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NEW REDUCTION FORMULAS

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The previously[1] developed method for the establishing of the relation between the matrix elements of local operators and experimentally observable quantities is generalized and strengthened in several directions. In the first place it is shown that the method is applicable not only to the energy-momentum tensor and the electromagnetic current but also to an arbitrary local quantity. In the second place the aggregate of the equations themselves, that establishes this connection, is significantly widened. In the third place it is shown that the well-known Lehmann-Symanzik-Zimmermann reduction formula[2] can also be derived on the basis of the method of dynamic moments developed in [1] and in the present work, this formula being a special case of the present approach.

1. INTRODUCTION

THE local quantities, such as fields and currents, are not subject to direct experimental measurement in microregions. On the other hand it is this kind of quantities that are important for the formulation of properties, such as microcausality, connected with the space-time description of relativistic quantum processes. It is therefore very important in modern quantum field theory to establish the maximum number of relations between matrix elements of local quantities and the matrix elements of the S matrix.

Until recently the only such relation consisted of the reduction formulas of Lehmann, Symanzik, and Zimmermann.[2] In these reduction formulas form factors of matrix elements of local operators on the mass shell only are related to S-matrix elements. The values of these form factors off the mass shell are usually tacitly understood to have no direct physical meaning. However, in [1] it was shown that there exist for the matrix elements of the local operators—the energy-momentum tensor $T_{\mu\nu}(x)$ and the electromagnetic current $j_\mu(x)$—an infinite number of relations, different from the reduction formula, that connect them to the S matrix when they are off the mass shell.

The purpose of the present work is to generalize and strengthen the results obtained in [1]. In Sec. 2 it is shown that the method of dynamic moments, developed in [1] for the operators $T_{\mu\nu}(x)$ and $j_\mu(x)$, can be utilized to establish an infinite number of relations connecting the S matrix with the off-the-mass-shell matrix elements of arbitrary local quantities, i.e., fields or currents. The main result of this section consists of the finding that from a given value of the Heisenberg matrix element $\langle 2|A(x)|2\rangle$ of a certain local operator $A(x)$ one can determine all the elastic scattering phase shifts but one (for example, all but the S-wave phase shift), and also the kinematic characteristics and transition amplitudes for all inelastic channels. Further strengthening of these results is achieved in Sec. 3, where we derive relations that allow one to determine from a given matrix element $\langle 2|A(x)|2\rangle$ the one-particle form factors of the field $A(x)$ (i.e., in the final analysis the quantities $\langle 1|A(x)|1\rangle$) for all particles produced in the inelastic channels.

In Sec. 4 we give a generalization of the method of dynamic moments which makes it possible to obtain the reduction formulas of Lehmann, Symanzik, and Zimmermann,[2] which are thus shown to be a special case of the method developed in [1] and in the present paper. In Sec. 5 the close connection is noted between the equations for the dynamic moments and the analytic properties of matrix elements of local operators. In the concluding Section we discuss the perspectives for the application and further generalization of the new reduction formulas.

2. THE METHOD OF DYNAMIC MOMENTS

Let $A(x)$ denote the operator referring to a certain local quantity. We assume for simplicity that $A(x)$ is a scalar and that all the particles are spinless. The generalization to the case of particles with spin and to local quantities with different tensorial transformation properties presents no difficulties of principle and may be carried out by means of the technique of invariant parametrization of matrix elements, as developed by Cheshkov and the author.[3] In the following we make no assumptions about the form, or even about the existence, of equations of motion but confine ourselves to the requirements of relativistic invariance, positiveness of the energy spectrum, the existence of the S matrix, and the condition of spatial boundedness of the particles which will be formulated below. We also have no need for the principle of causality in an explicit form since, in contrast to [2], we have no need for the introduction of T-products or retarded products of operators referring to local quantities.

