DETERMINATION OF THE INTERACTION BETWEEN NUCLEONS FROM ELECTRON SCATTERING BY NUCLEI

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Submitted to JETP editor April 26, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **45**, 1218-1224 (October, 1963)

The scattering of fast electrons by nuclei is investigated. The nucleus is regarded as being a Fermi liquid which consists of two kinds of particles with arbitrary interactions. It is shown that the interaction of the particles in the nucleus modifies the scattering cross section. The formula for the cross section contains the spherical harmonics of the forward-scattering amplitude of identical and of nonidentical quasiparticles on the Fermi surface.

1. In the theory of the Fermi fluid it was shown by Landau [1] that for an infinite system consisting of one kind of particles without pairing the spectrum of single particle excitations in the vicinity of the Fermi surface is characterized by a single constant, the effective mass, and that the spectrum of two-particle excitations (zero-sound) and the reaction of the system to external fields are determined by a single function, Γ^{ω}, the forward scattering amplitude for quasiparticles near the Fermi surface. The latter depends only on spin variables and on the angle between the quasi-particle momenta and is practically determined by one or two terms of the expansion in Legendre polynomials. The coefficients of the Legendre expansions are phenomenological constants to be determined from experiment.

This theory was extended by Migdal [2] to the case of two kinds of particles, and further, by Migdal and Larkin [2] to the case of finite systems. Two branches of single-particle spectra appear (for the protons and the neutrons) and two functions Γ^{ω}, the scattering amplitudes for identical and different particles.

In the present paper it is investigated whether it is possible to determine the amplitudes Γ^{ω} from the inelastic scattering of electrons by the nuclear Coulomb field.

2. It is possible to write the differential inelastic scattering cross sections of a relativistic electron in the nuclear Coulomb field averaged over initial and summed over final electron spin and summed over all possible excited nuclear states s, in first Born approximation, in the form [4]

$$d\sigma_{in} = \frac{2e^4}{k^4}\frac{p_f}{p_i}(\mathbf{p}_i\mathbf{p}_f + \varepsilon_i\varepsilon_f + m_e^2)\, I(\mathbf{k}, \omega)\, do\, d\varepsilon_f, \qquad (1)$$

$$I(\mathbf{k}, \omega) \equiv \sum_{s\neq 0}\left|\int d\mathbf{r}\, e^{i\mathbf{k}\mathbf{r}} n_{s0}(\mathbf{r})\right|^2 \delta(E_s - E_0 - \omega). \qquad (2)$$

Here (p_i, ϵ_i) and (p_f, ϵ_f) are the initial and the final four-momenta of the electron, $(\mathbf{k}, \omega)$ is the four-momentum transfer, and $n_{s0}(\mathbf{r})$ is the matrix element of the proton density operator between the ground and the excited state of the nucleus.

3. The correlation function (2) can be expressed in terms of the imaginary part of the proton polarizability operator $\mathcal{P}$, which is determined by the relation (see [2])

$$n'(x) = \int \mathcal{P}(x, x')\, A(x')\, d^4x', \qquad (3)$$

where n' is the change of the ground state expectation value of the nuclear density under the influence of an external scalar field $A(x)$.

When the scalar field $A(x)$ is applied to the system, the Hamiltonian acquires an addition:

$$H = H_0 + H', \qquad H'(t') = \int d\mathbf{r}'\, n(x')\, A(x'). \qquad (4)$$

Then

$$n'(x) = \langle \tilde{n}(x)\rangle - \langle n(x)\rangle, \qquad (5)$$

where $\langle\ldots\rangle$ denotes the ground state expectation value, $\tilde{n}(x)$ is the density operator in the Heisenberg representation of the Hamiltonian H, and $n(x)$ is the same for the Hamiltonian H_0.

