DAMPING OF A NONLINEAR LONGITUDINAL MONOCHROMATIC WAVE IN A WEAKLY IONIZED PLASMA

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We consider the damping of a nonlinear longitudinal monochromatic wave due to collisions between electrons and neutrals. The damping factor and the distribution functions for the resonance particles are computed.

INTRODUCTION

AN effective method of analyzing the dynamics of low-amplitude plasma waves is the quasilinear theory developed by Vedenov, Velikhov, and Sagdeev. [1,2] This theory describes damping or growth of oscillations with account of the feedback effect of the wave on the averaged velocity distribution of the particles, which is assumed to be uniform in space. Because of this assumption only rather wide wave packets can be analyzed. A plane longitudinal plasma wave does not satisfy the condition of spatial uniformity and thus cannot be analyzed by the quasilinear theory.

For this reason it is desirable to carry out a specific analysis of the dynamics of a monochromatic longitudinal wave. At low amplitudes, even if the dispersion is nonlinear, such a wave can propagate in a plasma almost without distortion due to nonlinearity. [1] It is well known that the damping of a wave of this kind is due primarily to its interaction with resonance particles, that is to say, electrons and ions whose projected velocity in the direction of the wave vector is approximately the same as the phase velocity of the wave. However, in the absence of collisions the particle-wave interaction quickly establishes a distribution of resonance particles such that the energy exchange between particles and wave is terminated and the wave propagates in the plasma without absorption. [3] Collisions distort this distribution and lead to damping. The damping factor will therefore depend on frequency and on the collision mechanism.

In contrast with the case of wave packets, for monochromatic waves it is necessary to consider two groups of resonance particles: the trapped particles, whose motion in the wave reference system remains finite, and the untrapped particles, whose motion is infinite. The distribution functions are usually different for the trapped and untrapped particles and the contribution of each group to wave damping must be treated separately.

The effect of collisions on damping of a longitudinal nonlinear monochromatic wave was first treated by Platzman and Buchsbaum. [4] In this work the authors used the usual expression for the imaginary part of the frequency ω, obtained from the linear theory, but substituted in it the derivative of the distribution function for the resonance particles obtained by solving the nonlinear kinetic equation with a simulated collision integral. It will be shown in the present work that this approach gives an incorrect result at low collision frequencies.

Zakharov and Karpman [5] investigated damping of a monochromatic wave in a highly ionized plasma using the Coulomb integral. However, the distribution of trapped particles was treated improperly in this work because the authors assumed that it was symmetric in velocity, an assumption that in general can only be justified in the absence of collisions.

In the present work, using the same method as in [5] we consider damping of a longitudinal monochromatic wave in a weakly ionized plasma; because of the simple form of the collision integral that is used the problem can be investigated quite completely.

1. DISTRIBUTION OF RESONANCE PARTICLES

We consider a longitudinal plasma wave with phase velocity much greater than the thermal velocity of the ions. In this case the number of resonance ions is exponentially small and the wave damping is determined by the interaction with resonance electrons. [6]

The electron distribution function $f(\mathbf{r}, \mathbf{v})$ is given by the kinetic equation

$$\frac{\partial f}{\partial t} + v_z\frac{\partial f}{\partial z} + \frac{e}{m}\frac{\partial \varphi}{\partial z}\frac{\partial f}{\partial v_z} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \tag{1}$$

(the wave vector is in the direction of the z-axis).

In a weakly ionized plasma elastic collisions between electrons and neutrals represent the dominant collisional effect. Since the neutral velocity is much smaller than the electron velocity the neutrals can be regarded as fixed. The cross-section for scattering of an electron on an atom, q, will be assumed to be isotropic and independent of v. With these assumptions the collision integral becomes

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = 4\pi q n_0 v\left[\frac{1}{4\pi}\int f(\mathbf{v}')\,d\Omega - f(\mathbf{v})\right], \tag{2}$$

where n_0 is the density of atoms, v is the electron velocity after collision, v′ is the electron velocity before collision and dΩ is the element of solid angle. In the collision model we have adopted the absolute magnitude of the electron velocity is conserved after the collision and there is an equal probability for scattering into any angle.