In the Heisenberg picture the local quantity A(x) is characterized by the matrix elements $\langle a | A(x) | b\rangle$, where $|a\rangle$, $|b\rangle$ are free-particle states at minus infinity in time (in-states). The technique developed in this Sec. is based on the properties of the one-particle matrix element $\langle \mathbf{p} | A(x) | \mathbf{p}'\rangle$ of the operator A(x). This element is parametrized, i.e., expressed in terms of the invariant form factors in the following way:

$$\langle \mathbf{p} | A(x) | \mathbf{p}'\rangle = \frac{e^{-ix(p-p')}}{(2\pi)^3 \sqrt{4EE'}} f(s), \tag{1}$$

where

$$E = \sqrt{\mathbf{p}^2 + M^2}, \quad E' = \sqrt{\mathbf{p}'^2 + M^2}, \quad \mathbf{xp} \equiv xp - Et,$$
$$s = -(p-p')^2 \equiv (E-E')^2 - (\mathbf{p}-\mathbf{p}')^2,$$

M, E, p are respectively the mass, energy and momentum of the particle.

We now construct the one-particle matrix elements of the operators D(t), $D_i(t)$, $D_{ij}(t)$, ... (i, j = 1, 2, 3) defined as follows:

$$D(t) = \int d^3x A(x), \qquad D_i(t) = \int d^3x x_i \frac{\partial A(x)}{\partial t},$$
$$D_{ij}(t) = \frac{1}{2}\int d^3x x_i x_j \frac{\partial^2 A(x)}{\partial t^2},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$D_{i\ldots i_n}(t) = \frac{1}{n!}\int d^3x x_i, \ldots, x_{i_n} \frac{\partial^n A(x)}{\partial t^n}. \tag{2}$$

Taking the one-particle matrix elements of the operators (2) and making use of (1) we find that all these elements are diagonal and have the form

$$\langle \mathbf{p} | D(t) | \mathbf{p}'\rangle = \frac{1}{2E} f(0)\, \delta(\mathbf{p}-\mathbf{p}'),$$
$$\langle \mathbf{p} | D_i(t) | \mathbf{p}'\rangle = \frac{1}{2E}\frac{p_i}{E} f(0)\, \delta(\mathbf{p}-\mathbf{p}') = \frac{f(0)}{2E} v_i \delta(\mathbf{p}-\mathbf{p}'),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\langle \mathbf{p} | D_{i,\ldots,i_n}(t) | \mathbf{p}\rangle = \frac{f(0)}{2E} v_{i_1} \ldots v_{i_n} \delta(\mathbf{p}-\mathbf{p}'), \tag{3}$$

where $v_i = p_i/M$ stands for the particle velocity. Consequently the quantities $D_{i,\ldots i_n}$ represent velocity moments of the form factor at zero f(0), divided by 2E. They may therefore be called the dynamic moments of the operator A(x).[1] Let us emphasize that according to (3) all the dynamic moments are exact integrals of the motion for the one-particle state.

In a system of several free particles the matrix element $\langle \mathbf{p}_1, \ldots, \mathbf{p}_N | A(x) | \mathbf{p}_1', \ldots, \mathbf{p}_N'\rangle$ of the operator A(x) is expressed as the sum of N elements of the type (1). Accordingly the dynamic moments also reduce to sums over particles and remain integrals of the motion:

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_N | D^0_{i,\ldots,i_n}(t) | \mathbf{p}_1' \ldots \mathbf{p}_N'\rangle$$
$$= \sum_N \frac{f_N(0)}{2E_N} v^{(N)}_{i_1} \ldots v^{(N)}_{i_n} \Pi\delta(\mathbf{p}-\mathbf{p}'). \tag{4}$$

In the presence of interaction the dynamic moments will no longer be, in general, integrals of the motion. In that case the matrix elements of the operator $D_{i\ldots}(t)$ of any one of the dynamic moments, taken between states at minus infinity in time, may be expressed in the form

$$\langle \mathbf{p}_{in} | D_{i\ldots}(t) | \mathbf{p}'_{in}\rangle = \langle \mathbf{p}_{in} | D^0_{i\ldots} | \mathbf{p}'_{in}\rangle + \langle \mathbf{p}_{in} | D^{int}_{i\ldots}(t) | \mathbf{p}'_{in}\rangle, \tag{5}$$

where D_i^0 is the sum of dynamic moments of free particles defined in (4) and p_{in} denotes the aggregate $p_1 \ldots p_N$ of the moments of the particles at minus infinity in time.

