INFINITE TIME FORMALISM IN QUANTUM FIELD THEORY

Yu. D. USACHEV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor January 23, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 187-195 (January, 1964)

A new type of infinite time formalism (i.t.f.) is proposed which contains all three representations, including the Schrödinger representation. The complete equivalence of the three representations is demonstrated. In the new i.t.f. the twofold time dependence of the field operators is avoided. The equations of motion of the field operators in the i.t.f. are derived. The connection with the usual S matrix is discussed.

1. INTRODUCTION

GENERALLY speaking, there exist at present two different formulations of the infinite time formalism (i.t.f.) in quantum field theory: that of Tomonaga[1] (T) and that of Schwinger[2] (Schw.). The difference between the two is easily seen in the analysis of the fundamental T and Schw. equations for the state vector F and for the field operators φ written in the interaction and Heisenberg representations with the help of the i.t.f. The basic representations in the two formulations are different: in T it is the interaction representation, while in Schw. it is the Heisenberg representation. As a result, the equations for the field operators are different in T and Schw. (as written with the help of the i.t.f.) for the two representations. In the Heisenberg representation of the Schw. formulation there are no consistency conditions, since the equations for F and φ are in this case trivial in the i.t.f. However, in the interaction representation one can obtain consistency conditions for both these equations.

In other words, the interaction and Heisenberg representations in the Schw. formulation are not completely mathematically equivalent in the sense that there is no unitary connection between the representations for the consistency condition, although it exists for all other relations. In the T formulation the situation is different. Here all relations, including the consistency conditions, in the interaction and Heisenberg representations are related by a unitary transformation, as they should be.

This difference between the two formulations does not, of course, affect the numerical values of the physical quantities calculated in the two formalisms. Also, both formulations conserve the correspondence with the single time formalism (s.t.f.) in the equations for the state vector and the field operators in both representations. However, the above-mentioned difference may be of importance in the analysis of different types of nonlocal field theories.

Both the T and Schw. formulations of the i.t.f. involve a number of difficulties. Let us discuss these briefly.

a) The Schrödinger representation in the i.t.f. is formulated in neither the papers of T nor of Schw. This omission is not accidental. Attempts to formulate the Schrödinger representation would meet with the difficulties of solving complicated functional equations for the operator quantities. It should be noted in this connection that the published attempts (see, for example, [3]) of formulating the Schrödinger representation in the i.t.f. have, unfortunately, been unsuccessful: the postulated equations do not fulfil the consistency conditions.

b) Schwinger's field operator in the interaction representation $\varphi_S^{\text{int}}[x; \sigma]$ and Tomonaga's field operator in the Heisenberg representation $\varphi_T^H[x; \sigma]$ depend on the time in a twofold way: they are ordinary functions of the fourdimensional space-time coordinate and functionals of the space-like hypersurface σ through this point. The evolution in the time t of these operators is connected with the change of the parameter τ ("curvilinear time") of a one-parameter family of hypersurfaces [$t = t(\mathbf{x}, \tau)$ is the equation for σ solved explicitly for t]. Thus, in particular, the natural transition to curvilinear coordinates (i.e., choosing the hypersurfaces τ = const instead of the hyperplanes t = const) is complicated for the operators $\varphi_S^{\text{int}}[x; \sigma]$ and $\varphi_T^H[x; \sigma]$ on account of their twofold time dependence. This gives rise to even further difficulties.

c) It has already been noted in the literature[4] that the different representations should be completely equivalent—none should have more physical significance than the others. But it was precisely the absence of the consistency conditions in the Heisenberg representation of the Schw. formulation which facilitated the introduction of nonlocal interactions in this representation (see, e.g., [5]), which thus seems to embody more physical possibilities than the interaction representaion, in which this cannot be done owing to the consistency conditions. In the subsequent development of the nonlocal theory [6] arguments have appeared in favor of a violation of the consistency conditions (in their usual form) and causality. However, this does not remedy the above-mentioned defect of the Schw. formulation, in which the Heisenberg and interaction representations are not completely equivalent. Therefore, in the Heisenberg representation, where the solution of the equations of motion involves in general the solution of a Cauchy boundary value problem, one needs either a special prescription for the commutator of Heisenberg operators on space-like σ [7] or consistency conditions of the type used in the T formulation.

All these features of the T and Schw. formulations, which are inessential in the usual local theory, may become of paramount importance in nonlocal field theories.