At low amplitudes (small values of φ_0) the effect of the wave on the motion of nonresonance particles is inconsequential and the distribution of these particles is essentially Maxwellian. If $v_{ph} \gg \sqrt{e\varphi_0/\Theta}$ the resonance particles occupy a small region in velocity space. This allows us to simplify the collision integral in the kinetic equation for the resonance particles: we assume that before the collisions these particles are nonresonant and characterized by a Maxwellian distribution. The equation is then written in the form

$$\frac{\partial f}{\partial t} + v_z\frac{\partial f}{\partial z} + \frac{e}{m}\frac{\partial \varphi}{\partial z}\frac{\partial f}{\partial v_z} = 4\pi q n_0 v\,[f_0(\mathbf{v}) - f(\mathbf{v})], \tag{3}$$

$$f_0(v) = n\,(m/2\pi\Theta)^{3/2}\exp[-mv^2/2\Theta]. \tag{4}$$

We integrate Eq. (3) over v_x and v_y having replaced v by v_{ph} as an approximation in the collision integral. Everywhere below we shall understand f to mean the integrated distribution function. Thus

$$\frac{\partial f}{\partial t} + v_z\frac{\partial f}{\partial z} + \frac{e}{m}\frac{\partial \varphi}{\partial z}\frac{\partial f}{\partial v_z} = \frac{f_0(v_z) - f}{\tau},$$

$$\tau \sim 1/4\pi q n_0 v_{\text{ph}}, \qquad f_0(v_z) = n\,(m/2\pi\Theta)^{1/2}\exp(-mv_z^2/2\Theta). \tag{5}$$

In the form in which it appears in Eq. (5) the collision integral has been used many times as the simplest model for solving kinetic problems. However, it should be emphasized that for resonance particles whose velocities are appreciably greater than the thermal velocities, this is not a simulated collision integral; it has rather been obtained from the exact collision integral as a result of certain justifiable simplifications.

Introducing $u = v_z - v_{ph}$ we write Eq. (5) in the wave system and convert to the variable $\epsilon = mu^2/2 - e\varphi$. Then

$$\frac{\partial f^\pm}{\partial z} = \mp\frac{1}{\tau\sqrt{2(\varepsilon + e\varphi)/m}}\left[f^\pm(\varepsilon, z) - f_0\left(v_\Phi \pm \sqrt{\frac{2}{m}(\varepsilon + e\varphi)}\right)\right]. \tag{6}$$

Assuming that the damping is small we neglect the partial derivative ∂f/∂t [cf. Eq. (26)]. The upper and lower signs refer respectively to electrons for which u > 0 and u < 0. We introduce the dimensionless coordinate y = kz, where k is the wave number, and use the notation

$$\frac{\varepsilon}{\Theta} = \mathscr{E}, \quad \frac{e\varphi}{\Theta} = \Phi, \quad \frac{v_\Phi}{\sqrt{2\Theta/m}} = \alpha, \quad \frac{1}{\tau k\sqrt{2\Theta/m}} = \nu. \tag{7}$$

In this notation the kinetic equation becomes

$$\frac{\partial f^\pm}{\partial y} = \mp\frac{\nu}{\sqrt{\mathscr{E} + \Phi}}\,[f^\pm(\mathscr{E}, y) - f_0(\alpha \pm \sqrt{\mathscr{E} + \Phi})]. \tag{8}$$

Below we shall consider the case of "strong" waves:[5]

$$\nu \ll \sqrt{e\varphi_0/\Theta}, \tag{9}$$

in which the distribution of particles in the resonance region is determined by their interaction with the wave. In this case the solution of Eq. (8) can be written as an expansion in powers of ν:[5,8]

$$f^\pm = f^{\pm(0)}(\mathscr{E}) + \nu f^{\pm(1)}(\mathscr{E}, y) + \ldots . \tag{10}$$

Substituting Eq. (10) in Eq. (8), we have to a first approximation

$$\frac{\partial f^{+(1)}}{\partial y} = -\frac{1}{\sqrt{\mathscr{E} + \Phi}}\,[f^{+(0)}(\mathscr{E}) - f_0(\alpha + \sqrt{\mathscr{E} + \Phi})], \tag{11}$$

$$\frac{\partial f^{-(1)}}{\partial y} = \frac{1}{\sqrt{\mathscr{E} + \Phi}}\,[f^{-(0)}(\mathscr{E}) - f_0(\alpha - \sqrt{\mathscr{E} + \Phi})]. \tag{12}$$

In solving the equations we must treat trapped particles and untrapped particles separately.

