THE DOUBLY LOGARITHMIC APPROXIMATION IN QUANTUM ELECTRODYNAMICS

G. A. MILEKHIN* and E. S. FRADKIN

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor May 18, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 45, 1926-1939 (December, 1963)

A perturbation theory in the number of hard quanta is developed within the framework of the continual integration method. The contribution of soft photons responsible for the appearance of the doubly logarithmic terms is taken into account exactly to all orders in e^2. The method is used to obtain various cross sections for high energy particle interactions with soft photon radiation taken into account.

In connection with the proposed colliding-beam experiments that are intended to test electrodynamics at small distances, it is important to improve the precision of the perturbation-theory formulas for the cross sections.

The same problem is also of interest from the point of view of methodology, since it is here possible to establish a connection between results of calculations, tested with the help of a dynamical principle, and general results on the behavior of cross sections at high energies, obtained recently on the basis of accumulation of moving poles of the scattering amplitude as function of angular momentum—the so called Regge poles.[1] In addition, as will be shown below, a study of this problem provides a beautiful demonstration of the effectiveness and generality of the functional method of field theory.

As is known,[2] the so called doubly logarithmic terms play a leading role in radiative corrections at high energies in quantum electrodynamics. For a whole series of processes in a definite region of angles these leading terms were first summed by Sudakov,[3] Abrikosov,[2] and then by Baĭer and Kheĭfets,[4] by means of directly summing the corresponding perturbation theory diagrams,[1)] and also by Blank[5] by means of an approximate solution of the Schrödinger equation by the proper time method. We propose below a simple general method for the calculation of cross sections at high energies, within the framework of the continual integration method.

Various cross sections are obtained for the interaction of two fermions at high energies accompanied by the emission of an arbitrary number of soft photons. It is shown that at high energies the cross section acquires a "Regge-like" form only if there exists a bound state in the annihilation channel of the corresponding process.

1. METHOD OF CALCULATION

As is known (see, for example,[7,8]), the problem of finding the S matrix by the continual integration method reduces to that of finding the Green's functions in an arbitrary external field and to the subsequent functional integration with appropriate weight.

It is not possible to find a solution in closed form for the Green's function in an arbitrary field in the general case. In the case of interest to us the behavior of the matrix elements at high energies in electrodynamics is determined by so called doubly logarithmic terms; the appearance of these terms is due to soft photons.[2] This circumstance somewhat simplifies the problem of finding the Green's functions in an external field, since it is only necessary to find a closed form expression for the low-frequency (corresponding to soft photons) part of this field. In the work of one of the authors[9] a method was developed which takes into account exactly the contributions due to soft photons within the framework of the continual integration method; that is, a closed-form expression was obtained for the Green's function in an external field and the functional integration over the external fields was carried out, and thus the S matrix was found for the case when only soft photons are present.

It is not hard to generalize this method to take into account along with the soft photons a finite number of hard photons;[2)] to this end it is neces-

*Deceased.

1)The most complete exposition of the problem and references to the literature are, apparently, to be found in the work of Yennie et al.[6]

2)A method similar to that described here, but less exact, has also been proposed by Mahanthappa.[10]

sary to first generalize the corresponding solution for the Green's function in the external field. Consider the equation for the Green's function in an external electromagnetic field:

$$(i\hat{\partial} - m + e\hat{A}(x))\, G(x, x' \mid A) = -\delta(x - x'). \qquad (1)$$

We divide the electromagnetic field into two parts: low-frequency A_1 and high-frequency A_2 ($A = A_1 + A_2$) and develop a perturbation theory in the high-frequency part of the electromagnetic field. The structure of the solution for G can be seen from the following integral form of the equation for G:

$$G(x, x' \mid A) = G_0(x, x' \mid A_1) - e\int G_0(x, y \mid A_1)\hat{A}_2(y)\, G(y, x' \mid A)\, d^4y, \qquad (1')$$

where $G_0(x, x'|A_1)$ is the exact Green's function in the presence of the low-frequency part A_1 (soft photons) only.

It is convenient to make use of an operator solution of Eq. (1), which may be written in the form

$$G(x, x' \mid A) = -[(i\hat{\partial} - m + e\hat{A}_1)(1 + (i\hat{\partial} - m + e\hat{A}_1)^{-1} e\hat{A}_2)]^{-1}\delta(x - x') = \sum_{n=0}^{\infty}[(-i\hat{\partial} + m - e\hat{A}_1)^{-1} e\hat{A}_2]^n \times (i\hat{\partial} + m - e\hat{A}_1)^{-1}\delta(x - x'). \qquad (2)$$

The first term in this sum, not containing $\hat{A}_2$, has been calculated explicitly previously.[9]

Consider the second term, containing $\hat{A}_2$ linearly. It may be written in the form

$$G^{(1)}(x, x' \mid A) = (i\hat{\partial} + m)\frac{1}{-\partial^2 + m^2 - e\hat{A}_1(i\hat{\partial} + m)} \times e\hat{A}_2 \frac{1}{-\partial^2 + m^2 - (i\hat{\partial} + m)e\hat{A}_1}(i\hat{\partial} + m)\,\delta(x - x'). \qquad (3)$$

Let us Fourier-transform in the difference $x - x'$. If we also expand $\hat{A}_2(x)$ in a Fourier integral and make use of the Fock integral representation for the inverse operators we obtain from (3)

$$G^{(1)}(x, p \mid A) = -\frac{1}{(2\pi)^2}\int d^4l\, e^{-ilx}(i\hat{\partial} + \hat{p} + \hat{l} + m)\int_0^\infty ds_1 \int_0^\infty ds_2 \times \exp\{is_2[m^2 + (i\hat{\partial} + \hat{p} + \hat{l})^2 + e\hat{A}_1(i\hat{\partial} + \hat{p} + \hat{l} + m)]\}\, e\hat{A}_2(l) \times \exp\{is_1[-m^2 + (i\hat{\partial} + \hat{p})^2 + (i\hat{\partial} + \hat{p} + m)e\hat{A}_1]\}(\hat{p} + m), \qquad (4)$$

where

$$G^{(1)}(x, x' \mid A) = \frac{1}{(2\pi)^4}\int G(x, p \mid A)\, e^{-ip(x-x')}\, d^4p. \qquad (4')$$

In this expression p (p is the electron's momentum, as will be seen in the following) and l (the momentum of the hard photon) are large. Ignoring in (4) all terms not containing the large quantities p and l we find

$$G^{(1)}(x, p \mid A) = -\frac{1}{(2\pi)^2}\int d^4l\, e^{-ilx}(i\hat{\partial} + \hat{q} + m)\int_0^\infty ds_1 \int_0^\infty ds_2 \times \exp\{is_2[-q^2 - m^2 - 2iq\partial + e\hat{A}_1(\hat{q} + m)]\}\, eA_2(l) \times \exp\{is_1[-p^2 - m^2 - 2ip\partial + (\hat{p} + m)e\hat{A}_1]\}(\hat{p} + m), \qquad (5)$$

where $q = p + l$.

