THE SPACE-TIME STRUCTURE OF THE RELATIVISTIC SCATTERING MATRIX

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A general, relativistically invariant method is proposed for the construction of the S matrix in field theory in which T products are not employed. Based on this method two different representations of the S matrix are constructed. It is shown that the problem of construction of the S matrix may be reduced to the solution of a certain set of integral equations. The equivalence of the method of construction here proposed with the conventional field theory formulation is proven. The unitarity of the S matrix is demonstrated explicitly.

1. INTRODUCTION

THE conventional method of construction of the S matrix in quantum field theory, which makes use of the T product

$$S = T\exp\int L(x)d^4x \qquad (1.1)$$

[here and in the following we simplify the notation by assuming the interaction Lagrangian density to be an antihermitian operator: $L^+(x) = -L(x)$], contains the relativistically not invariant time ordering of the operators. The relativistic invariance of the T products is insured by the condition

$$[L(x), L(y)] = 0 \text{ for } (x-y)^2 < 0, \qquad (1.2)$$

which reflects the property of microcausality in field theory. However the condition (1.2) does not explicitly enter into the definition of the T product, and this makes it difficult to study the "causal" properties of the S matrix written in the form (1.1). It is the purpose of the present work to construct other representations of the S matrix, in which the "causal" properties of the theory are more explicitly displayed.

In the following we shall make great use of the concept of a configuration, understanding by a configuration

$$A^n = \{x_1, x_2, \ldots, x_n\}$$

a set of n points of 4-space. Let us note that the configuration is defined by the choice of the points x_i independent of their order. Sometimes for notational convenience we shall make use (purely formally) of the "empty" configuration A^0, which contains no points at all.

The difficulty in the relativistic definition of the T product lies in the fact that the configuration A^n cannot be fully ordered in a natural relativistically invariant way. The T product

$$T\{L(x_1)L(x_2)\ldots L(x_n)\} \equiv T\{L(A^n)\} \qquad (1.3)$$

is a symmetric function of the coordinates x_i and therefore depends only on the configuration A^n. This circumstance is reflected in the manner in which the quantity (1.3) is written.

Let us consider the n-th term in the expansion of the S matrix, Eq. (1.1):

$$S_n = \frac{1}{n!}\int T\{L(A^n)\}\,d\Gamma_n \qquad (1.4)$$

($d\Gamma_n = d4x_1 \ldots d4x_n$ denotes the 4n-dimensional volume element). The contribution of each configuration A^n to the integral (1.4) is repeated n! times. We may therefore write symbolically S_n in the form

$$S_n = \sum_{(A^n)} T\{L(A^n)\}. \qquad (1.5)$$

The "summation" in Eq. (1.5) is over all different configurations A^n.

Expressions of the type (1.5) will be called configurational sums. Each configuration is defined by certain discrete parameters, which characterize the configuration type, and certain continuous parameters—the positions of the points x_i that vary within a specified region (see Sec. 3). Consequently the summation in (1.5) also includes the integration over certain continuous variables that define the configuration. In concrete cases, the integration may be extended over all of space without difficulty by making use of the symmetry of the integrand function. With the help of the relation (1.5) it is easy to express the complete S matrix, Eq. (1.1), in the form of the configurational sum

$$S = \sum_{(A)} T\{L(A)\}. \tag{1.6}$$

The sum (1.6) extends over all possible configurations.

2. THE STRUCTURE OF A CONFIGURATION CONSIDERED AS A PARTIALLY ORDERED SET

For points of 4-space one may define a relativistically invariant partial ordering with the help of the relation

$$x > y, \text{ if } (x-y)^2 \geqslant 0, \quad x_0 - y_0 > 0. \tag{2.1}$$

When x and y satisfy the condition (2.1) we say that x is later than y or that x follows y. It is clear that the ordering condition (2.1) is transitive, i.e. if $x > y$, $y > z$, then $x > z$. The condition (2.1) is not satisfied for an arbitrary pair of points x, y (and thus constitutes partial ordering).

It is also convenient to define the relation

$$x \sim y, \text{ if } (x-y)^2 < 0. \tag{2.2}$$

If $x > y$ then the event x is causally connected with the event y, and if $x \sim y$ then the events x and y are not causally connected with each other.

