium which does not deform during the course of the process under consideration.

*[uH] = u × H.

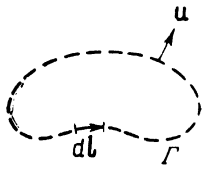

FIG. 1

tour is conserved, while in the latter the conserved flux is through the stationary contour. In the general case it can be stated that the Hall effect makes it possible for the field in the medium not only to be diffused but also to be transported.

As can be seen from (1.1), the field is strongly frozen in, when

$$Q = kH/c = eH\tau/mc = \omega_e\tau \gg 1. \qquad (1.7)$$

Here m is the carrier mass, τ^{-1} the collision frequency, and ω_e the Larmor frequency of the carriers. At present there are several semiconductor materials in which the mobility k is quite large. Thus, in InSb $k \approx 2.5 \times 10^7$ absolute units and consequently condition (1.7) is already well satisfied in a field $H \sim 10^4$ G.

In ordinary hydrodynamics the degree of freezing-in is determined, as is well known, by the magnetic Reynolds number

$$R_m = vL/\nu_m = 4\pi\sigma vL/c^2, \qquad (1.8)$$

where v is the characteristic velocity of the flow and L the characteristic scale. It is possible to reduce condition (1.7) to the same form by substituting in it the value of k from (1.2) and $H \sim 4\pi c^{-1}enuL$. As a result we obtain

$$Q \sim 4n\sigma uL/c^2. \qquad (1.9)$$

It must be noted that much research has been done on fields in solids in the presence of the Hall effect. However, most investigations either concern special problems connected with the operation of Hall transducers [1,2], or are devoted to the propagation of linear waves in a homogeneous medium or in waveguides under strongly nonstationary conditions [3,4], or else concern the behavior of an electron-hole plasma in modes when the number of carriers varies [5-7].

The purpose of the present paper is to clarify several general properties of nonlinear quasistationary processes in a homogeneous time-invariant conductor.

2. DIRECT CURRENT

If direct current flows in the medium, then (1.1) assumes the form

$$\nabla\varphi = c^{-1}[\mathbf{uH}] - \mathbf{u}/k, \qquad \mathbf{E} = -\nabla\varphi. \qquad (2.1)$$

In the case when $|k| \to \infty$, (2.1) goes over formally into the equation

$$\nabla p = c^{-1}[\mathbf{jH}], \qquad p = en\varphi, \qquad (2.2)$$

which coincides with the equation for equilibrium plasma configurations, and the role of the pressure is played in this case by the electrostatic polarization of the medium.

Consequently, whereas in an ideally conducting medium without the Hall effect a field of arbitrary configuration can exist for an arbitrarily long time, in an ideally conducting medium with the Hall effect only equilibrium configurations can exist for an unlimited time. These configurations can be either detached from the walls of the conductor under consideration (Fig. 2a) or bear against the walls (Fig. 2b), with the electric circuit closed outside the given conductor.

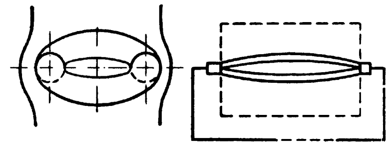

FIG. 2

Methods of calculating equilibrium configurations satisfying Eq. (2.2) have been developed in extensive detail [8,9] and will therefore not be dealt with here. We proceed to consider very simple examples satisfying (2.1). It is obvious that if the configuration is bounded in all three dimensions and the Poynting-vector flux (at infinity) in a volume containing this configuration is equal to zero, then the configuration becomes gradually destroyed, i.e., it is not stationary. Consequently only configurations of the type shown in Fig. 2b can be stationary. Let us consider two examples of such configurations.

A. Plane Flow

Let the magnetic field have only one component $H_z = H$. Then the lines H = const are stream lines of the carriers and

$$[\mathbf{uH}] = -\frac{1}{en}\nabla\frac{H^2}{8\pi}. \qquad (2.3)$$

Consequently, as is clear from (1.4), the field satisfies the Laplace equation

$$\Delta H = 0, \qquad (2.4)$$

i.e., the Hall effect does not appear in the equation. This does not mean, however, that it cannot appear at all in this case. Indeed, in addition to (2.4), the field should satisfy also the boundary conditions. On the side walls of the conductor the condition is (Fig. 3)

$$\mathbf{un}|_b = 0. \tag{2.5}$$

This condition is not sensitive to the Hall effect.