It follows from our assumption that the S-matrix exists, that at infinity in time all particles are free so that in the limit as $t \to \pm\infty$ the dynamic moments go over into additive in the particles integrals of the motion, i.e., they become unitarily equivalent to (4). For the in-basis (5) this equivalence transformation is equal to unity as $t \to -\infty$, and to the S matrix as $t \to +\infty$, i.e.,

$$\lim_{t\to-\infty} \langle \mathbf{p}_{in} | D_{i\ldots}(t) | \mathbf{p}'_{in}\rangle = \langle \mathbf{p}_{in} | D^0_{i\ldots} | \mathbf{p}'_{in}\rangle, \tag{6}$$
$$\lim_{t\to+\infty} \langle \mathbf{p}_{out} | D_{i\ldots}(t) | \mathbf{p}'_{out}\rangle = \langle \mathbf{p}_{out} | S^{-1} D^0_{i\ldots} S | \mathbf{p}'_{out}\rangle. \tag{7}$$

By combining the relations (6) and (7) the sought for system of relations connecting the matrix elements of the local operator A(x) to the S matrix may be written in the operator form:

$$D_{i\ldots}(+\infty) = S^{-1} D_{i\ldots}(-\infty) S. \tag{8}$$

We investigate now the question of precisely what information about the scattering matrix can be obtained from relations (8) for certain matrix elements of the operator $A(x)$. It is clear that from the matrix elements of the type $\langle 0|A(x)|0\rangle$, $\langle 1|A(x)|0\rangle$, $\langle 1|A(x)|1\rangle$ one cannot with the help of relations (8) obtain information about the scattering matrix, because these elements are taken between states for which the scattering matrix is equal to unity. Similarly nothing is learned from the use of matrix elements of the type $\langle 1|A(x)|2\rangle$ (and in general $\langle 1|A(x)|n\rangle$) since for such elements the dynamic moments vanish. Therefore the simplest elements that contain information about the scattering matrix turn out to be

$$\langle 2|A(x)|2\rangle \equiv \langle \mathbf{p}, \mathbf{k}|A(x)|\mathbf{p}', \mathbf{k}'\rangle,$$

where $\mathbf{p}$ and $\mathbf{k}$ are the momenta of the first and second particle, respectively.

We confine ourselves to the study of this simplest element, since the method of construction of the reduction formulas remains unchanged on going over to more complicated cases. The element $\langle 2|A(x)|2\rangle$ is parametrized, i.e., expressed in terms of invariant form factors, in the following manner:

$$\langle \mathbf{p}, \mathbf{k}|A(x)|\mathbf{p}, \mathbf{k}\rangle = \frac{e^{-ix(p-p')}f_1(s_3)}{(2\pi)^3\sqrt{4EE'}}\,\delta(\mathbf{k}-\mathbf{k}') + \frac{e^{-ix(k-k')}f_2(s_4)}{(2\pi)^3\sqrt{4\omega\omega'}} + \frac{e^{-ix(p+k-p'-k')}f_{int}(s_1,\ldots,s_6)}{(2\pi)^3\sqrt{16EE'\omega\omega'}}, \quad (9)$$

where $\omega = \sqrt{k^2+m^2}$, $\omega' = \sqrt{k'^2+m^2}$; m is the mass of the second particle,

$$s_1 = -(p+k)^2, \quad s_2 = -(p'+k')^2, \quad s_3 = -(p-p')^2,$$
$$s_4 = (k-k')^2, \quad s_5 = -(p-k')^2, \quad s_6 = -(p'-k)^2;$$

f_1, f_2 are the one-particle form factors of the first and second particle respectively, f_{int} is the form factor describing the interaction. We emphasize that the decomposition of the matrix element (9) into three terms does not constitute a new assumption but follows solely from considerations of relativistic invariance. The condition of spatial separation of the particles leads to the result that each of the one-particle form factors, f_1 and f_2, depends on only one of the three possible invariant variables.