The relation (2.1) partially orders any configuration of points A. This partial ordering makes it possible to define the T product $T\{L(A)\}$: if $x > y$ then the operator $L(x)$ stands to the left of the operator $L(y)$; and if $x \sim y$ then the order of the operators $L(x)$ and $L(y)$ in the T product is irrelevant. Such a definition of the T product, although fully relativistically invariant, is of little use for practical purposes. In order to make it more effective it is necessary to study in more detail the structure of an arbitrary configuration from the point of view of the ordering relation (2.1).

Let us consider first certain special configurations.

Layer—a configuration $C^{\alpha} = \{x_1, \ldots x_{\alpha}\}$ no points of which are causally connected with each other, i.e., every pair of points satisfies the relation $x_i \sim x_j$. In other words all points of a layer may be distributed over some space-like hypersurface σ.

Chain—a configuration $B^n = \{x_1, \ldots x_n\}$ whose points satisfy the condition $x_1 > x_2 > \ldots > x_n$. The number n will be called the length of the chain.

Let us introduce now certain ordering relations for point configurations which in a certain sense generalize the relations (2.1) and (2.2). Let A and B be two configurations. We shall denote by a an arbitrary point from the configuration A, by b an arbitrary point of the configuration B. Let us define the following relations between the configurations A and B:

$$A \sim B, \quad \text{if} \quad a \sim b, \tag{2.3}$$

$$A \gtrsim B, \quad \text{if} \quad a > b, \quad \text{or} \quad a \sim b. \tag{2.4}$$

Relations (2.3) and (2.4) are not transitive. Let us define now two transitive relations:

$$A \overset{\text{I}}{>} B, \tag{2.5}$$

if $A \gtrsim B$ and every point b proceeds at least one point a;

$$A \overset{\text{II}}{>} B, \tag{2.6}$$

if $A \gtrsim B$ and each point a follows at least one point b.

The transitive property of relations (2.5) and (2.6) is simply proved. Let, for example, $A \overset{\text{I}}{>} B$, $B \overset{\text{I}}{>} C$, then $A \overset{\text{I}}{>} C$. First of all $A \gtrsim C$, because if that were not true then there would exist a pair of points a and c satisfying the condition $c > a$. However since $B \overset{\text{I}}{>} C$ one can find a point $b > c$, and consequently $b > a$, but that is impossible in view of the condition $A \overset{\text{I}}{>} B$. Let us consider next an arbitrary point c from C. Since $B \overset{\text{I}}{>} C$ one can find a point $b > c$, and since $A \overset{\text{I}}{>} B$ one can find a point $a > b$. That point a satisfies the condition $a > c$. Thus the configurations A and C satisfy all the conditions of the definition (2.5). One proves analogously the transitivity of the definition (2.6).

The ordering relations (2.5) and (2.6) make it possible to consider any system of configurations as a partially ordered set. At that the orderings according to (2.5) and (2.6) are in general different from each other.

If A and B are two configurations then their sum A + B is defined as the configuration obtained from the union of the points entering into A and B. The definition of the sum is generalized in a natural way to an arbitrary finite number of configurations.

The fundamental result of the present section consists of the following theorem on decomposition.

An arbitrary configuration A may be expressed in two (generally speaking distinct) ways as a sum of layers. The first decomposition

$$A = C_1 + C_2 + \ldots + C_k \tag{2.7}$$

satisfies the condition

$$C_1 \overset{I}{>} C_2 \overset{I}{>} \ldots \overset{I}{>} C_k; \tag{2.8}$$

the second decomposition

$$A = C_1' + C_2' + \ldots + C_k' \tag{2.9}$$

satisfies the condition

$$C_1' \overset{II}{>} C_2' \overset{II}{>} \ldots \overset{II}{>} C_k'. \tag{2.10}$$

The decompositions (2.7) and (2.9) are uniquely determined by the configuration A. The number of terms in these decompositions is the same and equal to the maximum length of a chain contained in A.

The proof of this assertion is carried out as follows. We extract from the configuration A a set of points C_1 for which there are no later points [in the sense of (2.1)] in A. Obviously C_1 constitutes a layer. From the remaining points we extract the set of points C_2, for which there are no later points in the configuration A, excepting the points C_1. It is clear that every point from C_2 precedes some point from C_1. Consequently $C_1 \overset{I}{>} C_2$. Continuing with this process we obtain the decomposition (2.7). The decomposition (2.9) is constructed in an analogous fashion except that one starts with the lowest layer C_k', defined as that set of points from A that have no points preceding them. The uniqueness of the decompositions (2.7) and (2.9) follows directly from the method of their construction. It is obvious that each point of the layer C_k in the decomposition (2.7) is the terminal point of a chain of length k, and each point of the layer C_1' in the decomposition (2.9) is the initial point of a chain of length k. The configuration A can not contain chains that are longer because if that were the case then not all of the terms in the decompositions (2.7) and (2.9) could be layers.