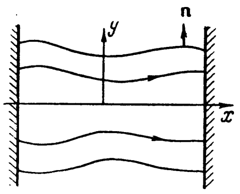

FIG. 3

However, if the end walls of the conductor in question come in contact with metal, we must equate to zero the tangential component of the electric field on the end walls

$$E_t|_{\text{end}} = 0. \tag{2.6}$$

In the absence of the Hall effect this means that $u_t|_{\text{end}} = 0$, whereas in the presence of the Hall effect we should have

$$(u_t k^{-1} - c^{-1}[\mathbf{uH}]_t)|_{\text{end}} = 0, \tag{2.7}$$

and the influence of the Hall effect is obvious[2)]. This question is analyzed in detail in the theory of Hall-effect transducers [2]. If we assume that $|k| = \infty$, then (1.4) is satisfied by any function $H = H(x, y)$. Inasmuch as the carrier stream lines are the lines $H = \text{const}$, the function H on the side boundaries of the conductor should be constant, by virtue of (2.5). It is easy to see that when $|k| = \infty$ it is impossible to satisfy condition (2.6) on the ends.

B. Axially-symmetrical flow with $H_r = H_z = 0$

If only the component $H_\varphi(r, z, t)$ differs from zero, then by introducing the function

$$I = rH_\varphi, \tag{2.8}$$

which is a function of the current-carrier flux, we obtain with the aid of (1.4) the equation

$$\frac{\partial^2 I}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial I}{\partial r}\right) = \frac{k}{c}\left(\gamma\frac{\partial I}{\partial t} + \frac{2}{r^2} I\frac{\partial I}{\partial z}\right) \tag{2.9}$$

or in the stationary case the equation

$$\frac{\partial^2 I}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial I}{\partial r}\right) = \frac{2k}{cr^2} I\frac{\partial I}{\partial z}. \tag{2.10}$$

If $\partial I/\partial z = 0$ and the field is regular on the axis, then the solution of (2.10) will be

$$H_\varphi = a_0 r/2. \tag{2.11}$$

This is the field of a straight cylindrical conductor with constant current density, and the field does not "feel" the Hall effect.

The situation is entirely different if $\partial I/\partial z \neq 0$. The latter condition occurs practically always both because of the irregular form of the conductor and because of the end effects; theoretically this is simplest to realize by corrugating the surface of an infinitely long conductor.

Equation (2.10) shows that as $|k| \to \infty$ there is no limiting transition to equation (2.2) if $\partial I/\partial z \neq 0$. In connection with the difficulty of exact solution of (2.10), we consider two particular cases.

1. Case of poor conductivity ($k \to 0$). If k is small, then we can seek a solution in the form of a series in powers of k. Simple calculations show that this series has the following first terms:

$$I = \frac{a_0 r^2}{2} + \frac{k}{c}\frac{a_0 a_0'}{16} r^4 + \dots \tag{2.12}$$

Here a_0 is an arbitrary function of z. In order for the terms of the series to decrease, it is necessary to have

$$\xi = k a_0' R^2/8c \ll 1 \tag{2.13}$$

(R is the characteristic radius of the conductor). If we denote by L_z the period of the corrugation of the conductor and introduce the thickness δ of the skin layer with the aid of the relation

$$\delta^2 = c^2 L_z/2\pi\sigma u,$$

where u is the carrier velocity, then condition (2.13) can be written in the form

$$\xi = R^2/16\pi\delta^2 \ll 1. \tag{2.13a}$$

Thus, if the thickness of the skin layer is appreciably larger than the radius of the conductor, then the influence of the Hall effect is relatively small and leads to some additional corrugation of the flux.