As is well known [4]

$$\tilde{n}(x) = S^{-1}(t)\, n(x)\, S(t), \qquad (6)$$

$$S(t) = T\exp\left\{-i\int_{-\infty}^{t} H'(t')\, dt'\right\}. \qquad (7)$$

Considering the field $A(x)$ to be small and limiting ourselves for the S matrix to two terms of the expansion of the exponential, we obtain from (4)—(7)

$$n'(x) = i\int_{-\infty}^{t} dt' \int d\mathbf{r}' \langle n(x')\, n(x) - n(x)\, n(x')\rangle A(x'). \tag{8}$$

We shall employ the well known relation

$$\int_0^\infty e^{ist}\,dt = \frac{i}{s+i\gamma} = i\mathrm{P}\frac{1}{s} + \pi\delta(s), \quad \gamma \to +0. \tag{9}$$

Going over to the Schrödinger representation for $n(\mathbf{r})$ and taking into account that the operator $\mathcal{P}$ depends only on the time interval $t - t'$, we easily obtain from (3) and (8) the expression

$$\mathcal{P}(\mathbf{r},\mathbf{r}',\omega) = \sum_s n_{0s}(\mathbf{r}')\, n_{s0}(\mathbf{r}) \times \left\{\frac{1}{\omega - E_s + E_0 + i\gamma} - \frac{1}{\omega + E_s - E_0 + i\gamma}\right\}. \tag{10}$$

The term $s = 0$ is absent from the sum over s. Comparing (2) and (10) we find

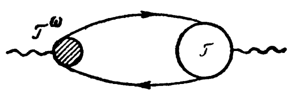
$$I(\mathbf{k},\omega) = -\frac{1}{\pi}\,\mathrm{Im}\int d\mathbf{r}\,d\mathbf{r}'\,e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\mathcal{P}(\mathbf{r},\mathbf{r}',\omega). \tag{11}$$

Here we took it into account that E_0 is the ground-state energy of the nucleus and only one δ-function will correspond to the process in which one quantum of frequency ω gets absorbed.

All the conclusions of this section remain valid also for nuclei which consist of two kinds of particles, if it is assumed that the field $A(x)$ acts only on the protons.

4. We shall consider a finite system without Cooper pairing correlations.

The polarizability operator $\mathcal{P}$ which describes the change of the density under the influence of an external scalar field [3], is given by the formula

$\mathcal{P} =$  $= (\mathcal{T}^\omega GG\mathcal{T})$ (12)

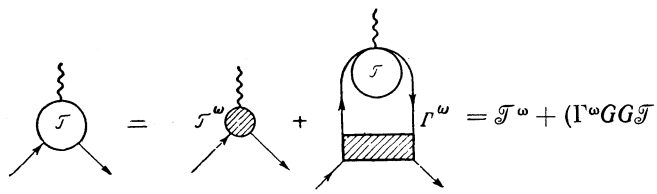
Here G is the pole part of the Green's function without pairing and the parentheses denote the necessary integrations.

As shown in [2,3] the vertex function $\mathcal{T}^\omega$ does not depend on the momenta of the particles and the field and it has in the coordinate representation the form

$$\mathcal{T}^\omega(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}) = a^{-1}\delta(\mathbf{r}_1 - \mathbf{r}_2)\,\delta(\mathbf{r} - \mathbf{r}_1), \tag{13}$$

where a is the Green's function renormalization.

The vertex function T for a scalar field [3] obeys the equation

 $= \mathcal{T}^\omega + (\Gamma^\omega GG\mathcal{T})$

where G is the pole part of the Green's function and Γ^ω is the forward scattering amplitude near the Fermi surface, introduced by Landau. [1] The latter depends only on the spin variables and on the angle between the momenta of the quasiparticles.

We limit ourselves to the spherical harmonics of the spinless part of the amplitude Γ^{ω} 1)

$$\Gamma^\omega(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3,\mathbf{r}_4) \cong \Gamma_0^\omega\,\delta(\mathbf{r}_1-\mathbf{r}_2)\,\delta(\mathbf{r}_3-\mathbf{r}_4)\,\delta(\mathbf{r}_1-\mathbf{r}_3) \tag{15}$$

and we shall attempt to obtain the vertex $\mathcal{T}$ in the form

$$\mathcal{T}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}') = \Phi(\mathbf{r}_1,\mathbf{r}')\,\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{16}$$

We now go to the functions $\varphi_\lambda(\mathbf{r})$ [5,3] which diagonalize the Green's function near the Fermi surface:

$$G(\mathbf{r},\mathbf{r}',\varepsilon) = \sum_{\lambda\lambda'} G_{\lambda\lambda'}(\varepsilon)\,\varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r}'),$$

where the pole term of the Green's function is

$$G_{\lambda\lambda'}(\varepsilon) = a\delta_{\lambda\lambda'}/(\varepsilon - \varepsilon_\lambda + i\delta). \tag{17}$$