In order to get a better idea of the structure of a configuration one may make use of diagrams. An example of a diagram is shown in Fig. 1. The points of the configuration are represented by points in the plane. If two points satisfy the relation a > b then they are connected with a line directed from b to a. At that if a > b > c, then we omit the line ca since the point c is connected with the point a by the line cba. In Fig. 1 the layers corresponding to the decomposition (2.7) are shown as solid lines, and the layers of the decomposition (2.9) as dashed lines (these lines should not be confused with the directed lines that define the ordering!).

Let us note one more important property of configurations which is easily seen in Fig. 1. A

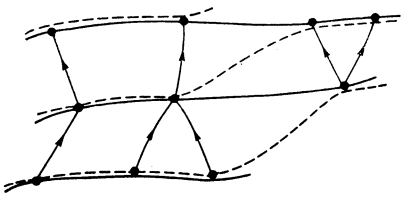

FIG. 1

configuration will be called connected if it cannot be decomposed into two parts that are not causally joined. A disconnected configuration can be decomposed into a number of connected components, with the various components no longer causally related to each other. The configuration shown in Fig. 1 is decomposable into three connected components.

3. CHARACTERISTIC FUNCTIONS

For a quantitative expression of the structural relations discussed in Sec. 2 it is convenient to introduce the so called characteristic functions. In general a characteristic function of some property (relation) is defined as a function $f(x_1, \ldots x_n)$ which is equal to 1 when all its arguments x_i satisfy the property in question, and equal to zero otherwise. Therefore in defining one or another characteristic function it is only necessary to specify the range of variation of its arguments for which the function is equal to 1.

In the following we shall need the following characteristic functions, which depend on one or two configurations and that correspond to the relations defined in Sec. 2:

$$\Lambda_0(A) = 1, \quad \text{if} \quad A \text{ is a layer} \tag{3.1}$$
$$\lambda_0(A,B) = 1, \quad \text{if} \quad A \sim B, \tag{3.2}$$
$$\Lambda(A,B) = 1, \quad \text{if} \quad A \gtrless B, \tag{3.3}$$
$$\Lambda_+^I(A,B) = 1, \quad \text{if} \quad A \overset{I}{>} B, \tag{3.4}$$
$$\Lambda_+^{II}(A,B) = 1, \quad \text{if} \quad A \overset{II}{>} B. \tag{3.5}$$

A large number of different relations holds between the functions (3.1)—(3.5). For example, directly from the definition of layer and the relation (2.3) we obtain for the function (3.1) the relation

$$\Lambda_0(A+B) = \Lambda_0(A)\Lambda_0(B)\lambda_0(A,B). \tag{3.6}$$

A function of a configuration will be called multiplicative if it satisfies the condition

$$f(A+B) = f(A)\,f(B). \tag{3.7}$$

In the Table are shown the multiplicative properties of the characteristic functions (3.2)—(3.5) that follow directly from their definitions. In addition there are given in the third and fourth

$f(A,B)$	Multiplicative in	$f(A^0,A^n)$	$f(A^n,A^0)$
$\lambda_0(A,B)$	A,B	1	1
$\Lambda(A,B)$	A,B	1	1
$\Lambda_+^{I}(A,B)$	B	δ_{n0}	1
$\Lambda_+^{II}(A,B)$	A	1	δ_{n0}

columns of the Table the values of the functions in the case when one of the arguments stands for the "empty" configuration A^0. (We use A^n to denote a configuration of n points.) It follows from Eq. (3.7) that for any multiplicative function $f(A^0) = 1$. The remaining values of the functions are obtained from Eqs. (3.13) and (3.14).