2. FORMULATION OF THE PROBLEM AND NOTATION

Merely the existence of the two versions (T and Schw.) of the i.t.f. suggests, at least from a methodological point of view, the search for another variant of this very useful tool of quantum field theory which would be free of the above-mentioned defects. We therefore pose ourselves the task of formulating a version of the i.t.f. in which 1) the Schrödinger representation can be written down on an equal footing with the other two representations, 2) the field operators do not have a twofold time dependence, 3) the equations of motion for the field operators can be written in terms of densities—the Hamiltonian density and the variational derivative on the hypersurface σ —on the same basis as the equations for the state vector, 4) the transition to the s.t.f. or to curvilinear coordinates in the equations of motion and for the state vector is effected by a simple integration of the equations in the variational derivatives with respect to the corresponding hypersurfaces, and 5) the three representations are completely equivalent, i.e., the forms of any relation in the three different representations are related by a unitary transformation.

We shall use the following notation: the Schrödinger representation will be assigned the index 1, the interaction representation the index 2, and the Heisenberg representation the index 3. In this connection the symbol V^{l}_{ik} $(i, k, l = 1, 2, 3)$, for example, will denote the unitary transformation which effects the transition from the representation k to the representation i, written in terms of operators in the representation l. The state vectors in all representations will be written as F^i $(i = 1, 2, 3)$ and the field operators as φ^i $(i = 1, 2, 3)$. In general, the upper index i indicates in which of the three representations a given quantity is written.

Furthermore, we enclose the arguments of functionals in square brackets and the arguments of ordinary functions in parentheses. If a functional is also an ordinary function of some variable, both arguments will be enclosed in square brackets, the functional argument standing to the right of the semicolon. For example, $\varphi^3[\mathbf{x}; t(\mathbf{x}, \tau)]$ denotes a Heisenberg operator which is an ordinary function of x and a functional of $t(\mathbf{x}, \tau)$. In the following, we shall for brevity use the notation t_1 for $t(\mathbf{x}_1, \tau)$ in the variational derivatives $[\delta/\delta t_1 \equiv \delta/\delta t(\mathbf{x}_1, \tau)]$ and t instead of $t(\mathbf{x}, \tau)$ in the functional arguments.

3. SCHRÖDINGER REPRESENTATION IN THE I.T.F.

Our first basic assumption is that the Schrödinger operator $\varphi^{\text{Schr}}(\mathbf{x}) \equiv \varphi^1(\mathbf{x})$ does not depend on the hypersurface:

$$\delta\varphi^1(\mathbf{x})/\delta t(\mathbf{x}_0, \tau) = 0 \tag{1}$$

(cf. the postulates $\delta\varphi^2_T(\mathbf{x}, t)/\delta t(\mathbf{x}_0, \tau) = 0$ in [1] and $\delta\varphi^3_S(\mathbf{x}, t)/\delta t(\mathbf{x}_0, t) = 0$ in [2]). Here $t = t(\mathbf{x}, \tau)$ is the equation of a one-parameter family of spacelike hypersurfaces σ which never intersect; τ is a parameter. We assume that (1) holds for arbitrary τ. This is also in accordance with the well-known equation of motion for the Schrödinger operator in the s.t.f.

$$\frac{\partial\varphi^1(\mathbf{x})}{\partial t} = \int\limits_{t=\text{const}} d^3x_0 \frac{\delta\varphi^1(\mathbf{x})}{\delta t(\mathbf{x}_0, \tau)}\Big|_{\sigma\to t=\text{const}} \equiv 0. \tag{2}$$

The integration in (2) goes over the equal time hypersurface t = const; $\sigma \to t = \text{const}$ indicates the straightening-out of the hypersurface.