Let us suppose now that the form factors f_1, f_2, and f_{int} are given and ask what information about the S matrix may be obtained with the help of Eqs. (8). To this end it is necessary to take dynamic moments (2) of the matrix element (9), pass to the limit $t \to \pm\infty$ and substitute these limit values of the dynamic moments into the matrix element of Eqs. (8), taken between two-particle in-states. In the in-basis the matrix elements of the operators $D^{int}(t)$ should vanish as $t \to -\infty$, and should become integrals of the motion as $t \to +\infty$. At energies below the thresholds of all inelastic channels the only unknown part remaining in (8) will be the elastic scattering S matrix, which may then be determined from these equations accurate up to a phase. At energies above inelastic channel thresholds there appear on the right sides limit values of matrix elements of dynamic moments in inelastic channels. In the in-basis all these matrix elements will be simply diagonal in the number of particles and in the kinematic variables sums of one-particle dynamic moments (3). The appearance of the contributions from the inelastic channels does not weaken but strengthens the method. Indeed, the number of dynamic moments, and therefore the number of Eqs. (8), is infinite. Therefore by taking a sufficient number of Eqs. (8) at a given energy we can determine not only the elastic scattering cross section, but also the inelastic scattering cross section, and also masses and values of form factors at zero for all particles in inelastic channels. Here too just one elastic scattering phase will remain undetermined (for example S-wave).[1)]

To clarify matters with an example, we consider the simplest inelastic channel corresponding to the transformation of the S state of the incident particle pair into the S state of the system of two particles with masses M', m' and form factors at zero equal to $F_1(0)$ and $F_2(0)$ respectively. In such a case it is convenient to carry out the partial wave expansion and consider just the S wave in the c.m. system, where $\mathbf{p} = -\mathbf{k}$. Both the S matrix and the dynamic moments will be diagonal in the single continuous variable—the energy, so that the system of Eqs. (8) becomes purely algebraic. It is sufficient to confine oneself to the consideration of the scalar dynamic moments D, D_{ii}, $D_{ii\,jj}\ldots$, whose form in the elastic channel at minus infinity is

$$D = D^{(1)} + D^{(2)} \equiv f_1(0)/2E + f_2(0)/2\omega,$$
$$D_{ii} = p^2\{f_1(0)/2E^3 + f_2(0)/2\omega^3\}, \ldots. \quad (10)$$

Analogous expressions for the matrix elements of D', D_{ii}, $D_{ii\,jj}$ in the inelastic channel are obtained by the substitutions

$$M \to M', \quad m \to m', \quad f_1(0) \to F_1(0);$$
$$f_2(0) \to F_2(0).$$

[1)]As shown by Lonskii and the author[4], the matrix element $\langle 2|A(x)|2\rangle$ also contains information about the S-wave elastic scattering phase shift.

These expressions must be substituted in the right side of Eq. (8). In the left side must be substituted expressions (10) with the addition of appropriate $D^{int}_{(+\infty)}$, considered as known for the elastic channel. Now, denoting by S_{ee} and S_{ie} the elements of the scattering matrix referring respectively to the elastic and inelastic channels, the desired equations from the system (8) for each of the scalar dynamic moments of the type (10) can be rewritten in the form

$$D^{(1)}_{\dots} + D^{(2)}_{\dots} + D^{int}_{\dots} = (1 - |S_{ie}|^2)(D^{(1)}_{\dots} + D^{(2)}_{\dots}) + |S_{ie}|^2 (D'^{(1)}_{\dots} + D'^{(2)}_{\dots}). \quad (11)$$

The unitarity relation $|S_{ee}|^2 + |S_{ie}|^2 = 1$ has been taken into account in (11).