The characteristic functions (3.1)—(3.5) can be expressed in terms of the elementary characteristic functions corresponding to the relations (2.1) and (2.2). The elementary characteristic functions depend on two points x, y and are defined by the relations

$$\lambda_+(x, y) = 1, \text{ if } x > y, \tag{3.8}$$
$$\lambda_0(x, y) = 1, \text{ if } x \sim y. \tag{3.9}$$

The explicit form of these expressions is immediately clear from the definition of the functions (3.1)—(3.5) and from their multiplicative properties given in the Table:

$$\Lambda_0(A) = \prod \lambda_0(a, a'). \tag{3.10}$$

In Eq. (3.10) the product extends over all different pairs of points (a, a') from A:

$$\lambda_0(A, B) = \prod_a \prod_b \lambda_0(a, b), \tag{3.11}$$
$$\Lambda(A, B) = \prod_a \prod_b \{\lambda_+(a, b) + \lambda_0(a, b)\}, \tag{3.12}$$
$$\Lambda_+^{I}(A, B) = \prod_b \{\Lambda(A, b) - \lambda_0(A, b)\}, \tag{3.13}$$
$$\Lambda_+^{II}(A, B) = \prod_a \{\Lambda(a, B) - \lambda_0(a, B)\}. \tag{3.14}$$

Sometimes representations of a different form are useful, which express the functions in terms of sums over all possible decompositions of the given configuration. We give an example of such a representation for the function (3.3):

$$\Lambda(A, B) = \sum_{B'+B''=B} \Lambda_+^{I}(A, B')\,\lambda_0(A, B''). \tag{3.15}$$

The sum in Eq. (3.15) extends over all possible decompositions of the configuration B into two configurations B′ + B″, including the cases when one of the terms B′ or B″ is the "empty" configuration.

To prove Eq. (3.15) we make use of the relation (3.13) for the case when B consists of one point b. We get

$$\Lambda_+^{I}(A, b) = \Lambda(A, b) - \lambda_0(A, b)$$

or

$$\Lambda(A, b) = \Lambda_+^{I}(A, b) + \lambda_0(A, b). \tag{3.16}$$

From Eq. (3.16), making use of multiplicativeness, we find

$$\Lambda(A, b) = \prod_b \{\Lambda_+^{I}(A, b) + \lambda_0(A, b)\}. \tag{3.17}$$

Relation (3.15) follows by opening the bracket on the right side of Eq. (3.17) and making use of the multiplicativeness of the functions Λ_+^{I} and λ_0 in the argument B.

The main purpose of this Section is to produce an explicit representation of the quantity $T\{L(A)\}$. By making use of the decomposition theorem (Sec. 2) we may express an arbitrary configuration A in the form of a sum of layers of the type (2.7) or (2.9). The following construction applies equally to either representation. It is clear that in $T\{L(A)\}$ there will appear on the left operators $L(x)$ for points x that belong to the layer C_1 (in arbitrary order), then those for points that belong to the layer C_2, etc.

We shall give an expression for $T\{L(A)\}$ not for a given specific configuration A, but right away for an entire type of configurations A_k which according to the decomposition theorem, are decomposable into k layers. The number of points in such configurations can be arbitrary but no less than k. All configurations of the type A_k may be obtained by the following procedure. Consider a set of k layers $C_1, \ldots C_k$ satisfying the conditions

$$C_1 > C_2 > \ldots > C_k. \tag{3.18}$$

The sign $>$ in (3.18) stands for either $\overset{I}{>}$ or $\overset{II}{>}$. Accordingly we shall here and in the following sometimes use the function $\Lambda_+(A, B)$ (without an upper index) with the understanding that it stands for either the function Λ_+^{I} of Eq. (3.4) or the function Λ_+^{II} of Eq. (3.5). Wherever we write the function Λ_+ the corresponding formula is valid in both of the indicated cases.

It is clear that if condition (3.18) is satisfied then the configuration

$$A = C_1 + C_2 + \ldots + C_k \tag{3.19}$$

belongs to the type A_k and, conversely, all configurations of the type A_k can be constructed in this way. The characteristic of the condition (3.18) is equal to

$$\Lambda_0(C_1)\,\Lambda_+(C_1, C_2)\,\Lambda_0(C_2) \ldots \Lambda_0(C_k). \tag{3.20}$$

If the layers $C_1, \ldots C_k$ are treated as independent variables then the expression (3.20) will be the characteristic function of the configuration of the type A_k.

Let us define the operator $L(C)$ as depending on an arbitrary layer C:

$$L(C) = \Pi L(x). \tag{3.21}$$

The product in Eq. (3.21) extends over all points x of the layer C, at that the order of the factors in Eq. (3.21) is immaterial in view of the commutation relations (1.2). Taking into account the preceding remarks we may write the quantity $T\{L(A_k)\}$ in the form

$$T\{L(A_k)\} = L(C_1)\,\Lambda_+(C_1, C_2)\,L(C_2)\ldots L(C_k). \tag{3.22}$$

Let us emphasize again that the layers $C_1, \ldots C_k$ in expression (3.22) are treated as independent variables and that, according to Eq. (3.20), the quantity (3.22) is different from zero only if the sum (3.19) forms a configuration of type A_k. In the construction to follow the relation (3.22) will play a fundamental role.