Let us further introduce the functional $V^1_{13}[t] \equiv V^1_{13}[t(\mathbf{x}, \tau)]$:

$$V^1_{13}[t] = e^{-i\alpha/\hbar},$$

$$\alpha = \int d^3x\, t(\mathbf{x}, \tau)\, n_\mu(\mathbf{x})\, n_\nu(\mathbf{x})\, T^{1,\mu\nu}(\mathbf{x}). \tag{3}$$

Here $n_\mu(\mathbf{x}) \equiv n_\mu(\mathbf{x}, \tau)$ are the covariant components[1] of the unit vector normal to the hypersurface $t = t(\mathbf{x}, \tau)$:

$$n_\mu(\mathbf{x}) = \frac{\delta_{\mu 0} - \partial t(\mathbf{x}, \tau)/\partial x^\mu}{\sqrt{1 - (\partial t/\partial x^k)^2}}; \qquad n_\mu(\mathbf{x})\, n^\mu(\mathbf{x}) = 1; \tag{4}$$

$T^{1,\mu\nu}(\mathbf{x})$ in (1) is the total energy-momentum tensor of the system of interacting fields written in terms of Schrödinger operators.

In straightening out the hypersurface the functional $V^1_{13}[t]$ goes over into the well-known unitary transformation operator which connects the Schrödinger and Heisenberg representations in the s.t.f.,

$$V^1_{13}(t) = V^1_{13}[t]\,|_{\sigma\to t=\text{const}} = \exp\left(-\frac{i}{\hbar}H^1 t\right), \tag{5}$$

where $H^1 = \int d^3x T^{1,00}(\mathbf{x}) = \int d^3x H^1(\mathbf{x})$ is the total Hamiltonian of the system in the s.t.f. and $H^1(\mathbf{x})$ is the density of this Hamiltonian.

For a formulation of the basic equations in the Schrödinger representation of the i.t.f. we shall assume that (3) is a solution of the desired equation for the state vector and establish the form of this equation from the given solution. First of all, we define the generalized Hamiltonian $\mathcal{H}^1[\mathbf{x}; t(\mathbf{x}, \tau)] \equiv \mathcal{H}^1(\mathbf{x})$ for the desired equation in the form

$$\mathcal{H}^1[\mathbf{x}_0; t(\mathbf{x}_0, \tau)] \equiv \mathcal{H}^1(\mathbf{x}_0) = -\frac{\hbar}{i}\frac{\delta V^1_{13}[t(\mathbf{x}, \tau)]}{\delta t(\mathbf{x}_0, \tau)}\{V^1_{13}[t(\mathbf{x}, \tau)]\}^{-1}$$

$$= \frac{\delta\alpha}{\delta t(\mathbf{x}_0, \tau)} + \sum_{n=0}^{\infty}\frac{(-i/\hbar)^{n+1}}{(n+2)!}\Big[\underbrace{\alpha, \alpha, \ldots, \alpha, \frac{\delta\alpha}{\delta t(\mathbf{x}_0, \tau)}}_{n+2}\Big], \tag{6}$$

$$\frac{\delta\alpha}{\delta t(\mathbf{x}, \tau)} = n_\mu(\mathbf{x})\, n_\nu(\mathbf{x})\, T^{1,\mu\nu}(\mathbf{x})$$

$$+ 2\frac{\partial}{\partial x^k}[n_0(\mathbf{x})\, n_\nu(\mathbf{x})\, T^{1,k\nu}(\mathbf{x})\, t(\mathbf{x}, \tau)]$$

$$+ 2\frac{\partial}{\partial x^k}[n_0(\mathbf{x})\, n_k(\mathbf{x})\, n_\mu(\mathbf{x})\, n_\nu(\mathbf{x})\, T^{1,\mu\nu}(\mathbf{x})\, t(\mathbf{x}, \tau)]. \tag{7}$$

The expression $[\underbrace{a, b, c, \ldots, e, f}_{n}]$ in (6) denotes, here and in the following, the $(n-1)$ fold repeated commutator

$$[\underbrace{a, b, c, \ldots, e, f}_{n}] = [a, \underbrace{[b, [c, \ldots [e, f] \ldots]}_{n-1}].$$

With the help of (6), the desired equation for the Schrödinger state vector in the i.t.f. can be written in the form

$$\frac{\hbar}{i}\frac{\delta F^1[t(\mathbf{x}, \tau)]}{\delta t(\mathbf{x}_0, \tau)} = -\mathcal{H}^1(\mathbf{x}_0)\, F^1[t(\mathbf{x}, \tau)] \tag{8}$$

with the additional definitions (in correspondence with the s.t.f.)