We can then obtain from (12)—(17)

$$\Phi(\mathbf{r},\mathbf{r}',\omega) = a^{-1}\delta(\mathbf{r}-\mathbf{r}') + a^2\Gamma_0^\omega\sum_{\lambda\lambda'}\frac{n_\lambda - n_{\lambda'}}{\varepsilon_\lambda - \varepsilon_{\lambda'} - \omega - i\delta}\varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r})\int d\mathbf{r}_1\varphi^*_\lambda(\mathbf{r}_1) \times \Phi(\mathbf{r}_1,\mathbf{r}',\omega)\,\varphi_{\lambda'}(\mathbf{r}_1), \tag{18}$$

where $n_\lambda = 1$ for $\epsilon_\lambda < 0$ and $n_\lambda = 0$ for $\epsilon_\lambda > 0$ where ϵ_λ and $\epsilon_{\lambda'}$ are reckoned from the Fermi surface ϵ_0,

$$\mathcal{P}'(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{a\Gamma_0^\omega}\left[\Phi(\mathbf{r},\mathbf{r}',\omega) - \frac{1}{a}\delta(\mathbf{r}-\mathbf{r}')\right]. \tag{19}$$

Taking (18) and (19) into account, the correlation function (11) can be written as

$$I(\mathbf{k},\omega) = -\frac{1}{\pi}\frac{dn/d\mu}{f_0^\omega}\,\mathrm{Im}\int d\mathbf{r}\,e^{i\mathbf{k}\mathbf{r}}s(\mathbf{r},\mathbf{k},\omega), \tag{20}$$

where $s(\mathbf{r},\mathbf{k},\omega) \equiv a\int d\mathbf{r}'\,e^{-i\mathbf{k}\cdot\mathbf{r}'}\Phi(\mathbf{r},\mathbf{r}',\omega)$ obeys the integral equation

$$s(\mathbf{r},\mathbf{k},\omega) = e^{-i\mathbf{k}\mathbf{r}} + \int d\mathbf{r}'K(\mathbf{r},\mathbf{r}',\omega)\,s(\mathbf{r}',\mathbf{k},\omega) \tag{21}$$

with the kernel

$$K(\mathbf{r},\mathbf{r}',\omega) = \frac{f_0^\omega}{dn/d\mu}\sum_{\lambda\lambda'}\frac{n_\lambda - n_{\lambda'}}{\varepsilon_\lambda - \varepsilon_{\lambda'} - \omega - i\delta} \times \varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r})\,\varphi^*_\lambda(\mathbf{r}')\,\varphi_{\lambda'}(\mathbf{r}'). \tag{22}$$

1)It is easy to verify that the spin dependent part of the amplitude does not appear in the equations for the scalar vertex.

Here f_0^ω is a dimensionless amplitude which is independent on the spin operators [2] and which is given by $(dn/d\mu)\, a^2\Gamma_0^\omega \equiv f_0^\omega$, and μ is the chemical potential of the system. The equations for the scalar vertex and for the polarizability operator of a system consisting of two kinds of particles [2,3] have the form (14) and (12) if one considers the quantities $\mathcal{T}$, $\mathcal{T}^\omega$, Γ^ω, and G to be two-by-two matrices in isospin space. Here $\mathcal{T}$, $\mathcal{T}^\omega$, and G are diagonal while

$$\Gamma^\omega = \begin{pmatrix} \Gamma^\omega_{pp} & \Gamma^\omega_{pn} \\ \Gamma^\omega_{np} & \Gamma^\omega_{nn} \end{pmatrix}$$

has off-diagonal elements.

For a scalar field acting only on the protons, the bare vertices (see [2]) in the momentum representation are

$$\mathcal{T}^0_p = 1, \qquad \mathcal{T}^0_n = 0;$$

Then it follows from gauge invariance [2] that

$$\mathcal{T}^\omega_p = 1/a, \qquad \mathcal{T}^\omega_n = 0 \tag{23}$$

and thus

$$\mathcal{P} = (\mathcal{T}^\omega_p\,(GG)_p\,\mathcal{T}_p), \tag{24}$$

$\Gamma^\omega_{pn} = \Gamma^\omega_{np}$ and we can furthermore assume that owing to isospin invariance

$$\Gamma^\omega_{pp} = \Gamma^\omega_{nn}, \qquad (GG)_p \approx (GG)_n. \tag{25}$$

This corresponds to the neglect of the difference in the velocities of protons and neutrons at the Fermi surface.