If the matrix element (9) is considered known, then the unknowns in the system (11) consist of M', m', $F_1(0)$, $F_2(0)$ and $|S_{ie}|^2$, i.e., the cross section for inelastic scattering. Therefore if we write equations (11) for five scalar dynamic moments, for example for D, D_{ii}, D_{iijj}, D_{iijjkk}, $D_{iijjkkll}$, we will have obtained a system of equations sufficient for the determination of all the above enumerated unknown quantities. Substitution of all these quantities into Eq. (11) for any one of the subsequent scalar dynamic moments should, apparently, give rise to an identity. If an identity is not obtained then this must mean that an additional inelastic channel is open or that the existing inelastic channel is more complicated, for example it consists of three particles. In such a case one needs to write out a larger number of equations in order to obtain information about the inelastic channels.

Let us discuss now the possible reasons as a consequence of which the information about the final states and the transition amplitudes may turn out to be incomplete. In the first place this information may turn out to be incomplete if two or more of the particles in the final state possess vanishing form factors at zero. The origin of this nonuniqueness is obvious: it is not possible to keep track of the particles with the help of a field that they do not possess. It is precisely for this reason that the energy-momentum tensor $T_{\mu\nu}(x)$ occupies a special place among all local quantities, because its one-particle matrix elements are different from zero for all particles without exception. A second important property of the operator $T_{\mu\nu}(x)$ is that its existence is a direct consequence of very general assumptions about the structure of space-time.[1]

An exact solution of the system (8) may become complicated if an infinite number of particles is present in the final state. This difficulty is characteristic of "infrared catastrophes" and, presumably, may be overcome by techniques such as the Bloch-Nordsieck method. Without stopping to discuss this essentially technical problem we only note that complete information about the particles with nonzero mass may be obtained also in the presence of an infinite number of particles with zero mass. To that end it is necessary to introduce in the definitions (2) everywhere under the integral sign the additional factor $1 - (1/2)\mathbf{x}^2(\partial/\partial t)^2$ which will multiply every one-particle dynamic moment (3) by $1 - \mathbf{v}^2$, which vanishes for particles with zero mass; consequently in the corresponding equations of type (8) there will remain only the sum over the always finite number of particles with nonzero mass.

To conclude this section we emphasize the fact that a connection has been established here between the scattering matrix,and the matrix elements of local operators off the mass shell.

3. DYNAMIC MOMENTS OF HIGHER RANK

The aggregate of dynamic moments introduced by relations (2) depends, according to (4), on the values of the form factors at zero $f_N(0)$ of the free particles. Corresponding to this circumstance one can determine from a given matrix element $\langle 2|A(x)|2\rangle$ with the help of (8) the values $f_N(0)$ for all particles produced in the inelastic channels. The question naturally suggests itself whether it might not be possible to generalize the method of dynamic moments so as to utilize not only the form factors themselves but also their derivatives at zero, i.e., in the final analysis the complete one-particle form factor.

Such a generalization is indeed possible, based on the idea expressed in the previous article of the author.[1] We generalize the definition (2) by introducing the quantities $\widetilde{D}^m_{i_1\dots i_n}(t)$ (for the sake of brevity we shall also use the notation $\widetilde{D}^m_n(t)$):

$$\widetilde{D}^m_{i_1\dots i_n}(t) = \frac{1}{n!}\int d^3x x_{i_1}\dots x_{i_n}\left(\frac{\partial}{\partial t}\right)^{n-m} A(x) \quad (12)$$

which coincide with the $D_{i_1\dots i_n}(t)$ for $m = 0$. From the definition (12) it follows that

$$\widetilde{D}^0_n(t) = (\partial/\partial t)^m \widetilde{D}^m_n(t). \quad (13)$$

The one-particle matrix elements of the operator (13) $\widetilde{D}^0_n \equiv D_n$ are, according to (3), independent of time and therefore the one-particle matrix elements of the operators $\widetilde{D}^m_n$ are polynomials in t of degree no higher than m. If we denote by D^m_n

the time independent part of $\tilde{D}_n^m$, then one finds easily from the definition (12) that the entire polynomial will be of the form

$$\tilde{D}_n^m(t) = D_n^m + tD_n^{m-1} + \frac{t^2}{2!}D_n^{m-2} + \ldots + \frac{t^m}{m!}D_n^0. \quad (14)$$