Further manipulation of this expression is simplest accomplished by the techniques given in [11]. The result is

$$G^{(1)}(x, p \mid A) = \frac{1}{(2\pi)^2}\int d^4l\, e^{-ilx}(i\hat{\partial} + \hat{q} + m)\int_0^\infty ds_1 \int_0^\infty ds_2 \times \exp[-is_1(p^2 + m^2) - is_2(q^2 + m^2)] \times T^+\exp\left[ie\int_0^{s_2}\hat{A}_1(x + 2qs')(\hat{q} + m)\, ds'\right] e\hat{A}_2(l) \times T^+\exp\left[ie(\hat{p} + m)\int_0^{s_1}\hat{A}_1(x + 2qs_2 + 2ps')\, ds'\right](\hat{p} + m). \qquad (6)$$

In a number of important cases the expression (6) can be simplified. To this end we write the operators appearing inside the T^+ exponentials in the form:

$$\hat{A}_1(\hat{q} + m) = -2qA_1 - (\hat{q} - m)\hat{A}_1,$$
$$(\hat{p} + m)\hat{A}_1 = -2pA_1 - \hat{A}_1(\hat{p} - m). \qquad (7)$$

The last terms on the right sides of Eqs. (7) may be neglected if the momenta p and q lie on the mass shell or near it. In that case formula (6) becomes

$$G^{(1)}(x, p \mid A) = -\frac{1}{(2\pi)^2}\int d^4l\, e^{-ilx}(i\hat{\partial} + \hat{q}_2 + m)\, e\hat{A}_2(l)(\hat{p} + m) \times \int_0^\infty ds_1 \int_0^\infty ds_2 \exp\Big\{-i\Big[s_1(p^2 + m^2) + s_2(q_1^2 + m^2) + 2e\int_0^{s_1} pA(x + 2q_1s_2 + 2ps')\, ds' + 2e\int_0^{s_2} q_1A(x + 2q_1s')\, ds\Big]\Big\}. \qquad (8)$$

It can be shown that the condition for the applicability of this formula is given by the inequalities $|pq_1| \gg |p^2 + m^2|$, $|q_1^2 + m^2|$.

The following expression for $G^{(1)}(q, p|A)$ is easily obtained from (4), (4′), and (8) (using the fact that in $A_1(k)$ $k^2 < 2pk, 2qk$):

$$G^{(1)}(q, p|A) = -\frac{1}{(2\pi)^4}\int d^4x\, e^{i(p-q)x}(\hat{q}+m)e\hat{A}_2(x)(\hat{p}+m)$$
$$\times \int_0^\infty dt \int_0^\infty ds \exp[-it(q^2-m^2) - is(p^2+m^2)]$$
$$\times \exp\left\{-2ie\left[\int_0^t qA_1(x-2qt')\,dt' + \int_0^s pA_1(x+2ps')\,ds'\right]\right\},$$
$$\times\ G(q,p) = \frac{1}{(2\pi)^4}\int G(x,x')\,e^{-iqx+ipx'}\,d^4x\,d^4x'. \qquad (8')$$

An analogous procedure gives for the third term of the sum (2) bilinear in A_2, the expression

$$G^{(2)}(x, p|A) = -\frac{i}{(2\pi)^4}\int d^4l_1 d^4l_2\,(i\hat{\partial}+\hat{q}_1+m)\int_0^\infty ds_1\int_0^\infty ds_2$$
$$\times \exp\Big[-i\Big\{s_1(q_1^2+m^2)+s_2(r^2+m^2)+s_3(p^2+m^2) + 2e\int_0^{s_1} qA_1(x+2q_1s')\,ds'\Big\}\Big]e\hat{A}_2(l_2)(i\hat{\partial}+\hat{r}+m)$$
$$\times T^+\exp\Big[ie\int_0^{s_2} A_1(x+2q_1s_1+2rs')\,ds'(\hat{r}+m)\Big]e\hat{A}_2(l_1)$$
$$\times \exp\Big[-2ie\int_0^{s_3} pA_1(x+2q_1s_1+2rs_2+2ps')\,ds'\Big]$$
$$\times(\hat{p}+m), \qquad (9)$$

where $r = p + l_2$, $q_1 = p + l_1 + l_2$.

To the same accuracy we obtain for $G^{(2)}(q, p|A)$ the following expression:

$$G^{(2)}(q, p|A) = -\frac{i}{(2\pi)^4}\int d^4x e^{i(p-q)x}(\hat{q}+m)\int_0^\infty ds_1\int_0^\infty ds_2\int_0^\infty ds_3$$
$$\times \exp\Big[-i\Big\{s_1(q^2+m^2)+s_2(r^2+m^2)+s_3(p^2+m^2) + 2e\int_0^{s_1} qA_1(x-2qs')\,ds'\Big\}\Big]e\hat{A}_2(x)(\hat{r}+m)$$
$$\times T^+\exp\Big[ie\int_0^{s_2} A_1(x+2rs')(\hat{r}+m)\,ds'\Big]$$
$$\times e\hat{A}_2(x)\exp\Big[-2ie\int_0^{s_3} pA_1(x+2ps'+2rs_2)ds'\Big](\hat{p}+m). \qquad (9')$$

These formulas were obtained on the assumption that q and p lie near the mass shell. Knowing the Green's functions in an external field we can proceed to the calculation of various matrix elements for the interaction of two particles.