To conclude this section we shall give two more specialized relations for characteristic functions, which will be used in Sec. 8. Let us consider two layers C^α and C^β (in the following we shall denote by C^α a layer containing α points). Let us decompose the layer C^α in all possible ways into the sum of two sublayers: $C^\alpha = C^\kappa + C^\lambda$. One of the numbers κ or λ may, in particular, be equal to zero. Let us analogously decompose the layer $C^\beta = C^\mu + C^\nu$. The relations that interest us are of the form

$$\sum_{C^\kappa + C^\lambda = C^\alpha} \sum_{C^\mu + C^\nu = C^\beta} \Lambda_+^I(C^\kappa, C^\nu)\,\Lambda_+^I(C^\mu, C^\lambda)\,\lambda_0(C^\kappa, C^\mu) = 1, \tag{3.23}$$

$$\sum_{C^\kappa + C^\lambda = C^\alpha} \sum_{C^\mu + C^\nu = C^\beta} (-1)^\kappa \Lambda_+^I(C^\kappa, C^\mu)\,\Lambda_+^I(C^\lambda, C^\nu)\lambda_0(C^\kappa, C^\nu) = \delta_{\alpha 0}\delta_{\beta 0}. \tag{3.24}$$

It is relevant to the validity of these relations that C^α and C^β are layers and not some arbitrary configurations. For the proof of Eq. (3.23) let us note that the configuration $C^\alpha + C^\beta$ cannot contain a chain of length greater than two. Hence $C^\alpha + C^\beta$ is either a layer or can be decomposed into two layers. In the first case the only term in Eq. (3.23) that is different from zero (and equal to unity) is the term corresponding to $\lambda = \nu = 0$. In the general case the situation corresponding to an arbitrary relative orientation of the layers C^α and C^β is illustrated in Fig. 2a. In this case

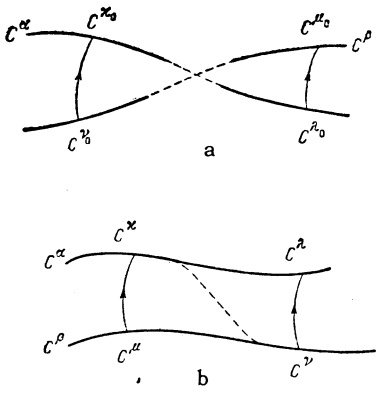

FIG. 2

there stands out unambiguously from the layer C^α the sublayer C^{λ_0} consisting of all points x from C^α, for which there exist later points in the layer C^β. We may express the layer C^α in the form of a sum: $C^\alpha = C^{\lambda_0} + C^{\kappa_0}$. Analogously, $C^\beta = C^{\nu_0} + C^{\mu_0}$.

It is easy to show that the relations $C^{\kappa_0} \overset{I}{>} C^{\nu_0}$, $C^{\mu_0} \overset{I}{>} C^{\lambda_0}$, $C^{\kappa_0} \sim C^{\mu_0}$ must hold, or, what is the same, that the equality

$$\Lambda_+^I(C^{\kappa_0}, C^{\nu_0})\,\Lambda_+^I(C^{\mu_0}, C^{\lambda_0})\,\lambda_0(C^{\kappa_0}, C^{\mu_0}) = 1$$

must hold. The validity of this equality follows from the fact that if it were violated then one could construct out of the configuration $C^\alpha + C^\beta$ a chain of length three, or chains of length two involving points belonging to one layer only, which is impossible. Thus we have shown that in the sum (3.23) one term is always equal to 1. In view of the uniqueness of the definition of the sublayers C^{λ_0} and C^{ν_0} all remaining terms in the sum on the left of Eq. (3.23) vanish, and this completes the proof of this relation.