Inverting the relations (14), for which it is sufficient to change the sign of t, one can express the operators D_n^m in terms of the $\tilde{D}_n^m$:

$$D_n^m = \sum_{l=0}^{m} \frac{(-1)^l t^l}{l!} \tilde{D}_n^{m-l}. \quad (15)$$

Let us accept now (15) as definition of the operators D_n^m meaningful for arbitrary, and not just one-particle, matrix elements. We shall refer to these operators as dynamic moments of m-th rank. The many-particle matrix elements of the operators D_n^m will in general depend on t, but will asymptotically turn into time independent operators as $t \to \pm\infty$, when all particles become free. Therefore equations of type (8) are valid for dynamic moments of higher rank:

$$D_n^m(+\infty) = S^{-1} D_n^m(-\infty) S. \quad (16)$$

In application to the given matrix element $\langle 2 | A(x) | 2 \rangle$, Eqs. (16) make it possible to obtain not only cross sections, masses and form factors at zero for elastic and inelastic channels, but also all derivatives of the form factors at zero, i.e., the one-particle form factors themselves for all particles in all channels. The latter can be seen from the fact that the dynamic moment of rank 2k contains the k-th derivative of the form factor at zero. For clarification we write out the simplest one-particle dynamic moments of first and second rank. Taking the one-particle matrix element of (15) and making use of (1) we find after some simple steps for n = 1, m = 1 and n = 2, m = 1, 2:

$$\langle \mathbf{p} | D_i^1 | \mathbf{p}' \rangle = \int d^3x x_i \left(1 - t\frac{\partial}{\partial t}\right) \langle \mathbf{p} | A(x) | \mathbf{p} \rangle$$

$$= \frac{f(0)}{\sqrt{4EE'}} i \frac{\partial}{\partial p_i} \delta(\mathbf{p} - \mathbf{p}'), \quad (17)$$

$$\langle \mathbf{p} | D_{ij}^1 | \mathbf{p}' \rangle = \frac{f(0)}{2\sqrt{4EE'}} \left(v_i' \frac{\partial}{\partial p_j} + v_j' \frac{\partial}{\partial p_i} \right) \delta(\mathbf{p} - \mathbf{p}'), \quad (18)$$

$$\langle \mathbf{p} | D_{ij}^2 | \mathbf{p}' \rangle = \frac{\delta(\mathbf{p} - \mathbf{p}')}{2E} f'(0) (\delta_{ij} - v_i v_j)$$

$$- \frac{f(0)}{2\sqrt{4EE'}} \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} \delta(\mathbf{p} - \mathbf{p}'), \quad (19)$$

where $f'(0) = \partial f(0)/\partial s$. It can be seen from (17)—(19) that the one-particle dynamic moments of higher rank, in contrast to those of zero rank, are no longer integrals of the motion. But these moments are time independent, so that the many-particle dynamic moments will tend to definite limits as $t \to \pm\infty$.

The dynamic moments of first rank do not depend on any constants. The simplest of these one-particle moments D_i^1 was first introduced in [4,5] for the purposes of getting the S-wave elastic scattering phase shift from the given matrix element $\langle 2 | A(x) | 2 \rangle$. As was indicated in [5], the operator $\langle p | D_i^1 | p' \rangle$ has the meaning of the particle coordinate at the zero instant of time. In that sense the subtraction procedure (15) corresponds to the subtraction of the product of time by velocity from all the coordinate operators. The one-particle dynamic moment of second rank $\langle p | D_{ij}^2 | p \rangle$ from (19) already contains a new constant $f'(0)$, whose meaning is that of the mean-square radius of the distribution of the field $A(x)$ in the particle. Consequently the substitution of the dynamic moments D_n^2 into the equation (16) indeed makes it possible to determine the quantities $f_N'(0)$ for all the particles produced in the reaction.

4. RELATION TO THE LEHMANN-SYMANZIK-ZIMMERMANN REDUCTION FORMULAS

The relations (16), which are the main result of the two preceding sections, are based on the properties of the one-particle matrix elements $\langle 1 | A(x) | 1 \rangle$, as is clear from (5) and (6). Let us show that one may analogously make use of matrix elements of the type $\langle 1 | A(x) | 0 \rangle$ and $\langle 0 | A(x) | 1 \rangle$ for the purposes of obtaining reduction formulas.