With these relations we can obtain in analogy to the case of one kind of particles

$$I(\mathbf{k}, \omega) = -\frac{1}{2\pi a^2}\,\mathrm{Im}\int d\mathbf{r}\, e^{i\mathbf{kr}}\left[\frac{s_+(\mathbf{r}, \mathbf{k}, \omega)}{(\Gamma_0^\omega)_+} + \frac{s_-(\mathbf{r}, \mathbf{k}, \omega)}{(\Gamma_0^\omega)_-}\right], \tag{26}$$

where $s_\pm(\mathbf{r}, \mathbf{k}, \omega)$ obey the equation

$$s_\pm(\mathbf{r}, \mathbf{k}, \omega) = e^{-i\mathbf{kr}} + \int d\mathbf{r}'\, K_\pm(\mathbf{r}, \mathbf{r}', \omega)\, s_\pm(\mathbf{r}', \mathbf{k}, \omega) \tag{27}$$

with the kernels

$$K_\pm(\mathbf{r}, \mathbf{r}', \omega) = a^2(\Gamma_0^\omega)_\pm \sum_{\lambda\lambda'} \frac{n_\lambda - n_{\lambda'}}{\varepsilon_\lambda - \varepsilon_{\lambda'} - \omega - i\delta}\,\varphi_\lambda(\mathbf{r})\,\varphi^*_{\lambda'}(\mathbf{r}) \times \varphi^*_\lambda(\mathbf{r}')\,\varphi_{\lambda'}(\mathbf{r}') \tag{28}$$

and where

$$(\Gamma_0^\omega)_\pm \equiv (\Gamma^\omega_{pp})_0 \pm (\Gamma^\omega_{pn})_0.$$

5. For the case when

$$kR \gg 1 \text{ and } \omega \gg v_0/R \sim \varepsilon_0/A^{1/3}, \tag{29}$$

where R is the nuclear radius, A is the mass number, and v_0 is the velocity of quasi-particles at the Fermi surface, we can consider the system to be infinite nuclear matter. Then the states λ are characterized by the momentum and spin of the quasi-particles and the wave functions $\varphi_\lambda(\mathbf{r})$ are plane waves.

The kernel (22) or (28) of the integral equation can then be written as

$$K(\mathbf{r}, \mathbf{r}', \omega) = \frac{f_0^\omega}{V}\, e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\int \frac{do}{4\pi}\,\frac{\mathbf{k}\mathbf{v}_0}{\omega - \mathbf{k}\mathbf{v}_0 + i\delta}$$
$$= -\frac{f_0^\omega}{V}\left[1 - \frac{1}{2}\xi\ln\left|\frac{1+\xi}{1-\xi}\right| + i\frac{\pi}{2}\xi\right]e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}, \tag{30}$$

where

$$f_0^\omega \equiv a^2 p_0 m^*\pi^{-2}\Gamma_0^\omega, \qquad \xi \equiv \omega/kv_0,$$

p_0 is the Fermi-momentum, m^* is the effective mass of the quasi-particles, and V is the volume of the system.

As expected, in an infinite system the kernel K depends on the difference $\mathbf{r} - \mathbf{r}'$. The integral equation (21) with the kernel (30) becomes an algebraic equation (see [2]) which has the solution

$$s(\mathbf{r}, \mathbf{k}, \omega) = e^{-i\mathbf{kr}}\left\{1 + f_0^\omega\left[1 - \frac{1}{2}\xi\ln\left|\frac{1+\xi}{1-\xi}\right| + i\frac{\pi}{2}\xi\right]\right\}^{-1}. \tag{31}$$

The correlation function (20) then has the form

$$I(\mathbf{k}, \omega) = \frac{1}{2}\pi^{-2}Vp_0m^*F(\xi), \tag{32}$$

$$F(\xi) = \xi\left\{\left[1 + f_0^\omega\left(1 - \frac{1}{2}\xi\ln\left|\frac{1+\xi}{1-\xi}\right|\right)\right]^2 + (f_0^\omega)^2\frac{\pi^2}{4}\xi^2\right\}^{-1}. \tag{33}$$

