# PHYSICAL REGION OF THE FIVE-POINT FUNCTION IN TERMS OF FIVE INVARIANTS

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The boundary of the physical region of the five-point function is obtained in explicit form in the space of the Lorentz-invariant variables that the amplitude depends on. The structure and relative location of the physical regions in various channels are discussed for an arbitrary inelastic process.

1. The dynamics of the interaction process between the particles determine the singularities of the invariant amplitude for that process (as opposed to kinematic singularities). Such an amplitude depends only on the Lorentz invariant combinations of 4-momenta of the particles $p_j$, $s_{ij} = (p_i \pm p_j)^2$. It is in terms of the $s_{ij}$ that one obtains, for example, the limitations on the masses and the fixed invariants for which it is possible to establish dispersion relations for the amplitude in perturbation theory [1]. The amplitude is related to experimentally observable quantities (except for the coupling constant) only in the physical region for the process. The unitarity condition, with which the analyticity properties must be combined is defined also only in the physical region. It is obvious that for these reasons it is necessary to know the boundary of the physical region in the space of the invariants. For the process corresponding to the transition of two particles into two particles (four-point function) this problem is trivial. [2] The well known diagram with triangular axes s, t, u represents the physical region in all three cross channels.

In this work we describe the physical region of the five-point function. In particular we present a physical interpretation of the equation for its boundaries. We also obtain a number of facts relating to the structure and the relative positions of the physical regions of various cross channels of an arbitrary inelastic process.

2. The physical region is determined by the requirements that inside it:

a) The 4-momenta of initial and final particles should be on the mass shell: $p_j^2 = m_j^2$.

b) The squares of the 3-momenta of all particles should be nonnegative.

c) The cosines of all angles between 3-momenta should be real and should not exceed unity in magnitude.

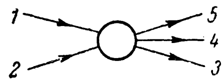


FIG. 1

The invariant amplitude for the process $p_1 + p_2 \to p_3 + p_4 + p_5$ (Fig. 1) depends on five independent Lorentz-invariant variables which can be chosen as $s_{12} = (p_1 + p_2)^2$, $s_{23} = (p_2 - p_3)^2$, $s_{34} = (p_3 + p_4)^2$, $s_{45} = (p_4 + p_5)^2$, and $s_{15} = (p_1 - p_5)^2$. The remaining $s_{ij}$, with $(i, j) = (1,3), (1,4), (2,4), (2,5), (3,5)$, can be expressed in terms of the chosen ones (see [3]). In order to fulfill the requirements b and c it is sufficient to require in the overall center of mass system ($\mathbf{p}_1 + \mathbf{p}_2 = 0$, $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 + \mathbf{p}_5$):

$$\mathbf{p}_1^2 \geqslant 0, \ \mathbf{p}_3^2 \geqslant 0, \ \mathbf{p}_5^2 \geqslant 0, \tag{1}$$

$$-1 \leqslant z_{ij} \leqslant +1, \quad (i, j) = (1,5), (2,3), (3,5), \tag{2}$$

where $z_{ij} = \cos(\mathbf{p}_i, \mathbf{p}_j)$. To determine the boundary of the physical region in the space of the invariants it is sufficient to substitute $\mathbf{p}_k^2$, $z_{ij}$, expressed in terms of the invariants, into equation (1), (2) and solve the resultant inequality. The expressions have the form

$$\mathbf{p}_1^2 = \mathbf{p}_2^2 = (4s_{12})^{-1}\Lambda(s_{12}, m_1^2, m_2^2) \equiv \Lambda_1/4s_{12},$$

$$\mathbf{p}_3^2 = (4s_{12})^{-1}\Lambda(s_{12}, m_3^2, s_{45}) \equiv \Lambda_3/4s_{12},$$

$$\mathbf{p}_5^2 = (4s_{12})^{-1}\Lambda(s_{12}, m_5^2, s_{34}) \equiv \Lambda_5/4s_{12} \tag{3}$$