2. Case of good conductivity ($|k| \to \infty$). In order to obtain an idea of this case, we consider flow in a hollow conductor, as shown in Fig. 4. In this case Eq. (2.10) can be linearized near the surface S:

$$I = I_0 + I_1 + I_2 + \dots, \tag{2.14}$$

where I_0 is a constant quantity due to the extraneous current flowing along the axis of the

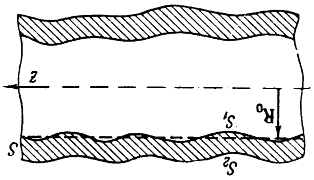

FIG. 4

[2)]In order to avoid this, it is necessary to use separated electrodes, on which condition (2.6) does not necessarily hold.

channel, while I_1 and I_2 are additions due to the currents flowing in the conductor when $r > R_0(S)$. Substituting (2.14) in (2.10) we obtain in the first approximation

$$\frac{\partial^2 I_1}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial I_1}{\partial r}\right) = \frac{2kI_0}{cr^2}\frac{\partial I_1}{\partial z}. \qquad (2.15)$$

Putting $I_1 = e^{i\alpha z} R(r)$, we get

$$\frac{\partial^2 R}{\partial r^2} - \frac{1}{r}\frac{\partial R}{\partial r} - \left(\alpha^2 + \frac{\lambda^2}{r^2}\right) R = 0. \qquad (2.16)$$

Here

$$\lambda^2 = i\frac{2kI_0}{c}\alpha = 4\pi i\omega_{e0}\tau\frac{R_0}{L}. \qquad (2.17)$$

If $\lambda^2 \to \infty$ then (2.16) has a solution of the form

$$R = C_1 r^\lambda + C_2 r^{-\lambda}, \qquad \mathrm{Re}\,\lambda > 0 \qquad (2.18)$$

and

$$I_1 = (C_1 r^\Lambda + C_2 r^{-\Lambda})\cos(\Lambda \ln r - \alpha z),$$

$$\Lambda = \sqrt{2\pi\omega_{e0}\tau R_0/L}. \qquad (2.19)$$

The constants C_1 and C_2 are determined by the form of the boundary surfaces s_1 and s_2[3)] and by the total current flowing in the conductor.

Expression (2.19) shows that when $\omega_{e0}\tau \to \infty$ the current has an oscillating structure with ever increasing amplitude. This ensures large dissipation even as $\sigma \to \infty$. If we take into account the next terms of the expansion, then they contain along with the oscillating part also terms that depend monotonically on r.

3. NONLINEAR QUASISTATIONARY FIELDS

Because of the difficulty of calculating the nonlinear quasistationary processes, we confine ourselves to three particular problems.

1. We ascertain under what conditions a limited configuration will retain its structure while collapsing in time. It is obvious that this can occur only if

$$\partial \mathbf{H}/\partial t = -q^2\mathbf{H}, \qquad q^2 > 0, \qquad (3.1)$$

for only in this case does the configuration current change in proportion. From (3.1) it follows that

$$\mathbf{H} = \mathbf{H}_0 e^{-q^2 t}. \qquad (3.2)$$

Substituting (3.2) in (1.4) we obtain

$$\mathrm{rot}\,[\mathbf{uH}] = 0 \qquad (3.3a)^*$$

[3)]It is assumed that the geometry of the surfaces admits of flows that can be described a single harmonic.

*rot = curl.

and consequently

$$\Delta\mathbf{H} = -(q^2/\nu_m)\,\mathbf{H}. \qquad (3.3b)$$

This condition, at least in the presence of symmetry, is satisfied only by force-free fields.

2. It is of definite interest to ascertain the conditions under which the Hall effect does not manifest itself in the field equations, in other words, when

$$[\mathbf{uH}] = \nabla\chi. \qquad (3.4)$$

It follows therefore that both the electric current lines and the magnetic force lines should lie on the surface χ = const, since

$$(\mathbf{u}\nabla)\chi = 0, \qquad (\mathbf{H}\nabla)\chi = 0. \qquad (3.4a)$$

These are the necessary but not the sufficient conditions, since the necessary and sufficient condition (3.4) breaks up into three conditions.

If the field has axial symmetry, then there are grounds for assuming that under sinusoidal time variation of the field the Hall effect will have no influence if and only if the field depends on r (see Sec. 2).