For the untrapped particles we have $f_u^\pm(\pi) = f_u^\pm(-\pi)$. Integrating with respect to y from $-\pi$ to π and assuming that $f^{\pm(0)}(\mathscr{E})$ is independent of y, in the zeroth approximation we have

$$f_u^{\pm(0)}(\mathscr{E}) = \frac{\int_{-\pi}^{\pi}\frac{f_0(\alpha \pm \sqrt{\mathscr{E} + \Phi})}{\sqrt{\mathscr{E} + \Phi}}\,dy}{\int_{-\pi}^{\pi}\frac{dy}{\sqrt{\mathscr{E} + \Phi}}}. \tag{13}$$

For the trapped particles, at the turning points $y_{1,2}$: $f_t^+(y_{1,2}) = f_t^-(y_{1,2})$. Thus

$$\int_{y_1}^{y_2}\frac{\partial f_t^{+(1)}}{\partial y}\,\partial y + \int_{y_2}^{y_1}\frac{\partial f_t^{-(1)}}{\partial y}\,dy = 0,$$

and furthermore $f_t^{+(0)} = f_t^{-(0)} = f_t^{(0)}$.[1)]

Thus, integrating Eqs. (11) and (12) and adding we have

[1)]In computing the distribution of trapped particles the relation $f_t^-(\mathscr{E}, y) = f_t^+(\mathscr{E}, y)$ was used in [5]. When collisions are taken into account this relation does not hold in general.

$$f_{\mathrm{t}}^{(0)} = \int_{y_1}^{y_2} \frac{1}{2\sqrt{\mathscr{E}+\Phi}} \times [f_0(\alpha + \sqrt{\mathscr{E}+\Phi}) - f_0(\alpha - \sqrt{\mathscr{E}+\Phi})] \cdot dy \Bigg/ \int_{y_1}^{y_2} \frac{dy}{\sqrt{\mathscr{E}+\Phi}}, \tag{14}$$

where $y_{1,2}$ is determined from the condition $\mathscr{E} + \Phi(y_{1,2}) = 0$.

For a sinusoidal wave $E = E_0 \sin y$ (correspondingly $\Phi = \Phi_0 \sin^2(y/2)$, where $\Phi_0 = e\varphi_0/\Theta$) substituting Eq. (3) in Eqs. (13) and (14) we find that

a) for the untrapped particles

$$f_{\mathrm{u}}^{\pm(0)} = \frac{A}{K(k)} \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2 x}} \times \exp\{-(\alpha \pm \sqrt{\mathscr{E}(1-k^2\sin^2 x)})^2\}\, dx, \tag{15}$$

where $k = \sqrt{\Phi_0/\mathscr{E}}$; $K(k)$ is a complete elliptic integral of the first kind, $A = n(m/2\pi\Theta)^{1/2}$.

For a low-amplitude wave ($\alpha\Phi_0 \ll 1$) we find

$$f_{\mathrm{u}}^{\pm(0)}(\mathscr{E}) = A \exp\{-(\alpha \pm \sqrt{\mathscr{E}})^2\}\Big[1 \pm \sqrt{\mathscr{E}}(\alpha \pm \sqrt{\mathscr{E}}) \times \Big(1 - E\Big(\sqrt{\tfrac{\Phi_0}{\mathscr{E}}}\Big) \Big/ K\Big(\sqrt{\tfrac{\Phi_0}{\mathscr{E}}}\Big)\Big)\Big], \tag{16}$$

where $E(k)$ is a complete elliptic integral of the second kind.