2. INTERACTION OF TWO FERMI PARTICLES

We shall consider such energies of the interacting particles for which the photon vacuum polarization may be neglected. In that case the Green's function for two Fermi particles with masses m_1 and m_2 (for example, electron and μ meson) is given by the following formula:[7]

$$G_{12}(x_1, x_2; x_3, x_4) = \int [G_{m_1}(x_1, x_3|A)\,G_{m_2}(x_2, x_4|A) - \delta_{m_1m_2} G(x_1, x_4)\,G(x_2, x_3)]$$
$$\times \exp\Big[-\frac{i}{2}\int A(x)\,\Box\,A(x)\,d^4x\Big]\Pi\, dA(x), \qquad (10)$$

where

$$\delta_{m_1m_2} = \begin{cases}1, & m_1 = m_2\\ 0, & m_1 \neq m_2\end{cases}.$$

In the following we consider for simplicity only the case of different masses since the generalization to the equal-mass case can be obtained from the final answer by antisymmetrization in the particle coordinates. Substituting expression (8′) into (10) we obtain the function that describes the scattering process of interest:

$$G(q_1, q_2; p_1, p_2) = -\frac{ie^2}{(2\pi)^4}\int d^4x_1\,d^4x_2 \exp[-i(q_1-p_1)x_1 - i(q_2-p_2)x_2](\hat{q}_1+m_1)D_{\mu\nu}(x_1-x_2)$$
$$\times(\hat{q}_2+m_2)\gamma_\nu(\hat{p}_2+m_2)$$
$$\times\int_0^\infty \prod_{n=1}^2 ds_n'\,dt_n' \exp\Big\{-i\sum_{n=1}^2 [s_n(p_n^2+m_n^2)+t_n(q_n^2+m_n^2) - 2e\int_0^{s_n} ds_n' p_nA(x_n+2ps_n') + 2e\int_0^{t_n} dt_n' q_nA(x_n-2qt_n)]\Big\}$$
$$\times\exp\Big[2ie^2\sum_{n,m=1}^2\Big\{\int_0^{s_n} ds_n'\int_0^{s_m} ds_m' p_n^\mu D_{\mu\nu}\,(2p_ns_n'-2p_ms_m'+x_n-x_m)\,p_m^\nu$$
$$+\int_0^{t_n} dt_n'\int_0^{t_m} dt_m' q_n^\mu D_{\mu\nu}\,(2q_mt_m'-2q_nt_n'+x_n-x_m)\,q_m^\nu$$
$$+2\int_0^{s_n} ds_n'\int_0^{t_n} dt_n' p_n^\mu D_{\mu\nu}\,(2p_ns_n'+2q_mt_m'+x_n-x_m)\,q_m^\nu\Big\}\Big],$$
$$D_{\mu\nu}(x) = \frac{\delta_{\mu\nu}}{(2\pi)^4}\int\frac{e^{-ikx}}{k^2-i\varepsilon}\,d^4k = \frac{i\delta_{\mu\nu}}{4\pi^2(x^2+i\varepsilon)}. \qquad (11)$$

As is easy to see by comparison with perturbation theory, the first two terms inside the exponential with $n = m$ describe radiative corrections to one-particle Green's functions, while the terms with $n = m$ in the last term inside the exponential correspond to the radiative additions to the vertex parts ("clothing" of the interaction knots Γ), and the remaining terms result from the exchange by the particles of soft quanta while scattering (so called radiative additions of type $d\Gamma/dA$).

We now take the Fourier-transform in the variable $x_1 - x_2$. This is done simplest within the accuracy of the present work by expanding the exponentials of (11) in powers of e and integrating over $x_1 - x_2$. Introducing the integral representation for the Green's function of the hard photon:

$$D(k) = \frac{1}{k^2 - i\varepsilon} = i\int_0^\infty d\alpha\, e^{-i\alpha k^2 - \varepsilon\alpha} \tag{12}$$

and discarding terms quadratic in the momenta of the soft photons, whose contribution is beyond the accuracy of our approximation, we obtain

$$G(q_1, q_2; p_1, p_2 | A) = \frac{e^2}{(2\pi)^4}\int dx^4\, e^{-i(q_2 - p_2 + q_1 - p_1)x}$$
$$\times (\hat{q}_1 + m_1)\gamma_\mu(\hat{p}_1 + m_1)(\hat{q}_2 + m_2)\gamma_\mu(p_2 + m_2)$$
$$\times \int_0^\infty d\alpha\, e^{-i\alpha l^2 - \varepsilon\alpha}\int_0^\infty \prod_{n=1}^2 ds_n\, dt_n \exp\Big[-i\sum_{n=1}^2 \Big\{s_n(p_n^2 + m_n^2)$$
$$+ t_n(q_n^2 + m_n^2)$$
$$+ 2e\int_0^{s_n} ds_n'\, p_n A(x + (-1)^n \alpha l + 2p_n s_n')$$
$$+ 2e\int_0^{t_n} dt_n'\, q_n\, A(x + (-1)^n \alpha l + 2q_n t_n')\Big\}\Big]$$
$$\times \exp\Big[2ie^2 \sum_{n;\, m=1}^2 \Big\{\int_0^{s_n} ds_n'\, p_n D(2p_n s_n' - 2p_m s_m'$$
$$+ 2(n - m)\alpha l)\, p_m$$
$$+ \int_0^{t_n} dt_n' \int_0^{t_m} dt_m'\, q_n D(2q_m t_m' - 2q_n t_n' + 2(n - m)\alpha l)\, q_m$$
$$+ 2\int_0^{s_n} ds_n' \int_0^{t_m} dt_m'\, p_n D(2p_n s_n' + 2q_m t_m'$$
$$+ 2(n - m)\alpha l)\, q_m\Big\}\Big]. \tag{13}$$

The integral over x yields 4-momentum conservation, and in the case when the external field is absent we have

$$q_2 - p_2 + q_1 - p_1 = 0,$$

$$l = \tfrac{1}{2}(q_1 - p_1 + p_2 - q_2) = q_1 - p_1 = p_2 - q_2. \tag{13'}$$

Calculation of the indicated integrals of the type $\iint ds'\, ds'' D(2ps' + 2qs'' + 2\alpha l)$ over s' and s'' with doubly logarithmic accuracy presents no particular difficulty, since in this case the problem reduces to the finding of the region of integration over s' and s'' in which the denominator of every term is proportional to $s's''$. This region has on a logarithmic scale the form of a polygon whose area gives the value of the integral in the doubly logarithmic approximation. Certain complication arise in going to the limit $l \to 0$, and also in evaluating the integrals connected with the vertex parts. In that case it is necessary for the determination of the lower boundary of the doubly logarithmic region (for small s' and s'') to take into account the quadratic in gradient terms that were omitted in passing from the exact formula (4) to formula (5) and correspondingly in the following formulas.