$$F^1[t(\mathbf{x}, \tau)] = V^1_{13}[t(\mathbf{x}, \tau)]\, F^3, \tag{9}$$

$$\frac{\hbar}{i}\frac{\delta F^3}{\delta t(\mathbf{x}, \tau)} = 0. \tag{10}$$

It is easy to show the correspondence between the s.t.f. and Eq. (8). For this purpose we deform the hypersurface $t = t(\mathbf{x}, \tau)$ into the equal-time hyperplane and integrate (8) over this plane with account of the structure of formulas (6), (7), and (3):

$$\int d^3x_0 \left\{\frac{\hbar}{i}\frac{\delta F^1[t(\mathbf{x}, \tau)]}{\delta t(\mathbf{x}_0, \tau)} + \mathcal{H}^1(\mathbf{x}_0)\, F^1[t(\mathbf{x}, \tau)]\right\}_{\sigma\to t=\text{const}}$$

$$= \frac{\hbar}{i}\frac{\partial F^1(t)}{\partial t} + H^1 F^1(t) = 0. \tag{11}$$

Thus (2), (11), (9), and (10) bear out the complete correspondence of our formulas with the s.t.f.

The method of obtaining (8) from the given solution (3) may be useless if the chosen solution has some defect. This may be the case, in particular, in working with nonlocal fields. Here the consistency conditions must be verified separately. The satisfaction of these conditions is indeed a criterion of the usefulness of the chosen solution.

Since the Hamiltonian (6) depends explicitly on the hypersurface, the consistency condition has the following form in the Schrödinger representation:

$$\frac{\delta^2 F^1[t]}{\delta t_2 \delta t_1} - \frac{\delta^2 F^1[t]}{\delta t_1 \delta t_2} = \frac{i}{\hbar}\left\{\frac{\delta\mathcal{H}^1(\mathbf{x}_2)}{\delta t_1}\right.$$

$$\left. - \frac{\delta\mathcal{H}^1(\mathbf{x}_1)}{\delta t_2} + \frac{i}{\hbar}[\mathcal{H}^1(\mathbf{x}_1), \mathcal{H}^1(\mathbf{x}_2)]\right\} F^1[t] = 0. \tag{12}$$

Substituting (6) in (12), we find after a simple but lengthy calculation the following consistency condition:

$$\delta^2\alpha / \delta t_1 \delta t_2 = \delta^2\alpha / \delta t_2 \delta t_1, \tag{13}$$

where the form of α has not yet been specified. It can be seen that if α has the form (3) and $T^{1,\mu\nu}(\mathbf{x})$ is a tensor density of local fields, relation (13) is satisfied. In the case of nonlocal fields α may not be a functional of the hypersurface $t = t(\mathbf{x}, \tau)$ and (13) may not be valid.

Using the operator V^1_{13}, we can easily construct the S matrix in the Schrödinger representation. First of all, let us define the operator

$$W^1_{11}[t(\mathbf{x}, \tau_2),\ t(\mathbf{x}, \tau_1)] = V^1_{13}[t(\mathbf{x}, \tau_2)]\,(V^1_{13}[t(\mathbf{x}, \tau_1)])^{-1}, \tag{14}$$

whose action on the Schrödinger state vector $F^1[t(\mathbf{x}, \tau_1)]$ is equivalent to the transformation from a state of the system on the hypersurface $t = t(\mathbf{x}, \tau_1)$ to a state on the hypersurface $t = t(\mathbf{x}, \tau_2)$:

[1] In this paper we use the metric $g_{11} = g_{22} = g_{33} = -g_{00} = -1$, $c = 1$; Greek indices run through the values 0, 1, 2, 3, Latin indices through 1, 2, 3.

$$F^1[t(\mathbf{x},\tau_2)] = W_{11}^1[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)]F^1[t(\mathbf{x},\tau_1)] \equiv V_{13}^1[t(\mathbf{x},\tau_2)]F^3. \tag{15}$$

From this it is clear that

$$S_{11}^1 = W_{11}^1[\infty, -\infty] = \lim_{\substack{\tau_1\to-\infty\\ \tau_2\to+\infty}} W_{11}^1[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)] \tag{16}$$

is the S matrix in the Schrödinger representation.