b) For the trapped particles

$$f_{\mathrm{t}}^{(0)}(\mathscr{E}) = A\Big[2\int_0^{\arcsin(1/k)} \frac{dx}{\sqrt{1-k^2\sin^2 x}}\Big]^{-1} \int_0^{\arcsin(1/k)} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \times \{\exp[-(\alpha + \sqrt{\mathscr{E}(1-k^2\sin^2 x)})^2] + \exp[-(\alpha - \sqrt{\mathscr{E}(1-k^2\sin^2 x)})^2]\}. \tag{17}$$

When $\alpha\Phi_0 \ll 1$:

$$f_{\mathrm{t}}^{(0)}(\mathscr{E}) = A\exp(-\alpha^2) \times \left\{1 + (2\alpha^2 - 1)\frac{\int_0^{\arcsin(1/k)} \sqrt{\mathscr{E} - \Phi_0\sin^2 x}\, dx}{\int_0^{\arcsin(1/k)} \frac{dx}{\sqrt{\mathscr{E}-\Phi_0\sin^2 x}}}\right\}. \tag{18}$$

2. WAVE DAMPING

In order to compute the damping we find $\overline{\dot{W}}$, the mean value of the time derivative of the energy density of the wave. On the basis of considerations similar to those given in [5] we have

$$\overline{\dot{W}} = \frac{\omega\Theta^{3/2}}{\pi\sqrt{8m}} \int_{-\pi}^{\pi} dy \int_{-\Phi}^{\infty} \frac{d\mathscr{E}}{\sqrt{\mathscr{E}+\Phi}} \frac{d\Phi}{dy}(f^+ + f^-). \tag{19}$$

We write $\overline{\dot{W}} = \overline{\dot{W}}_{\mathrm{u}} + \overline{\dot{W}}_{\mathrm{t}}$, i.e., the total effect is the sum of the contributions of trapped and untrapped particles. We first consider the untrapped particles

$$\overline{\dot{W}}_{\mathrm{u}} = \frac{\omega\Theta^{3/2}}{\pi\sqrt{8m}} \int_{-\pi}^{\pi} dy \int_{\Phi_0}^{\infty} \frac{d\mathscr{E}}{\sqrt{\mathscr{E}+\Phi}} \frac{d\Phi}{dy}(f^+ + f^-).$$

Changing the order of integration, we integrate by parts taking account of the periodicity of the distribution function:

$$\overline{\dot{W}}_{\mathrm{u}} = -\frac{\omega\Theta^{3/2}}{\pi\sqrt{2m}} \int_{\Phi_0}^{\infty} d\mathscr{E} \int_{-\pi}^{\pi} \sqrt{\mathscr{E}+\Phi}\, \frac{d}{dy}(f^+ + f^-)\, dy.$$

Substituting df/dy from Eqs. (11) and (12) we have

$$\overline{\dot{W}}_{\mathrm{u}} = -\nu\frac{\omega\Theta^{3/2}}{\pi\sqrt{2m}} \int_{\Phi_0}^{\infty} d\mathscr{E} \int_{-\pi}^{\pi} \{[f^{+(0)}(\mathscr{E}) - f^{-(0)}(\mathscr{E})] - [f_0(\alpha + \sqrt{\mathscr{E}+\Phi}) - f_0(\alpha - \sqrt{\mathscr{E}+\Phi})]\}\, dy. \tag{20}$$

By similar transformations we find the damping due to the trapped particles:

$$\overline{\dot{W}}_{\mathrm{t}} = \frac{\omega\Theta^{3/2}}{\pi\sqrt{8m}} \int_{-\pi}^{\pi} dy \int_{-\Phi}^{\Phi_0} \frac{d\mathscr{E}}{\sqrt{\mathscr{E}+\Phi}} \frac{d\Phi}{dy}(f^+ + f^-) = -\frac{\omega\Theta^{3/2}}{\pi\sqrt{2m}} \int_0^{\Phi_0} d\mathscr{E} \times \int_{-2\arcsin(1/k)}^{2\arcsin(1/k)} [f_0(\alpha - \sqrt{\mathscr{E}+\Phi}) - f_0(\alpha + \sqrt{\mathscr{E}+\Phi})]\, dy. \tag{21}$$