Indeed, in passing from (4) to (5) we have, in particular, neglected the squared gradient, i.e., ∇^2 in comparison with $p\nabla$; taking ∇^2 into account makes it possible to determine the lower boundary of the doubly logarithmic region.

We will not give here all the necessary steps but will only present a resume of the changes that must be made in the obtained expressions, namely: it is necessary to replace in formula (3) and all the following formulas the terms of the type

$$\int_0^s pA(x + 2ps')\, ds' \to \frac{1}{(2\pi)^{3/2}}\int\int_0^s pA(k)\, e^{ikx + 2ikps' - is'k^2} d^4k\, ds'$$

and correspondingly

$$D(2ps' \pm 2ps'' \pm 2\alpha l) \to \tilde{D}(2ps' \pm 2qs'' \pm 2\alpha l)$$
$$= \frac{ipq}{4\pi^2(2ps' \pm 2ps'' \pm 2\alpha l)^2}\left(1 - \exp\left[\frac{i(ps' \pm qs'' \pm \alpha l)^2}{s' + s'' + \alpha}\right]\right). \tag{14}$$

Comparison with perturbation theory shows that such a replacement corresponds to the taking into account in the denominators of the electron propagation functions of additional diagonal quadratic terms, i.e., terms of the type $\sum k_i^2$, and the ignoring of terms of type $\sum_{j \neq i} k_i k_j$, while up to now we were taking into account in the denominators of the Feynman diagrams

$$\int d^4k_1 \ldots d^4k_n \{[(p_1 + k_1 + k_2 + \ldots + k_n)^2 + m^2]$$
$$\ldots [(p_2 + k_1)^2 + m^2]\}^{-1}$$

only terms of the type $\sum_i pk_i$ and $p_i^2 + m^2$. Taking into account terms of the type $\sum_i (2pk_i + k_i^2)$ turns out to be sufficient to determine the lower boundary of the doubly logarithmic region in the integrals over s' and s''.

Indeed, as can be seen from Eq. (14), taking these terms into account has resulted in the appearance of the additional factor (for $\alpha l \to 0$) $1 - \exp\{i(ps' + qs'')^2/(s' + s'')\}$ which vanishes for $|(ps' + qs'')^2| \ll s' + s''$. Only in the case when

$$|(ps' + qs'')^2| \gg s' + s'', \tag{15}$$

is the additional factor equivalent to unity (since the exponential oscillates rapidly). It is the condition (15) that determines the lower boundary of the doubly logarithmic region for terms for which $\alpha l \to 0$ or $\alpha l = 0$.

With the help of the obtained Green's function we find the matrix element by the standard procedure of passing to the mass shell. However the matrix element for pure elastic scattering vanishes in that case. This is a well known situation due to the fact that the photon has zero rest mass (the so called "infrared catastrophe"), it reflects itself in experiments in the fact that an instrument with resolving power ΔE measures the interaction cross section for processes involving the emission of an arbitrary number of soft photons whose energy does not exceed ΔE. Therefore physical significance attaches only to "elastic" scattering of electrons with the emission of soft photons of energy ΔE (or not in excess of ΔE). This is achieved by applying to expression (13) the operator

$$\frac{1}{2} e_i \pi^{-3/2} (2w_i)^{-1/2}\, \delta/\delta A(k_i)$$

respectively for each emitted photon of momentum k_i and polarization e_i, after which one must set the external field A equal to zero.

Carrying out the indicated operations and going over in the standard way from the Green's function to the matrix element, we find for the matrix element for the scattering of particles with the production of n soft photons:

$$\begin{aligned} M_n = {} & e^2 \bar{u}_\rho(q_1)\,\gamma_\mu u_\nu(p_1)\,\bar{u}_\lambda(q_2)\,\gamma_\mu u_\sigma(p_2) \\ & \times \lim\, (p_1^2 + m_1^2)(q_1^2 + m_1^2)(p_2^2 + m_2^2)(q_2^2 + m_2^2)\Big|_{\substack{p_i^2 \to -m_i^2 \\ q_i^2 \to -m_i^2}} \\ & \times \int_0^\infty e^{-i\alpha k^2 - \varepsilon\alpha} d\alpha \int_0^\infty \prod_{n=1}^{2} ds_n dt_n \exp[-is_n(p_n^2 + m_n^2) \\ & - it_n(q_n^2 + m_n^2)]\Big| \exp\Big[2ie^2 \sum_{n;\, m=1}^{2} \Big\{\int_0^{s_n} ds_n' \int_0^{s_m} ds_m'' \, p_n D \\ & \times (2p_n s_n' - 2p_m s_m'' + 2(n-m)\alpha l)\; p_m \\ & + \int_0^{t_n} dt_n' \int_0^{t_m} dt_m''\, q_n D\, (2q_m t_m'' - 2q_n t_n' + 2(n-m)\alpha l)\; q_m \\ & + 2\int_0^{s_n} ds_n' \int_0^{t_m} dt_m'\, p_n D\, (2p_n s_n' + 2q_m t_m' + 2(n-m)\alpha l)\; q_m\Big\}\Big] \\ & \times \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} \frac{e_i}{(2\pi)^{3/2}} \frac{1}{\sqrt{2w_i}} \Big\{-2ieq_n \int_0^{t_n} dt_i \\ & \times \exp[-2it_i k_i q_n + i(-1)^n \alpha k_i l] \\ & - 2iep_n \int_0^{s_n} ds_i \exp[2is_i k_i p_n + i(-1)^n \alpha k_i l]\Big\}. \end{aligned} \tag{16}$$

The factor $1/\sqrt{n!}$ takes account of the identity of the emitted photons.