3. We have considered above the case of current flow with $H_r = H_z = 0$. Assuming ideal conductivity, we can investigate the behavior of the configuration for $\partial/\partial z \neq 0$. Indeed, in this case Eq. (2.9) for the current function I assumes the form

$$\gamma\frac{\partial I}{\partial t} + \frac{2}{r^2} I \frac{\partial I}{\partial z} = 0, \qquad (3.5)$$

hence

$$I = f(r,\, z - 2It/\gamma r^2). \qquad (3.6)$$

The function f is determined by the initial conditions. We see that in the presence of perturbations that depend on z, we have a flow of waves that travel with velocities

$$v_{\mathrm{ph}} = 2I/\gamma r^2, \qquad (3.7)$$

dependent on I and r. Substituting here order of magnitude values of I and γ: $I \sim 2enur^2/c$ and $\gamma = 4\pi en/c$, we obtain $v_{ph} \sim u$.

Thus, as expected on the basis of (1.4), the perturbations travel with the velocity of the order of that of the carriers. Unlike in plasma configurations, the superposition of sausage-type perturbation will not lead in this case to an exponential growth of the perturbation.

Equation (3.6) has one interesting feature. Namely, if the function I vanishes on some surface $r = R(z)$, then the flow is stationary on this surface, i.e., $\partial/\partial t = 0$. Then $f(r, z) = (r - R(z))\,g(r, z)$, where g is an arbitrary bounded function. Consequently, only that stream surface on which $I = 0$ can be rigidly fixed. If $I \neq 0$ on the fixed surface, then dissipative processes of the type considered above must develop.

4. PLANE LINEAR WAVES IN THE PRESENCE OF DC IN THE MEDIUM

Since an exact investigation of (1.4) is very complicated, it is natural to resort to linearization of these equations. In the present section we consider the propagation of waves in a medium in which homogeneous current flows with velocity $\mathbf{u}_0$ in the plane (y, z):

$$\mathbf{u}_0 = \{0, u_{0y}, u_{0z}\}. \quad (4.1)$$

The magnetic field of the current $\mathbf{H}_j$ will be assumed direct along the x axis

$$\mathbf{H}_j = \{\gamma (u_{0y} z - u_{0z} y), 0, 0\}. \quad (4.2)$$

In addition, we assume that along with the field $\mathbf{H}_j$ there is a certain arbitrarily oriented homogeneous field $\mathbf{H}_1$, so that the total unperturbed field is

$$\mathbf{H}_0 = \mathbf{H}_j + \mathbf{H}_1. \quad (4.3)$$

It is easy to verify that the field $\mathbf{H}_0$ satisfies Eq. (1.4). We now impose on the field $\mathbf{H}_0$ a small perturbation $\mathbf{h}$ such that

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}, \qquad \mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1. \quad (4.4)$$

Linearizing (1.4) we obtain, generally speaking, a system with coefficients that depend on y and z. However, if we confine ourselves to waves that propagate in a plane perpendicular to $\mathbf{H}_j$, then the coefficients of the equations remain constant. We direct the z axis along the direction of wave propagation (Fig. 5) and assume that the perturbation is of the form $\exp(ikz - i\omega t)$. Then by virtue of the equation $\operatorname{div} \mathbf{H} = 0$ the field component $h_z = 0$ and remaining components are determined by

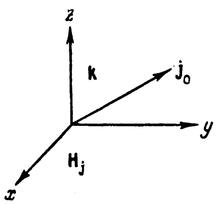

FIG. 5

$$(-i\omega/\nu_m + k^2)\, h_x - k\,(\mathbf{k}\,\boldsymbol{\Omega}\tau)\, h_y = 0,$$
$$k\,(\mathbf{k}\,\boldsymbol{\Omega}\tau)\, h_x + (-i\omega/\nu_m + k^2 + i\,(\mathbf{k}\mathbf{u}_0)/\nu_m)\, h_y = 0. \quad (4.5)$$

Here $\boldsymbol{\Omega} = e\mathbf{H}_0/mc$.

We are especially interested in two cases.