We shall discuss the first of these matrix elements; the procedure is exactly the same for the second one. The element $\langle 1 | A(x) | 0 \rangle$ is parametrized as follows:

$$\langle \mathbf{k} | A(x) | 0 \rangle = e^{-ikx} / (2\pi)^{3/2} \sqrt{2\omega}. \quad (20)$$

Strictly speaking there should appear on the right side certain numerical factor, which, however, can always be put equal to unity by an appropriate choice of normalization of the field itself.

We now wish to obtain from the matrix element (20) the totality of the quantities of the type of the dynamic moments. To this end we multiply (20) by $(2\pi)^{-3/2}(2\omega')^{-1/2}e^{ik'x}$, where $k'^2 = k^2 = -m^2$. As a result we obtain an expression analogous to (1) with a constant and equal to unity form factor. From that expression it is already an easy matter to obtain the quantities of the dynamic moment type. The first of these moments $d(t)$ is of the form

$$d(t) = \int d^3x \frac{e^{ik'x}}{(2\pi)^{3/2}\sqrt{2\omega'}} A(x). \quad (21)$$

Let us call $d(t)$ the dynamic moment with exponential. For the one-particle matrix element (20) the relation (21) becomes, in full analogy to (3),

$$\langle \mathbf{k} \mid d(t) \mid 0\rangle = \tfrac{1}{2}\omega^{-1}\delta(\mathbf{k}-\mathbf{k}'). \qquad (22)$$

From (22) one obtains reduction formulas in just the same way as formulas (8) were obtained from (3). As an example let us discuss the simplest nontrivial example, the matrix element $\langle \mathbf{pk} \mid A(x) \mid \mathbf{p}\rangle$. In full analogy to (9) the invariant parametrization of this matrix element has the form

$$\langle \mathbf{pk} \mid A(x) \mid \mathbf{p}'\rangle = \delta(\mathbf{p}-\mathbf{p}')\langle \mathbf{k} \mid A(x) \mid 0\rangle + \frac{\exp[-ix(p+k-p')]g_{int}(s_1, s_3, s_6)}{(2\pi)^{3/2}\sqrt{8EE'\omega}}. \qquad (23)$$

From the condition of separability of the particles at infinity in time, and from the fact that the matrix element (23) is taken in the in-basis, it follows that the dynamic moment (21) of the second term on the right side of (23) should tend to zero as $t \to -\infty$ and to a definite limit as $t \to +\infty$.

From here it follows that the form factor F_{int} should have a pole structure of the following form

$$g_{int} = \frac{-T^+(s_1, s_3, s_6)}{2\pi\{(p+k-p')^2+m^2+i\varepsilon\}}, \qquad (24)$$

where the function T is regular at the point $(p+k-p')^2 = -m^2$. Applying now to (23) the operation (21) and letting in turn $t \to -\infty$ and $t \to +\infty$ we obtain the expression (having taken into account the fact that $\langle \mathrm{in} \mid d(-\infty) \mid \mathrm{in}\rangle = \langle \mathrm{out} \mid d(+\infty) \mid \mathrm{out}\rangle$)

$$\langle \mathbf{pk} \mid S^{-1} \mid \mathbf{p}'\mathbf{k}'\rangle = \delta(\mathbf{p}-\mathbf{p}')\,\delta(\mathbf{k}-\mathbf{k}') + i\,\frac{\delta^4(p+k-p'-k')}{\sqrt{16EE'\omega\omega'}}\,T^+(s_1, s_3), \qquad (25)$$

from which it is clear that the hermitian adjoint quantity T^+ coincides with the invariant scattering amplitude.