Let us calculate the total cross section for scattering with the emission of soft photons whose energy in the center of mass system does not exceed Δ:

$$\begin{aligned} d\sigma = {} & \frac{1}{(2\pi)^2} \sum_{n=0}^{\infty} \int \theta\Big(\Delta - \sum_{i=1}^{n} w_i\Big) d^3k_1 \ldots d^3k_n\, |M_n|^2 \\ & \times \delta^4\Big(p_1 + p_2 - q_1 - q_2 - \sum_{i=1}^{n} k_i\Big) \frac{d^3q_1 d^3q_2}{J}; \end{aligned}$$

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}. \tag{17}$$

It is convenient to express the integral over the soft photons in the form

$$\begin{aligned} & \int \theta\Big(\Delta - \sum_{i=1}^{m} w_1\Big) d^3k_1 \ldots d^3k_n \\ & = \int_0^\Delta dw\, \delta\Big(w - \sum w_i\Big) \int d^3k_1 \ldots d^3k_n \\ & = \int d\tau \frac{e^{i\Delta\tau} - 1}{2\pi i\tau} \int d^3k_1 \ldots d^3k_n \exp\Big\{-i\tau \sum w_i\Big\}. \end{aligned} \tag{18}$$

The remaining calculations are performed under the assumption that the quantity Δ is small in comparison with the energies of the colliding particles, and that, therefore, one may neglect the particle recoil in the emission of a soft photon. Under that condition one may neglect the term Σk_i in the argument of the δ function in formula (17).

Summing over n in formula (17), with (18) taken into account and under the assumption made above with respect to Δ, we find the following expression for the cross section for the scattering of two particles accompanied by the production of soft photons whose total energy does not exceed Δ:

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_0}{d\Omega}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\Delta\tau}-1}{\tau}d\tau l^4\int_0^\infty d\alpha\int_0^\infty d\beta$$

$$\times\exp\left[i\beta\left(l^2+i\varepsilon\right)-i\alpha\left(l^2-i\varepsilon\right)\right]$$

$$\times\exp\left\{2\int_0^\infty ds\int_0^\infty ds'\left[F(\alpha)+F^*(\beta)-2\Phi(\tau)\right]\right\}; \quad (19)$$

$$F(\alpha)=ie^2\sum_{n;\,m=1}^{2}\{p_n\widetilde{D}\left[2p_ns-2p_ms'+\left((-1)^n-(-1)^m\right)\alpha l\right]p_m$$
$$+q_n\widetilde{D}\left[2q_ms-2q_ns'+\left((-1)^n-(-1)^m\right)\alpha l\right]q_m$$
$$+2p_n\widetilde{D}\left[2p_ns+2q_ms'+\left((-1)^n-(-1)^m\right)\alpha l\right]q_m\}; \quad (20)$$

$$\Phi(\tau)=ie^2\sum_{n;\,m=1}^{2}\{p_nD_+\left[2p_ns-2p_ms'+(-1)^m\alpha l-(-1)^m\beta l+\tau\right]p_m$$
$$+q_nD_+\left[2q_ms-2q_ns'+(-1)^n\alpha l-(-1)^m\beta l+\tau\right]q_m$$
$$+p_nD_+\left[2p_ms+2q_ms'+(-1)^n\alpha l\,(-1)^m\beta l+\tau\right]q_m$$
$$+q_nD_+\left[-2q_ns'-2p_ms'+(-1)^n\alpha l-(-1)^m\beta l+\tau\right]p_m\}, \quad (21)$$

where $d\sigma_0/d\Omega$ is the corresponding differential cross section in first approximation of perturbation theory; $\tau_4=\tau$, $\tau_{1,2,3}=0$;

$$D_+(x)=\frac{i}{2(2\pi)^3}\int e^{ikx}\frac{d^3k}{w}=\frac{i}{4\pi^2\left(\mathbf{x}^2-(t+i\varepsilon)^2\right)},$$

$$\widetilde{D}_{\mu\nu}(2ps\pm 2p_1s_1\pm 2\alpha l)=\frac{1}{(2\pi)^4}\int D^c_{\mu\nu}(k)$$
$$\times\exp\left[ik\left(2ps\pm 2p_1s_1\pm 2\alpha l\right)-ik^2\left(s+s_1+\alpha\right)\right]d^4k. \quad (22)$$

In the case of the Feynman gauge $D^c_{\mu\nu}=(k^2-i\epsilon)^{-1}\times\delta_{\mu\nu}$, and the expression (22) coincides with formula (14).

In conclusion we note that in the derivation of (19)—(21) we have made use of the following relation, valid for any finite function:

$$\lim_{p^2+m^2\to 0} i\int_0^\infty ds\exp\left[-is\left(p^2+m^2-i\varepsilon\right)+F(s)\right]$$
$$=\frac{1}{p^2+m^2}e^{F(\infty)}. \quad (23)$$

3. ASYMPTOTIC BEHAVIOR OF THE INTERACTION CROSS SECTION OF TWO FERMI PARTICLES

It is seen from the formulas for the cross section that the dependence of F and Φ on α and β enters into these formulas along with the scalar products of the corresponding momenta of the particles. Since we are interested in the behavior of the cross sections at high energies, the corresponding scalar products of the momenta will be large (excluding, generally speaking, the region of angles close to 0 or π) and it seems reasonable to ignore the dependence of F and Φ on α and β. In that approximation the formulas (19)—(21) simplify considerably and take on the form:

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_0}{d\Omega}\int_{-\infty}^{+\infty}\frac{e^{i\Delta\tau}-1}{2\pi i\tau}d\tau\, e^{2B(\tau)}. \quad (24)$$