(we have introduced the notation $\Lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$),

$$z_{15} = \{s_{12}^2 + s_{12}(2s_{15} - m_1^2 - m_5^2 - m_2^2 - s_{34}) + (m_1^2 - m_2^2)(m_5^2 - s_{34})\} \times \{\Lambda(s_{12}, m_1^2, m_2^2)\,\Lambda(s_{12}, m_5^2, s_{34})\}^{-1/2} \equiv M_{15}(\Lambda_1\Lambda_5)^{-1/2},$$

$$z_{23} = \{s_{12}^2 + s_{12}(2s_{23} - m_1^2 - m_3^2 - m_2^2 - s_{45})$$
$$+ (m_2^2 - m_1^2)(m_3^2 - s_{45})\}$$
$$\times \{\Lambda(s_{12}, m_1^2, m_2^2)\,\Lambda(s_{12}, m_3^2, s_{45})\}^{-1/2} \equiv M_{23}(\Lambda_2\Lambda_3)^{-1/2},$$
$$z_{35} = -\{s_{12}^2 + s_{12}(2m_4^2 - m_3^2 - m_5^2 - s_{34} - s_{45})$$
$$- (m_3^2 - s_{45})(m_5^2 - s_{34})\}$$
$$\times \{\Lambda(s_{12}, m_3^2, s_{45})\,\Lambda(s_{12}, m_5^2, s_{34})\}^{-1/2} \equiv M_{35}(\Lambda_3\Lambda_5)^{-1/2}. \tag{4}$$

The inequalities (1) give respectively

$$s_{12} \geqslant (m_1 + m_2)^2,\ s_{34} \leqslant (\sqrt{s_{12}} - m_5)^2,\ s_{45} \leqslant (\sqrt{s_{12}} - m_3)^2, \tag{5}$$

and (2) are equivalent to

$$-4s_{12}L_{15}(s_{12}, s_{15}, s_{34}) \equiv \Lambda_1\Lambda_5 - M_{15}^2 \geqslant 0, \tag{6a}$$
$$-4s_{12}L_{23}(s_{12}, s_{23}, s_{45}) \equiv \Lambda_2\Lambda_3 - M_{23}^2 \geqslant 0, \tag{6b}$$
$$-4s_{12}N(s_{12}, s_{34}, s_{45}) \equiv \Lambda_3\Lambda_5 - M_{35}^2 \geqslant 0. \tag{6c}$$

The curve $L_{15} = 0$ is nothing but the Landau curve which determines the singularities of the three-point functions of Fig. 2, a.[4] This is a hyperbola in the variables $s_{15}$, $s_{34}$ (Fig. 3, a); the region that is allowed by the inequalities (5) and (6a) corresponds to the inside of the lower branch that has as horizontal and vertical tangents $s_{34}'^{(-)} = (\sqrt{s_{12}} - m_5)^2$ and $s_{15}' = (m_1 - m_5)^2$ respectively. The regions allowed by the inequality (6b) in the variables $s_{23}$, $s_{45}$ is determined analogously. For that it is sufficient to replace in the formulas and diagrams the subscripts 1, 5, 3, 4 by 2, 3, 5, 4 respectively.

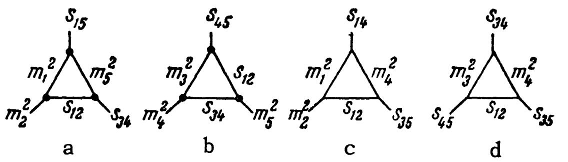

FIG. 2

The form of the curve $N = 0$ (6c) is easily found after use has been made of the following property:[5] if one sets

$$N(s_{34}, s_{45}) = As_{34}^2 + 2Bs_{34} + C = as_{15}^2 + 2bs_{15} + c,$$

then

$$B^2 - AC = \Lambda(s_{12}, m_3^2, s_{45})\,\Lambda(m_4^2, m_5^2, s_{45}),$$
$$b^2 - ac = \Lambda(s_{12}, m_5^2, s_{34})\,\Lambda(m_3^2, m_4^2, s_{34}).$$

It then follows that the horizontal tangents to $N = 0$ are given by $s_{45}''^{(\pm)} = (m_4 \pm m_5)$ and $s_{45}'^{(\pm)} = (\sqrt{s_{12}} \pm m_3)^2$, whereas the vertical tangents are given by interchanging the subscripts 3 and 5. In addition $N = 0$ has as asymptotes $s_{34} = 0$, $s_{45} = 0$, and $s_{34} + s_{45} = s_{12} - m_3^2 - m_4^2 - m_5^2$. The equation $N = 0$ gives the Landau curve of singularities for the three-point function of Fig. 2, b. Combining Eqs. (6a)–(6c) we obtain for the physical region in the $s_{34}$, $s_{45}$ plane (Fig. 3b) the shaded oval, provided that also

$$s_{12} \geqslant (m_3 + m_4 + m_5)^2. \tag{7}$$

3. The boundary of the physical region is given by a certain hypersurface $\Omega$ in the space of the five independent invariants, whose choice for the purposes of determining $\Omega$ is absolutely unimportant. In the preceding section $\Omega$ was determined by its projections on three spaces of the triplets of invariants: $s_{12}$, $s_{15}$, $s_{34}$; $s_{12}$, $s_{23}$, $s_{45}$; $s_{12}$, $s_{34}$, $s_{45}$. At that the invariant $s_{12}$ played a special role. To determine the physical region in channel 12, where $s_{12}$ is the square of the total energy, it was sufficient to require that Eqs. (1) and (2) be satisfied for those $p_k^2$ and $z_{ij}$ which are expressed in the simplest, standard manner in terms of the independent invariants chosen for that channel. In turn, $p_4^2$ and the remaining $z_{ij}$ are expressed in the same manner in terms of the $s_{kl}$ for a different