In the derivation of (25) we made use of the relation

$$\lim e^{i\omega t}(\omega - i\varepsilon)^{-1} = \begin{cases} 2\pi i\delta(\omega) & \text{as } t \to +\infty \\ 0 & \text{as } t \to -\infty \end{cases}. \qquad (26)$$

It is clear that the reduction formulas obtained by the substitution of (24) into (23) coincide with the corresponding relations of Lehmann, Symanzik, and Zimmermann.[2] In particular the formula (24) expresses the standard connection between the field and the current. By the same token the reduction formulas obtained in [2] are a special case of the method of dynamic moments.

In view of the fact that (21) is analogous to the first of the relations (2) it would seem that one could continue the analogy and introduce dynamic moments with exponential of the type $d_i(t)$ etc., by introducing in (21) in the integrand from the left factors of the type $x_i\partial/\partial t$ etc. In fact, however, the reduction formulas obtained in this way turn out to be identities for the moment $d(t)$, i.e., they contain no new information; in just the same way it makes no sense to introduce moments d_n^m of higher rank.

5. RELATION TO ANALYTIC PROPERTIES

The connection between the matrix elements of the S matrix and local operators established in the preceding sections opens up new possibilities for the exact, i.e., not perturbation theoretic, study of the analytic properties of various matrix elements. The basis for obtaining these properties lies in the physical assumption that the local quantities tend to zero sufficiently rapidly far away from the particles. Therefore the space-time distribution of an arbitrary local quantity for the process of two-particle collision has the following form: at minus infinity in time there are two cylinders with a smeared out boundary. At some finite instant of time the two cylinders unite forming a cloud bounded in space and time. Thereafter the cloud again comes apart into two or more cylinders with smeared out boundary corresponding to a given reaction channel.

These properties of the space-time distribution give rise in a natural way to definite analytic properties of the Fourier transforms. The relations (8) and (16) reflect analytic properties of this type, connected to the properties of the space-time distribution at infinity in time. For example according to (6) and (7) it follows from relation (8) for the first of the dynamic moments (2) that the form factor f_{int} should have a pole of the type $(s_1 - s_2 - i\epsilon)^{-1}$, starting with those values of the variables s_1 and s_2 for which at least one inelastic channel is open. Passage to the equations (8) for the remaining dynamic moments (6) will uncover more and more analytic properties of the form factor f_{int} from (9).

It is to be expected, and simplest trial calculations confirm this expectation, that all these singularities are present also in perturbation theory. The picture changes, however, as one goes over to the equations (16) for the dynamic moments of higher rank (15). The dynamic moments of higher rank correspond to the taking into account of the spatial extension of the particles participating in the reaction. It is therefore to be expected that the corresponding analytic singularities of the form factors will have no analogues in perturbation theory. The study of the analytic properties of the matrix element is made very difficult by the fact that the corresponding form factor depends on six invariant variables. For this reason it is simplest to confine oneself to the study of the case when there appear in the matrix element only S states as far as the relative motion in the center of mass system is concerned. In that case the interaction form factor depends on only three invariant variables.

6. DISCUSSION OF RESULTS

It is to be hoped that the substantial broadening of the class of formulas of the reduction type obtained in the third and fourth sections will make it possible to obtain a number of new nontrivial relations between the cross sections of various reactions, similar to the way in which it becomes possible on the basis of the conventional reduction formulas (25) to prove certain dispersion relations[6] and derive a number of other rigorous results. Even more interesting is the problem of obtaining a maximally complete set of relations between the matrix elements of local operators and the S matrix. The solution of this problem would make it possible to establish to how rigid an extent are local quantities defined through experimentally observable quantities. In essence we have here the mathematical formulation of the problem on the measurability of fields and other local quantities.[7] It might happen that the system of equations imposed on the local matrix element will turn out to be contradictory. This would indicate that there is a limit to the applicability of the concept of the corresponding local quantity. In application to the operator $T_{\mu\nu}(x)$ representing the energy-momentum tensor such a contradiction would be, according to [1], an indication of a change in the structure of space-time.

In practice it is very important that the matrix element under investigation be as simple as possible, i.e., that it depend on a minimal number of invariant variables. For this reason it would be very desirable to generalize the technique developed in this paper to the matrix elements of the type $\langle 2 | A(x) | 0 \rangle$, which depend on but one invariant variable.

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