The expression for $B(\tau)$ is easily obtained from (19)—(21) by setting $\alpha=\beta=0$ in F and Φ and performing the integration over s and s'. As a result we obtain an expression for the cross section that differs little from the analogous expression obtained by Yennie et al [6] by an entirely different method. To illustrate this it is sufficient to set $\alpha=\beta=0$ in (19)—(20) and pass to the momentum representation for $\widetilde{D}$ and D_+. Afterwards the integration over s and s' is elementary. As a result we obtain the following simple expression for $B(\tau)$:

$$B(\tau)=\frac{e^2}{32\pi^3}\sum_{n;\,m=1}^{2}\left\{\int\frac{d^3k}{w}e^{-iw\tau}\left(\frac{p_n}{kp_n}-\frac{q_n}{kq_n}\right)\left(\frac{p_m}{kp_m}-\frac{q_m}{kq_m}\right)\right.$$
$$-\operatorname{Re}\frac{4i}{\pi}\int d^4k\left(\frac{p_n}{k^2-2p_nk+i\varepsilon}+\frac{q_n}{k^2+2q_nk+i\varepsilon}\right)_\mu$$
$$\left.\times D^c_{\mu\nu}(k)\left(\frac{p_m}{k^2+2p_mk+i\varepsilon}+\frac{q_m}{k^2-2q_mk+i\varepsilon}\right)_\nu\right\}. \quad (25)$$

The integrations in (25) present no particular difficulties; they are carried out most conveniently with $D^c_{\mu\nu}$ expressed in the Feynman gauge, however for purposes of gauge invariance it is more convenient to use for $D^c_{\mu\nu}$ the transverse gauge, i.e.,

$$D^c_{\mu\nu}=\left(\delta_{\mu\nu}-k_\mu k_\nu k^{-2}\right)\left(k^2-i\varepsilon\right)^{-1}.$$

Omitting the details we give the asymptotic form of the expression for $B(\tau)$ for large τ:

$$B(\tau)=A_1+\widetilde{A}(m_1,m_2)\left(\ln|\tau|+\tfrac{1}{2}i\pi\operatorname{sign}\tau\right), \quad (26)$$

$$\widetilde{A}=\frac{e^2}{8\pi^2}\sum_{n;\,m=1}^{2}\left\{\frac{(p_np_m)}{\sqrt{(p_np_m)^2-p_n^2p_m^2}}\times\ln\frac{\left|-(p_np_m)+\sqrt{(p_np_m)^2-p_n^2p_m^2}\right|}{(p_n^2p_m^2)^{1/2}}\right.$$
$$+\frac{(q_nq_m)}{\sqrt{(q_nq_m)^2-q_n^2q_m^2}}\ln\frac{\left|-(q_nq_m)+\sqrt{(q_nq_m)^2-q_n^2q_m^2}\right|}{(q_n^2q_m^2)^{1/2}}$$
$$\left.-2\frac{(q_np_m)}{\sqrt{(q_np_m)^2-q_n^2p_m^2}}\ln\frac{\left|-(q_np_m)+\sqrt{(q_nq_m)^2-q_n^2p_m^2}\right|}{(q_n^2p_m^2)^{1/2}}\right\}. \quad (27)$$

The expression for A_1 is somewhat awkward and we therefore give only the asymptotic form in the c.m.s.:

$$A_1 = \sum_{i,j=1}^{2} \{a(p_i; p_j) = a(q_i; q_j) - 2a(-q_i; p_j)\};$$

$$a(p_i; p_j) = \frac{e^2}{4\pi^2}(p_i p_j)\int_{-1}^{1}\frac{\ln[{}^1/_2\,|E_i - E_j + x(E_i + E_j)|]}{(p_i - p_j + x(p_i + p_j))^2}dx$$

$$= -\frac{e^2}{16\pi^2}\frac{(p_i p_j)}{\sqrt{(p_i p_j)^2 - p_i^2 p_j^2}}$$

$$\times\Big[\ln\frac{1}{2}\Big|E_i - E_j + x_{ij}(E_i + E_j)\Big|\ln\frac{1 - x_{ij}}{1 + x_{ij}}$$

$$+ \Phi\Big(-\frac{(E_i + E_j)(1 - x_{ij})}{E_i - E_j + x_{ij}(E_i + E_j)}\Big)$$

$$- \Phi\Big(\frac{(E_i + E_j)(1 + x_{ij})}{E_i - E_j + x_{ij}(E_i + E_j)}\Big) + \binom{i \to j}{j \to i}\Big],$$

$$x_{ij} = \frac{p_i^2 - p_j^2 - 2\sqrt{(p_i p_j)^2 - p_i^2 p_j^2}}{-(p_i + p_j)^2},$$

$$\Phi(x) = \int_0^x \frac{dy}{y}\ln|1 - y|, \qquad (28)$$

where $E_i = \sqrt{m_i^2 + \mathbf{p}_i^2}$.

Integrating over τ we obtain

$$d\sigma/d\Omega = (d\sigma_0/d\Omega)\,\Delta^{-2\widetilde{A}}e^{2A_1}/\Gamma(1 - 2\widetilde{A}). \qquad (29)$$

Using standard procedures one can obtain from these formulas also the cross section for the scattering of particles with opposite signs of the charge. The formulas are also easily generalized to the case of interaction of identical particles. At that it is necessary to keep in mind that $d\sigma_0$ consists of two diagrams (direct and exchange) and the appropriate procedures must be applied to each diagram. Thus, in the approximation here considered the cross section has the asymptotic form

$$d\sigma/d\Omega = (d\sigma_0/d\Omega)(E_1/\Delta)^{A(m_1, m_2)}(E_2/\Delta)^{A(m_2, m_1)}, \qquad (30)$$

where E_1 and E_2 are the energies of the colliding particles in the c.m.s. At that in the c.m.s. of the corresponding processes A has the form:

1) for electron-electron scattering

$$A = -\frac{e^2}{2\pi^2}\Big[\frac{m^2 + 2p^2\sin^2(\theta/2)}{p\,|\sin(\theta/2)|\,\sqrt{m^2 + p^2\sin(\theta/2)}}$$

$$\times\ln\frac{p\,|\sin(\theta/2)| + \sqrt{m^2 + p^2\sin^2(\theta/2)}}{m}$$

$$-\frac{m^2 + 2p^2}{p\sqrt{m^2 + p^2}}\ln\frac{p + \sqrt{m^2 + p^2}}{m} - 1$$

$$+\frac{m^2 + 2p^2\cos^2(\theta/2)}{p\,|\cos(\theta/2)|\,\sqrt{m^2 + p^2\cos(\theta/2)}}$$

$$\times\ln\frac{p\,|\cos(\theta/2)| + \sqrt{m^2 + p^2\cos^2(\theta/2)}}{m}\Big]; \qquad (31)$$