4. HEISENBERG REPRESENTATION IN THE I.T.F.

The transformation to the Heisenberg representation is accomplished with the help of the operator $(V_{13}^1[t(\mathbf{x},\tau)])^{-1}$ [see (9) and (10)]:

$$F^3 = V_{31}^1[t(\mathbf{x},\tau)]F^1[t(\mathbf{x},\tau)], \qquad (V_{31}^1 = (V_{13}^1)^{-1}), \tag{17}$$

$$\varphi^3[\mathbf{x}; t(\mathbf{x},\tau)] = (V_{13}^1)^{-1}\varphi^1(\mathbf{x})V_{13}^1. \tag{18}$$

Equation (10) is the equation for the Heisenberg state vector in the i.t.f. and from (18) we find the equation of motion for the Heisenberg operator

$$\frac{\hbar}{i}\frac{\delta\varphi^3[\mathbf{x}; t(\mathbf{x},\tau)]}{\delta t(\mathbf{x}_0,\tau)} = \left[\mathcal{H}^3[\mathbf{x}_0; t(\mathbf{x},\tau)], \varphi^3[\mathbf{x}; t(\mathbf{x},\tau)]\right], \tag{19}$$

$$\mathcal{H}^3[\mathbf{x}; t(\mathbf{x},\tau)] = (V_{13}^1[t])^{-1}\mathcal{H}^1(\mathbf{x})V_{13}^1[t]. \tag{20}$$

The consistency conditions can be obtained from (19) alone:

$$\frac{\hbar}{i}\left(\frac{\delta^2\varphi^3[\mathbf{x};t]}{\delta t_1\delta t_2} - \frac{\delta^2\varphi^3[\mathbf{x};t]}{\delta t_2\delta t_1}\right) = \left[\left\{\frac{\delta\mathcal{H}^3[\mathbf{x}_2;t]}{\delta t_1} - \frac{\delta\mathcal{H}^3[\mathbf{x}_1;t]}{\delta t_2} - \frac{i}{\hbar}[\mathcal{H}^3[\mathbf{x}_1;t], \mathcal{H}^3[\mathbf{x}_2;t]]\right\}, \varphi^3[\mathbf{x};t]\right] = 0. \tag{21}$$

It is easily seen that consistency conditions in the Heisenberg representation [the expression in the curly brackets in (21) set equal to zero] and in the Schrödinger representation [see (12)] are connected by the unitary transformation operator $V_{13}^1[t]$.

The transition to the s.t.f. in (19) is made by straightening the hypersurface $t = t(\mathbf{x},\tau)$ into the equal-time hyperplane and integrating over this hyperplane. Using (5), we obtain

$$\frac{\hbar}{i}\frac{\partial\varphi^3(\mathbf{x},t)}{\partial t} = [H^3(t), \varphi^3(\mathbf{x},t)], \tag{22}$$

where $H^3(t)$ is the total Hamiltonian in the Heisenberg representation in the s.t.f.

5. INTERACTION REPRESENTATION IN THE I.T.F.

In order to go from the Schrödinger representation, which is the starting representation in our formulation, over to the interaction representation, we define the unitary operator $V_{12}^1[t(\mathbf{x},\tau)]$, which satisfies the equation

$$\frac{\hbar}{i}\frac{\delta V_{12}^1[t(\mathbf{x},\tau)]}{\delta t(\mathbf{x}_0,\tau)} = -\mathcal{H}_0^1(\mathbf{x}_0)V_{12}^1[t(\mathbf{x},\tau)]. \tag{23}$$

Here the operator $\mathcal{H}_0^1(\mathbf{x}) \equiv \mathcal{H}_0^1[\mathbf{x}; t(\mathbf{x},\tau)]$ is a generalization of the free field Hamiltonian defined by the relations

$$\mathcal{H}_0^1(\mathbf{x}) = \frac{\delta\alpha_0}{\delta t(\mathbf{x},\tau)} + \sum_{n=0}^{\infty}\frac{(-i/\hbar)^{n+1}}{(n+2)!}\Big[\underbrace{\alpha_0, \alpha_0, \ldots, \alpha_0, \frac{\delta\alpha_0}{\delta t(\mathbf{x},\tau)}}_{n+2}\Big], \tag{24}$$

$$\alpha_0 \equiv \alpha_0[t(\mathbf{x},\tau)] = \int d^3x\, t(\mathbf{x},\tau)n_\mu(\mathbf{x})n_\nu(\mathbf{x})T_0^{1,\mu\nu}(\mathbf{x}). \tag{25}$$

In (25), $T_0^{1,\mu\nu}(\mathbf{x})$ is the tensor density of the energy-momentum of the free fields expressed in terms of Schrödinger operators. The functional $\delta\alpha_0/\delta t(\mathbf{x},\tau)$ can be written exactly as in (7) with $T^{1,\mu\nu}$ replaced by $T_0^{1,\nu\mu}$.