The function $F(\xi)$ has the form (33) for $|\xi| \le 1$. If $\xi \to 1$ from the left then $F(\xi) \to +0$; $F(\xi) = 0$ for $\xi > 1$, since then the imaginary part in the integral (30) vanishes. The latter characteristic of $F(\xi)$ has the following physical meaning: if $\xi > 1$, i.e., $\omega > kv_0$, the external field cannot create a real particle and hole, since $\omega = \mathbf{k}\cdot\mathbf{v}_0$ for a real transition. Therefore the imaginary part of the polarizability operator (12) which corresponds to real transitions, vanishes.

One can obtain the correlation function (20) of the ideal Fermi gas from (32) and (33) if one puts $m^* = m$, where m is the nucleon mass, and $f_0^\omega = 0$, i.e.,

$$I_0(\mathbf{k}, \omega) = \frac{1}{2}\pi^{-2}Vp_0m\xi.$$

Thus the nucleon interaction in the nucleus changes considerably the correlation function (20) and, consequently, also the cross section of reaction (1).

As one can easily verify, the correlation function (26) for two kinds of particles has the form

$$I(\mathbf{k}, \omega) = \frac{1}{4}\pi^{-2}Vp_0m^* [F_+(\xi) + F_-(\xi)], \quad (34)$$

where $F_\pm(\xi)$ are given by (33) if one changes f_0^{ω} to $(f_0^{\omega})_\pm \equiv a^2m^*p_0\pi^{-2}(\Gamma_0^{\omega})_\pm$.

The method employed for the evaluation of the vertex function and of the polarizability operator is applicable for small perturbations of a system with a large number of particles A. The four-momentum transfer $q = (\mathbf{k}, \omega)$ therefore is limited to

$$|\mathbf{k}| \ll p_0 (\sim 300 \text{ MeV}/c); \ \omega \ll \varepsilon_0 (\sim 40 \text{ MeV}), \quad (35)$$

where p_0 and ϵ_0 are the Fermi momentum and energy respectively.

From (29) and (35) we get $\epsilon_0 A^{-1/3} \ll kv_0 \ll \epsilon_0$. We shall assume that the electron is highly relativistic before and after the scattering, i.e., $\epsilon_i \gg m_e c^2$, $\epsilon_i \gg \omega$. Then the conditions (29) and (35) impose limitations on ϑ, the scattering angle of the electron

$$A^{-1/3} p_0 c/\varepsilon_i \ll \vartheta \ll p_0 c/\varepsilon_i. \quad (36)$$

Here one can take $k \approx 2p_i \sin(\vartheta/2)$. Then the differential cross section (1) together with (34) can be written as

$$\frac{d\sigma_{in}}{do\, d\varepsilon_f} = \frac{e^4}{8\pi^2}\frac{V p_0 m^*}{\varepsilon_i^2}\frac{\cos^2(\vartheta/2)}{\sin^4(\vartheta/2)} \cdot \frac{1}{2}[F_+(\xi) + F_-(\xi)], \quad (37)$$

where $F_\pm(\xi)$ is given by (33) if one changes f_0^{ω} to $(f_0^{\omega})_\pm$.

In this way the amplitudes $(f_0^{\omega})_\pm \equiv \{f_{pp}^{\omega}\}_0 \pm \{f_{pn}^{\omega}\}_0$, which characterize the interaction between the nucleons in the nucleus, can be determined from inelastic electron scattering on nuclei when the excited levels lie close to the Fermi surface.

One should expect that the higher harmonics in the amplitudes $(f^{\omega})_\pm$ will give small corrections to the obtained result. In our approximation of infinite nuclear matter, pairing is inessential because the energy transfer must obey the condition (29):

$$\omega \gg \varepsilon_0 A^{-1/3} \gg \Delta (\sim 1 \text{ MeV}),$$

where Δ is the energy gap in the single particle spectrum.

In order to improve the obtained results one must consider the finite size of the nucleus and take into account the pairing correlations.[6,3]

In conclusion the author expresses his deep gratitude to A. B. Migdal for guidance in this work, and also thanks B. Medvedev who participated in the initial stages of the work and É. Sapershteĭn for a number of valuable comments.

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Translated by M. Danos
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