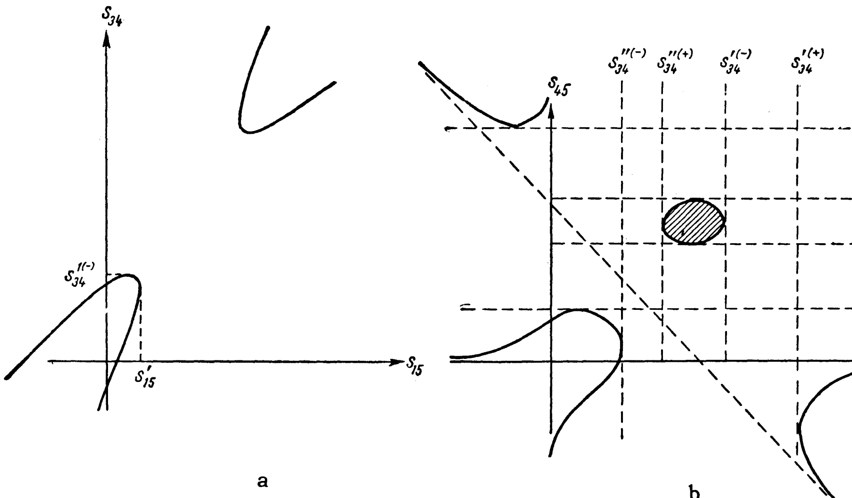

FIG. 3

choice of the independent invariants.

For example, $p_4^2 = (4s_{12})^{-1}\Lambda(s_{12}, m_4^2, s_{35})$, $s_{35} = (p_3 + p_5)^2$, and the equations $z_{14}^2 = 1$ and $z_{34}^2 = 1$ are respectively the Landau singularities equations for the three-point functions (see Figs. 2c, d). At that the new equations determine the same hypersurface $\Omega$, except in a different system of coordinates in the space of the five independent invariants. The transformation from one system of coordinates to another may be accomplished with the help of the five known relations between the various invariants:

$$s_{12} + s_{23} + s_{13} = s_{4\bullet} + m_1^2 + m_2^2 + m_3^2$$

(the remaining four relations are obtained from the above by cyclic permutations of the indices 1, 2, 3, 4, 5).

Having noted this fact, we chose for all crossed channels the same five invariants as the independent ones. In the Feynman diagram corresponding to the process in question the particles are divided into two groups: incoming and outgoing. If for a given crossed channel the invariant $s_{ijk...} = (p_i \pm p_j \pm p_k \pm \ldots)^2$ is formed out of the 4-momenta of particles from just one group then we shall refer to that invariant as being of the energy type; and if the particles belong to different groups then the invariant will be called of the momentum-transfer type.

It follows from the results of the preceding section that for the energy-type invariants one has in the physical region of the given channel

$$s_{ij} \geqslant (m_i + m_j)^2, \tag{8a}$$

and for the momentum-transfer-type

$$s_{ij} \leqslant (m_i - m_j)^2. \tag{8b}$$

Points of the strip $(m_i - m_j)^2 < s_{ij} < (m_i + m_j)^2$, consequently, do not belong in the physical region of any channel.

The distribution of the invariants between the energy and momentum-transfer types changes when the crossing symmetry operation transfers some one particle from one group to the other. By means of just such crossing operations one obtains ten different crossed channels for the five-point function (they have as the square of the total energy in the c.m.s. respectively the invariants with the subscripts 12, 13, 14, 15, 23, 24, 25, 34, 35, and 45). Consequently, the values of at least one of the independent invariants allowed in the physical regions of any two crossed channels are separated from each other by the forbidden strip. Consequently the physical regions of any two crossed channels of the five-point function have no common points in the space of the five independent invariants. (The degenerate case when some of the particles are photons is an exception.) This result is in full agreement with the well-known assertion for the four-point function: the physical regions of any two (out of the possible three) crossed channels have no points in common in the plane of the two independent invariants.