The solution of (23) is known beforehand, as is clear from the discussion of Sec. 3, and is written in the form

$$V_{12}^1[t(\mathbf{x},\tau)] = e^{-i\alpha_0/\hbar}. \tag{26}$$

Defining further the state vector in the interaction representation $F^2[t(\mathbf{x},\tau)]$ with the help of the relation

$$F^2[t(\mathbf{x},\tau)] = V_{21}^1[t(\mathbf{x},\tau)]F^1[t(\mathbf{x},\tau)], \quad (V_{21}^1 = (V_{12}^1)^{-1}), \tag{27}$$

we easily find the generalization of the Schrödinger equation in the interaction representation of the i.t.f.:

$$\frac{\hbar}{i}\frac{\delta F^2[t(\mathbf{x},\tau)]}{\delta t(\mathbf{x}_0,\tau)} = -\mathcal{H}_{int}^2[\mathbf{x}_0; t(\mathbf{x},\tau)]F^2[t(\mathbf{x},\tau)], \tag{28}$$

$$\mathcal{H}_{int}^2[\mathbf{x}; t(\mathbf{x},\tau)] = (V_{12}^1[t])^{-1}\{\mathcal{H}^1(\mathbf{x}) - \mathcal{H}_0^1(\mathbf{x})\}V_{12}^1[t]. \tag{29}$$

In the s.t.f., Eq. (28) goes over into the usual equation

$$\frac{\hbar}{i}\frac{\partial F^2(t)}{\partial t} = H_{int}^2(t)F^2(t). \tag{30}$$

The equation of motion for the field operators in the interaction representation

$$\varphi^2[\mathbf{x}; t(\mathbf{x},\tau)] = (V_{12}^1[t(\mathbf{x},\tau)])^{-1}\varphi^1(\mathbf{x})V_{12}^1[t(\mathbf{x},\tau)] \tag{31}$$

is obtained by variational differentiation of relation (31):

$$\frac{\hbar}{i}\frac{\delta\varphi^2[\mathbf{x};t]}{\delta t(\mathbf{x}_0,\tau)} = [\mathcal{H}_0^2[\mathbf{x}_0; t(\mathbf{x},\tau)], \varphi^2[\mathbf{x};t]]. \tag{32}$$

In going over to the s.t.f., this last equation leads to the equations of motion for the free field

$$\frac{\hbar}{i}\frac{\partial\varphi^2(\mathbf{x},t)}{\partial t} = [H_0^2(t), \varphi^2(\mathbf{x},t)]. \tag{33}$$

We note, incidentally, that the transition to curvilinear coordinates in the equations for F and φ is effected simply by integrating (28) and (32) over the hypersurfaces $t = t(\mathbf{x}, \tau)$, i.e.,

$$\frac{\hbar}{i}\frac{\partial F^2(\tau)}{\partial\tau} \equiv \frac{\hbar}{i}\int_\sigma \frac{\delta F^2[t(\mathbf{x},\tau)]}{\delta t(\mathbf{x}_0,\tau)} d\sigma_0 = -\int_\sigma \mathcal{H}^2_{int}[\mathbf{x}_0; t(\mathbf{x},\tau)]\, d\sigma_0 \cdot F^2(\tau), \tag{34}$$

$$\frac{\hbar}{i}\frac{\partial \varphi^2(\mathbf{x},\tau)}{\partial\tau} \equiv \frac{\hbar}{i}\int_\sigma \frac{\delta\varphi^2[\mathbf{x}; t(\mathbf{x},\tau)]}{\delta t(\mathbf{x}_0,\tau)} d\sigma_0 = \int_\sigma [\mathcal{H}^2_0[\mathbf{x}_0; t(\mathbf{x},\tau)], \varphi^2[\mathbf{x}; t(\mathbf{x},\tau)]]\, d\sigma_0. \tag{35}$$

The connection between the derivative with respect to τ and the variational derivative is analogous to the connection (2) in the transition to the s.t.f.[2] Analogous formulas hold in the other representations.

We emphasize also that the consistency conditions obtained from (28) and (32) can be related to the consistency conditions (12) in the Schrödinger representation by a unitary transformation.