1. If $\mathbf{u}_0 = 0$ but $\mathbf{k}^2 \cdot \mathbf{H}_0 \neq 0$, then we have ordinary low-frequency waves with rotating plane of polarization [10]. In this case the dispersion equation takes the form

$$i\omega/\nu_m - k^2 \pm ik\,(\mathbf{k}\,\boldsymbol{\Omega}\tau) = 0. \quad (4.6)$$

If $\nu_m \to 0$ and $\Omega_z \tau \to \infty$, then

$$k \approx \sqrt{\omega/\nu_m \Omega_z \tau}\,(1 + i/2\Omega_z\tau). \quad (4.7)$$

2. If $\mathbf{k} \cdot \mathbf{H}_0 = 0$ and $\mathbf{k} \cdot \mathbf{u}_0 \neq 0$, then two linearly polarized waves are possible, of which one ($h_y = 0$, $h_x \neq 0$) does not "feel" the Hall effect and attenuates within the thickness of ordinary skin layer, whereas the second ($h_y \neq 0$, $h_x = 0$) 'feels' the Hall effect and as $\nu_m \to 0$ it propagates with velocity u_{0z} and attenuates weakly.

5. WAVES IN A CYLINDRICAL CONDUCTOR

We have considered above the propagation of waves in a cylindrical ideal conductor. Now, using the method of linearization, we consider the propagation of waves with small amplitude in cylindrical flow and for arbitrary conductivity. For simplicity we shall assume that the perturbation $\mathbf{h}$ has only an azimuthal component. Equation (1.4) has in this case the form

$$\frac{\partial h}{\partial t} = \nu_m \Delta h + \frac{cH}{2\pi ner}\frac{\partial h}{\partial z} - \nu_m \frac{h}{r^2}. \quad (5.1)$$

Here H—azimuthal component of the main field. The form of (5.1) does not depend on whether volume current flow, or whether the field H is produced by currents flowing near the axis and remains potential in the remaining part of the conductor.

If the field H is produced by volume currents, then $H \sim r$ and

$$\partial h/\partial t = \nu_m (\Delta h - h/r^2) + u_1\, \partial h/\partial z. \quad (5.2)$$

On the other hand, if it is produced by a current J_1 flowing over the surface $r = r_1$ of the inner cylinder (see Fig. 6), then

$$\partial h/\partial t = \nu_m (\Delta h - h/r^2) + u_2 (r_1/r)^2\, \partial h/\partial z. \quad (5.3)$$

In formula (5.2) we have introduced the carrier

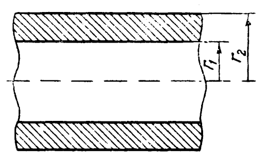

FIG. 6

velocity u_1, while in (5.3) we have the "effective" carrier velocity $u_2 = J_1/enr_1^2$.

The solution of (5.2) and (5.3) can be sought in the form $f(r)\exp(ikz - i\omega t)$. Then $f(r)$ is expressed in terms of cylindrical functions with rather complicated indices and arguments. We therefore confine ourselves here to the propagation of a wave between two separated cylinders[4)], the distance $\delta r = r_2 - r_1$ between which is small ($\delta r/r_1 \ll 1$). In this case we can neglect, for example, the r-dependence in (5.3), and we obtain

$$\partial h/\partial t = \nu_m \partial^2 h/\partial z^2 + u_2 \partial h/\partial z. \qquad (5.4)$$

Putting $h \sim \exp(ikz - i\omega t)$ we obtain the dispersion equation

$$i(\omega + u_2 k) - \nu_m k^2 = 0. \qquad (5.5)$$

If $\nu_m k^2 \ll |u_2 k|$, then the solution of (5.4) can be represented in the form

$$h = h_0 \exp\left\{\frac{\nu_m \omega^2}{u_2^3} z - i\omega\left(t + \frac{z}{u_2}\right)\right\}. \qquad (5.6)$$

We see that the characteristic damping $\delta \sim u_2^3/\nu_m\omega^2$ increases with increasing conductivity. The phase velocity of the wave is $v_{ph} = -u_2$.

It is curious to note that in the 'plane' case as shown above, the wave polarized along the field $\mathbf{H}_j$ attenuates in the usual fashion. Here, however, this wave is transported with velocity u_2.

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4)In order to avoid difficulties with the boundary condition $E_t \mid b = 0$.

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