2) for electron-positron scattering

$$A = -\frac{e^2}{2\pi^2}\Big[\frac{m^2 + 2p^2\sin^2(\theta/2)}{p\,|\sin(\theta/2)|\,\sqrt{m^2 + p^2\sin^2(\theta/2)}}$$

$$\times\ln\frac{p\,|\sin(\theta/2)| + \sqrt{m^2 + p^2\sin^2(\theta/2)}}{m}$$

$$-1 + \frac{m^2 + 2p^2}{p\sqrt{m^2 + p^2}}\ln\frac{p + \sqrt{p^2 + m^2}}{m}$$

$$-\frac{m^2 + 2p^2\cos(\theta/2)}{p\,|\cos(\theta/2)|\,\sqrt{m^2 + p^2\cos^2(\theta/2)}}$$

$$\times\ln\frac{p\,|\cos(\theta/2)| + \sqrt{m^2 + p^2\cos^2(\theta/2)}}{m}\Big]; \qquad (32)$$

3) for electron-μ^- meson scattering

$$A(m, M) = -\frac{e^2}{2\pi^2}\Big[\frac{m^2 + 2p^2\sin^2(\theta/2)}{p\,|\sin(\theta/2)|\,\sqrt{m^2 + p^2\sin^2(\theta/2)}}$$

$$\times\ln\frac{p\,|\sin(\theta/2)| + \sqrt{m^2 + p^2\sin^2(\theta/2)}}{m}$$

$$-1 + \frac{p^2\cos\theta + E_1E_2}{\sqrt{(p^2\cos\theta + E_1E_2)^2 - m^2M^2}}$$

$$\times\ln\frac{p^2\cos\theta + E_1E_2 + \sqrt{(p^2\cos\theta + E_1E_2)^2 - m^2M^2}}{mM}$$

$$-\frac{p^2 + E_1E_2}{p(E_1 + E_2)}\ln\frac{p^2 + E_1E_2 + pE_1 + pE_2}{mM}\Big],$$

$$E_1 = \sqrt{p^2 + m^2}, \qquad E_2 = \sqrt{p^2 + M^2}; \qquad (33)$$

4) for electron-μ^+ meson scattering

$$A(m, M) = -\frac{e^2}{2\pi^2}\Big[\frac{m^2 + 2p^2\sin^2(\theta/2)}{p\,|\sin(\theta/2)|\,\sqrt{m^2 + p^2\sin^2(\theta/2)}}$$

$$\times\ln\frac{p\,|\sin(\theta/2)| + \sqrt{m^2 + p^2\sin^2(\theta/2)}}{m}$$

$$-1 + \frac{p^2 + E_1E_2}{p(E_1 + E_2)}\ln\frac{p^2 + E_1E_2 + pE_1 + pE_2}{mM}$$

$$-\frac{p^2\cos\theta + E_1E_2}{\sqrt{(p^2\cos\theta + E_1E_2)^2 - m^2M^2}}$$

$$\times\ln\frac{p^2\cos\theta + E_1E_2 + \sqrt{(p^2\cos\theta + E_1E_2)^2 - m^2M^2}}{mM}; \qquad (34)$$

5) for annihilation of an electron pair into a μ meson pair

$$A(m, M) = -\frac{e^2}{2\pi^2}\Big[\frac{E^2 + p^2}{2Ep}\ln\frac{E + p}{m} + \frac{E^2 + q^2}{2Eq}\ln\frac{E + q}{M}$$

$$+\frac{E^2 - pq\cos\theta}{\sqrt{E^2(p^2 - 2pq\cos\theta + q^2) - (pq\cos\theta)^2}}$$

$$\times\ln\frac{E^2 + pq\cos\theta + \sqrt{E^2(p^2 - 2pq\cos\theta + q^2) - (pq\cos\theta)^2}}{mM}$$

$$-1 - \frac{E^2 + pq\cos\theta}{\sqrt{E^2(p^2 + 2pq\cos\theta + q^2) - (pq\sin\theta)^2}}$$

$$\times\ln\frac{E^2 + pq\cos\theta + \sqrt{E^2(p^2 - 2pq\cos\theta + p^2) - (pq\sin\theta)^2}}{mM}\Big], \qquad (35$$

where $q^2 + M^2 = p^2 + m^2$.

In the case when $\cos\theta \approx 1$ the formulas (26)–(28) coincide with the results of Baĭer and Kheĭfets.[4]

process	$\theta \sim \pi$	$\theta \sim 0$
$e + e \to e + e$	0	0
$e^+ + e^- \to e^+ + e^-$	$-2\ln^2(E/m)$	0
$\mu^- + e^- \to \mu^- + e^-$	$^1/_2 \ln^2[(E_1+E_2)^2/mM]$	0
$\mu^+ + e^- \to \mu^+ + e^-$	$-^1/_2 \ln^2[(E_1+E_2)^2/mM]$	0
$e^- + e^+ \to \mu^- + \mu^+$	$-^1/_2 \ln^2(E^2/mM)$	$^1/_2 \ln^2(E^2/mM)$

Let us note that the cross sections obtained in that approximation have the characteristic "Regge-like" form. Indeed if we go over to the Mandelstam variables s, t, and u we obtain for electron-electron scattering, for example,

$$\frac{d\sigma}{d\Omega} \approx \frac{d\sigma_0}{d\Omega} s^{2[\alpha(t)-\alpha(s)+\alpha(u)]}, \tag{36}$$

where $s = -(p_1 + p_2)^2$ is the energy in the c.m.s.; $u = -(p_1 - q_2)^2$; $t = -(p_1 - q_1)^2$ is the momentum transfer in the c.m.s. At that if we continue the initial formulas analytically in t we find

$$\alpha(t) = \frac{e^2}{4\pi^2}\left\{1 + \frac{(t-2m^2)}{\sqrt{t(t-4m^2)}}\left[\ln\left|\frac{2m^2 - t + \sqrt{t(t-4m^2)}}{2m^2}\right|\right] + i\pi\theta(t-4m^2)\right\}, \tag{37}$$

when $t < 0$ and $t > 4m^2$;

$$\alpha(t) = \frac{e^2}{4\pi^2}\left\{1 + \frac{2(t-2m^2)}{\sqrt{t(4m^2-t)}}\tan^{-1}\sqrt{\frac{t}{4m^2-t}}\right\}, \tag{38}$$

when $0 < t < 4m^2$.