In the interaction representation it is also easy to find the operator $W^1_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)]$ which transforms the state vector of the system on the hypersurface $t = t(\mathbf{x},\tau_1)$ into the state vector on the hypersurface $t = t(\mathbf{x},\tau_2)$:

$$F^2[t(\mathbf{x},\tau_2)] = W^1_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)]F^2[t(\mathbf{x},\tau_1)]; \tag{36}$$

$$\begin{aligned} W^1_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)] &= (V^1_{12}[t(\mathbf{x},\tau_2)])^{-1}V^1_{13}[t(\mathbf{x},\tau_2)] \\ &\times (V^1_{13}[t(\mathbf{x},\tau_1)])^{-1}V^1_{12}[t(\mathbf{x},\tau_1)] \\ &\equiv V^1_{21}[t(\mathbf{x},\tau_2)]W^1_{11}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)]V^1_{12}[t(\mathbf{x}.\tau_1)]. \end{aligned} \tag{37}$$

The relation (37) gives the connection between the operators W^1_{22} and W^1_{11} in the interaction and Schrödinger representations. The S matrix in the interaction representation can be defined as

$$S^1_{22} = \lim_{\substack{\tau_2\to+\infty \\ \tau_1\to-\infty}} W^1_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)]. \tag{38}$$

The expression for the S matrix becomes closer to the usual one if we introduce the operator $W^2_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)]$, which is a solution of the equation

$$\frac{\hbar}{i}\frac{\delta W^2_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)]}{\delta t(\mathbf{x}_0,\tau_2)} = -\mathcal{H}^2_{int}[\mathbf{x}_0; t(\mathbf{x},\tau_2)]\, W^2_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)] \tag{39}$$

or of the integral equation

$$W^2_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)] = 1 - \frac{i}{\hbar}\int_{\sigma_1}^{\sigma_2} d^4x'\mathcal{H}^2_{int}[\mathbf{x}';\sigma']\, W^2_{22}[\sigma', t(\mathbf{x},\tau_1)] \tag{40}$$

which is completely equivalent to (39).

It can be shown with the help of (40) that the variational derivative of W^2_{22} with respect to any of the intermediate hypersurfaces σ' between σ_1 and σ_2 is equal to zero. In other words, the quantity W^2_{22} is not a functional of σ' and, hence, does not depend on the choice of the system of intermediate hypersurfaces. (Otherwise all physical quantities would depend on the character of the intermediate surfaces σ', which would be disastrous for our apparatus.)

The S matrix can be obtained from (40) by the usual procedure of integrating and letting the boundary surfaces σ_1 and σ_2 tend to infinity:

$$S^2_{22} = \lim_{\substack{\tau_1\to-\infty \\ \tau_2\to+\infty}} W^2_{22}[t(\mathbf{x},\tau_2), t(\mathbf{x},\tau_1)]. \tag{41}$$

Comparison of (41) and (38) shows that the two S matrices differ from each other by the structure of the operators involved: S^2_{22} in (41) is expressed in terms of operators in the interaction representation, while S^1_{22} is expressed in terms of Schrödinger operators.

In deforming the hypersurfaces in (41) into equal-time hyperplanes we obtain the usual expression for the S matrix, as has already been demonstrated in analogous cases. Since the S matrix is not a functional of the intermediate σ', its final expression will not depend on the way in which it has been calculated.

We note that the S matrix can be a functional of the intermediate σ' in a nonlocal field theory, which is one of the indications of the inadequacies of a given version of the nonlocal theory.

6. CONCLUSION

The existence of yet another version of the i.t.f., discussed above, bears witness to the fact that besides the known T and Schw. variants, others may also be put forward. However, it is easy to see that other generalized Hamiltonians may differ from each other and from (6) only by terms which have the character of a divergence when the hypersurface is deformed into an equal-time hyperplane (otherwise there would be no correspondence with the s.t.f.). Quantities describing observable effects computed with the help of the S matrix as well as the character of the equations of motion in the s.t.f. are not changed.

In conclusion I regard it as my pleasant duty to express my deep gratitude to Prof. M. A. Markov

[2]This question, as well as a number of others, including the question of the relativistic invariance of the formalism proposed, will be considered in a subsequent paper.

for his constant interest in this work and fruitful discussions and D. A. Kirthnits for valuable advice and comments.

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Translated by R. Lipperheide
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