We shall limit ourselves to the case of a low-amplitude wave that satisfies the condition $\alpha\Phi_0 \ll 1$. Substituting Eqs. (3) and (15) in Eqs. (20) and (21), expanding in powers of the small parameters Φ_0 and $\alpha\Phi_0$, and carrying out the integration and other calculations (not presented here) we find

$$\overline{\dot{W}}_{\mathrm{u}} = 0.2\, A\omega\Theta^{3/2}\Phi_0^{3/2} m^{-1/2}\nu\alpha e^{-\alpha^2},$$
$$\overline{\dot{W}}_{\mathrm{t}} = 1.86\, A\omega\Theta^{3/2}\Phi_0^{3/2} m^{-1/2}\nu\alpha e^{-\alpha^2}. \tag{22}$$

Thus, expressing A in terms of n and Θ and adding we have

$$\overline{\dot{W}} = 0.83 n\omega\Theta\nu\Phi_0^{3/2}\alpha e^{-\alpha^2}. \tag{23}$$

We now find the damping factor for the wave $\gamma = \overline{\dot{W}}/2\overline{W}$ ($\overline{W}$ is obviously $E_0^2/16\pi$):

$$\gamma = \frac{3.3}{\tau}\left(\frac{e\varphi_0}{\Theta}\right)^{-1/2}\left(\frac{v_{\mathrm{ph}}}{v_T}\right)^4 \exp\left[-\left(\frac{v_{\mathrm{ph}}}{2v_T}\right)^2\right], \tag{24}$$

where $v_T = \sqrt{\Theta/m}$. In contrast with a highly ionized plasma, where $\gamma \sim \varphi_0^{-3/2}$ because of Coulomb collisions,[5] here γ depends on the wave amplitude as $\varphi_0^{-1/2}$.

We compare this result with the damping factor in the linear theory

$$\gamma_L = \sqrt{\frac{\pi}{8}}\,\omega_0 \left(\frac{v_{ph}}{v_T}\right)^3 \exp\left[-\left(\frac{v_{ph}}{2v_T}\right)^2\right]. \qquad (25)$$

Taking account of Eq. (9) we find

$$\frac{\gamma}{\gamma_L} = 5{,}2\,\frac{v_{ph}}{\omega_0 \tau v_T}\left(\frac{\Theta}{e\varphi_0}\right)^{1/2} \ll 1, \qquad (26)$$

that is to say, a nonlinear monochromatic wave damps much more slowly than a wave described by the linear Landau theory.

In simplifying the kinetic equation we have neglected the partial derivative of the distribution function with respect to time $\partial f/\partial t$ [cf. Eq. (6)]. This procedure is valid if $\tau\gamma \ll 1$.[5] Expressing $\tau\gamma$ by means of Eq. (26) we get

$$\tau\gamma \sim \gamma_L / k\sqrt{e\varphi_0/m} \ll 1. \qquad (27)$$

In other words, the wave amplitude must be large enough so that the oscillation period of the electron in the potential well of the wave is much smaller than the damping time for the linear wave.

In addition to the damping considered here, which arises as a consequence of the interaction of resonance particles with the wave, there is also the usual damping, given by $\gamma_0 \sim (1/\tau)(v_T/v_{ph})$; this damping is associated directly with the loss of ordered electron momentum in collisions with electronic atoms [cf. for example [4]].

Taking account of the relation in (27) we obtain the following condition for the applicability of the formulas that have been obtained:

$$v_T/v_{ph} \ll \gamma_L / k\sqrt{e\varphi_0/m} \ll 1. \qquad (28)$$

In conclusion we wish to note that Platzman and Buchsbaum [4] have considered essentially the same problem but have obtained a different result because of a number of errors. In particular, there is some doubt as to the applicability of the dispersion equation obtained in the linear approximation in an analysis of damping with few collisions inasmuch as the distribution function for the trapped particles is not analytic, as is evident from Eq. (18) (cf. also [3]).

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