It is seen from (36) that the cross section has the "Regge-like" form $s^{\alpha(t)}(s^{\alpha(u)})$ for small t (or u); for $s \approx t \approx u$ the cross section has the form $s^{\alpha(s)}$. At that for the general case of the interaction of two particles the behavior of the cross section at large energies s and small t (or u) is determined by the presence or absence of bound states in the t (or u) channel. Only in the case when a bound state exists for the corresponding process in the t (or u) channel does the cross section have the "Regge-like" form $s^{\alpha(t)}$ (or $s^{\alpha(u)}$) for small t (or u).

Thus for the $e^+ + e^+$ scattering process we have a bound state (positronium) in the t as well as in the u channels and the cross section has correspondingly Regge-like behavior for both forward and backward scattering. In the case of electron-positron scattering there exists a bound state in the t channel only and consequently only for small angle scattering (t small) does the cross section behave Regge-like; scattering backward has instead the form $s^{\alpha(s)}$. In particular for the process of annihilation of an electron pair into a μ meson pair we have a bound state in the t channel $(\mu^+ + e^-)$ and, correspondingly the cross section has Regge-like behavior for small angles. Indeed from (35) we get

$$\frac{d\sigma}{d\Omega} \approx \frac{d\sigma_0}{d\Omega} s^{\alpha(t)}, \ \alpha(t) = \frac{e^2 t}{4\pi^2}\int_{(m+M)^2}^{\infty} \frac{t' - m^2 - M^2}{\sqrt{(t'-(M-m)^2(t'-(M+m)^2)}} \frac{dt'}{t'(t'-t-i\varepsilon)}. \tag{39}$$

With the help of (38) and (39) it is possible to obtain the spectrum of the corresponding Regge trajectories (see also [12,13]), however in contrast to the real spectrum in our case l enters (in the expression for the spectrum) in the combination $(l + \nu)$, where ν has various values for the various processes (it is not equal to unity as is the case for the real spectrum).

We shall not give here the details of calculations with the more accurate formulas (19)—(21), except to note that the expressions for the corresponding diagrams obtained with the help of (19)—(21) are, generally speaking, substantially different from the analogous results obtained from (25) and represent more correctly the situation for the usual Feynman diagrams. It is relevant, however, that in the final expressions for the cross sections the differences in the "direct" and "exchange" diagrams mutually compensate each other and for the greater part of the angular region we arrive at the previous result for the cross sections. Differences occur only for certain processes for backward ($\theta \approx \pi - m/E$) and forward ($\theta \approx m/E$) scattering. At that, in order to obtain more precise formulas in that region of angles it is necessary to slightly correct also formulas (20) and (21).

To put it succinctly, the situation is such that with the help of the approximate formulas (24)—(25) we obtain the doubly logarithmic terms that are due to the photon pole ($k^2 = 0$) only; in formulas (20)—(21) the fermion pole[3)] ($k^2 + m^2 = 0$) contributes along with the photon pole, it therefore only remains to take more precisely into account the

[3)]As shown by Abrikosov[2] the fermion pole gives rise to an additional doubly logarithmic term for backward $e^- + e^+$ scattering.

spinor dependence of the fermion pole in these formulas [i.e., to take approximately into account the operator $\hat{\partial}$ (i.e., $\hat{k}$) along with $\hat{p}$ in the starting formula (4)].

Details of the more accurate cross sections reach beyond the framework of this article and will be given in a separate paper of one of the authors (E.F.). Here we shall only give for reference a table of the necessary corrections for the cross sections (31)—(35) for forward and backward scattering

$$d\sigma/d\Omega \approx (d\sigma^{(n)}/d\Omega)\, e^{e^2\delta/4\pi^2}, \quad (40)$$

where $d\sigma^{(n)}/d\Omega$ is given by the formulas (31)—(35), and the quantity δ is given in the table.

[1] T. Regge, Nuovo cimento **14**, 951 (1959). V. N. Gribov, JETP **41**, 667, 1962 (1961), Soviet Phys. JETP **14**, 478, 1395 (1962). G. F. Chew and S. C. Frautschi, Phys. Rev. Lett. **7**, 394 (1961).

[2] A. A. Abrikosov, JETP **30**, 96, 386, 544 (1956), Soviet Phys. JETP **3**, 71, 474, 379 (1956).

[3] V. V. Sudakov, JETP **30**, 87 (1956), Soviet Phys. JETP **3**, 65 (1956).

[4] V. N. Baĭer and S. A. Kheĭfets, JETP **40**, 613, 715 (1961), Soviet Phys. JETP **13**, 428, 500 (1961).

[5] V. Z. Blank, DAN SSSR **107**, 383 (1956).

[6] Yennie, Frautschi, and Suura, Ann. of Phys. **13**, 379 (1961).

[7] E. S. Fradkin, DAN SSSR **98**, 47 (1954).

[8] N. N. Bogolyubov and D. V. Shirokov, Vvedenie v teoriyu kvantovannykh poleĭ (Introduction to Quantum Field Theory), (1957).

[9] E. S. Fradkin, Dissertation, Inst. Theor. Exptl. Phys. (1960).

[10] K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962).

[11] G. A. Milekhin, JETP **43**, 1012 (1962), Soviet Phys. JETP **16**, 715 (1963).

[12] M. Levy, Phys. Rev. Lett. **9**, 235 (1962).

[13] Arbuzov, Logunov, Tavkhelidze, Faustov, and Filippov, preprint Joint Inst. Nuc. Res. E-1061. L. D. Solov'ev, preprint JINR, R-1110.

Translated by A. M. Bincer
302