4. The last conclusion remains in force for any many-point function. It we do not take into account the "geometrical conditions" of Asribekov,[3] which lower the number of independent invariants of the n-point function for $n \geq 6$, then a trivial repetition of the preceding arguments will bring us to the assertion that the restrictions (8) remain in force for the two types of invariants in any channel. Since, as has been noted, the choice of independent invariants is at that immaterial, and cosines of angles are simplest expressed in terms of triple, quadruple, etc, invariants, it is convenient to use these latter when determining the boundaries of the physical regions. The role of the geometrical conditions reduces to the circumstance that only part of the "physical regions" of the crossed channels obtained with these conditions ignored will constitute the true physical regions; namely the part lying on the corresponding hypersurfaces whose equations are the geometrical conditions.

5. The above picture makes it possible to draw conclusions about the structure of dispersion relations for inelastic amplitudes. We consider a one-dimensional dispersion relation in the square of the total energy in the c.m.s. of some one channel (for example, in $s_{12}$ in channel 12). The remaining four independent invariants (the case of the five-point function) are fixed in the physical region of that channel. The antihermitian part of the amplitude consists of a sum of six terms. Each term may be interpreted as the contribution from the absorptive part of the amplitude of a definite crossed channel, since in any case it differs from zero only when the total energy in the c.m.s. of that channel is above threshold. For example, to the right side of the dispersion relation in $s_{12}$ contribute the absorptive parts of channels 12, 13, 14, 25, 35, and 45.

Recalling that the four independent invariants were fixed in the physical region of channel 12 we see that the value of at least one of them for every channel, except the 12 channel, will be separated by the forbidden strip from values that are physical in that channel. This means that the right side of the dispersion relation will require a not always trivial analytic continuation of absorptive parts from the values determined by the unitarity conditions in the physical region of the corresponding channel.

This circumstance, not present in the utilization of analytic properties of the four-point function, may, on the other hand, significantly simplify the situation with anomalous thresholds for the five-point function. The point is that, at least for some of the crossed channels, the analytic continuation in question is in the direction of decreasing invariants of the energy type in the given channel. It is known from perturbation theory studies [1] that the presence of anomalous thresholds is connected with "unstable" values of such invariants. Therefore at least a definite class of diagrams will contribute to the absorptive part in a manner free from anomalous cuts after analytic continuation.

6. In estimating the contribution to the absorptive part of partial amplitudes from different crossed channels, the correspondence of the equations of the boundary of the physical region to the Landau singularities of various three-point functions turns out to be useful. It is customary to take into account the contributions from the lowest few partial waves only. The choice of the angular dependence of the partial-wave decomposition depends on the angular momentum coupling scheme of the three particles. For the diagram of Fig. 1 one usually selects the dependence on $z_{15}$ and $z_{34}$ (or $z_{23}$): this is convenient when particles 1 and 5 are fermions. The angular momenta conjugate to the arguments of these cosines will be denoted by $J$ and $l$. Wishing to preserve the scheme of quantization for the crossed channels we must select, together with $z_{15}$, successively $z_{ij}$ with $(i,j) = (2,3), (2,4), (3,4)$. The contribution to the $l$-th partial wave from the $l'$-th partial wave in the crossed channel is then proportional to the integral over the absorptive part of the amplitude with the Legendre functions $Q_{l'}(z_{ij})$ and $Q_l(z_{34})$. The singularities of this integral arise both from the singularities of the absorptive part itself and from the branch points of the Legendre functions. The latter, when expressed in terms of the invariants, represent the Landau singularities for the corresponding three-point functions of Fig. 3.

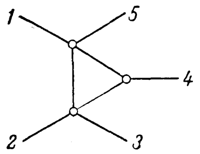

FIG. 4

7. If the stability conditions for the masses at each vertex of the original diagram of Fig. 1 are satisfied, then $s'_{15} \le s^0_{15}$, $s'_{23} \le s^0_{23}$ and, on the contrary, $s''^{(+)}_{34} \ge s^0_{34}$ or $s''^{(+)}_{45} \ge s^0_{45}$, where $s^0_{ij}$ stands for the normal threshold in the dispersion relation in $s_{ij}$. This means that in the physical region of the five-point function the "stability conditions" [1] are always fulfilled for the invariants $s_{15}$ and $s_{23}$ and always violated for $s_{34}$ or $s_{45}$. Consequently dispersion relations without complex cuts exist in the physical region of the five-point function only if its amplitude depends on the variables $s_{15}$ and $s_{23}$. An example of the corresponding diagram is given in Fig. 4. There is, incidentally, also the case when the dependence on the variables $s_{34}$ and $s_{45}$ separates from the dependence on the remaining variables.

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