The proof of Eq. (3.24) proceeds somewhat differently. The corresponding situation is illustrated in Fig. 2b. Applying the approach used in proving Eq. (3.15) and taking into account the multiplicativeness properties we may express the inner sum in Eq. (3.24) in the form

$$\sum_{C^\mu + C^\nu = C^\beta} \Lambda_+^I(C^\kappa, C^\mu)\,\Lambda_+^I(C^\lambda, C^\nu)\,\lambda_0(C^\kappa, C^\nu) = \prod_x \{\Lambda_+^I(C^\kappa, x) + \Lambda_+^I(C^\lambda, x)\,\lambda_0(C^\kappa, x)\}. \tag{3.25}$$

The product in Eq. (3.25) extends over all points of the layer C^β. But it is easy to show that if the layer $C^\alpha = C^\kappa + C^\lambda$, then for an arbitrary point x the equality

$$\Lambda_+^I(C^\kappa, x) + \Lambda_+^I(C^\lambda, x)\,\lambda_0(C^\kappa, x) = \Lambda_+^I(C^\alpha, x) \tag{3.26}$$

is valid.

It is relevant for the validity of Eq. (3.26) that the configuration C^α is a layer. In that case the function $\Lambda_+^I(C^\alpha, x) = 1$ if and only if each point x precedes at least one point from the layer C^α This is clear from the fact that if x precedes some one point from C^α, then x cannot follow any other point from C^α either. Utilizing this comment and taking into account the fact that no two terms on the left side of Eq. (3.26) can simultaneously equal unity we easily verify the validity of Eq. (3.26). Making use of Eq. (3.26) we may express the sum (3.25) in the form $\Lambda_+^I(C^\alpha, C^\beta)$ and then Eq. (3.24) takes on the form

$$\sum_{C^\varkappa + C^\lambda = C^\alpha} (-1)^\varkappa \Lambda_+^I(C^\alpha, C^\beta) = \Lambda_+^I(C^\alpha C^\beta) \sum_{\varkappa=0} (-1)^\varkappa \binom{\alpha}{\varkappa}$$

$$= \delta_{\alpha 0}\Lambda_+^I(C^0, C^\beta) = \delta_{\alpha 0}\delta_{\beta 0}.$$

We have used here known properties of the binomial coefficients $\binom{\alpha}{\varkappa}$ and the value of the function $\Lambda_+^I(C^0, C^\beta)$ given in the Table.

4. FIRST REPRESENTATION OF THE S MATRIX

We shall obtain explicit expressions for the S matrix starting from the expression for it in terms of a configurational sum, Eq. (1.6). These expressions will differ by the method used to classify the configurations. Let us note that the conventional method of representing the S matrix in the form of Eq. (1.1) corresponds to the classification of configurations by the number of points and is most inconvenient since for an n-point configuration A^n no simple way is known for representing the quantity $T\{L(A^n)\}$.

In Sec. 3 we discussed the classification of configurations by types A_k. The configurations of type A_k, containing k layers, may be decomposed into subtypes $A_k(\alpha_1, \ldots \alpha_k)$, where α_1 denotes the number of points in the upper layer C_1, etc. Making use of this classification we may rewrite the configurational sum (1.6) in the form

$$S = 1 + \sum_{\alpha=1}^{\infty} \sum T\{L(A_1(\alpha))\}$$

$$+ \sum_{\alpha,\beta=1}^{\infty} \sum T\{L(A_2(\alpha, \beta))\} + \ldots . \qquad (4.1)$$

The inner sums in the decomposition (4.1) symbolize integration over points of the configuration belonging to the given type. The corresponding quantities $T\{L(A)\}$ are defined by Eq. (3.22).

Taking into account that the expression (3.22) is symmetric with respect to the points within any one layer and does not possess any higher symmetry, we may extend the integration in Eq. (4.1) to the entire space and introduce at the same time compensating factorials. The S matrix, Eq. (4.1), takes on the form

$$S = 1 + \sum_{\alpha=1}^{\infty} \frac{1}{\alpha!} \int L(C^\alpha)\, d\Gamma_\alpha$$

$$+ \sum_{\alpha,\beta=1}^{\infty} \frac{1}{\alpha!\,\beta!} \int L(C^\alpha)\, \Lambda_+(C^\alpha, C^\beta)\, L(C^\beta)\, d\Gamma_\alpha\, d\Gamma_\beta + \ldots . \qquad (4.2)$$

Let us remark that the series, Eq. (4.2), is far from being an expansion in powers of the coupling constant. Additional transformations and simplifications of Eq. (4.2) will be discussed in Sec. 6.

5. SECOND REPRESENTATION OF THE S MATRIX

Let us consider another representation of the S matrix based on a classification of configurations by their connectedness. Let us denote by $A^{(q)}$ the class of configurations that can be broken up into q connected components (see Sec. 2). The sum (1.6) may be written in the form

$$S = 1 + \sum_{q=1}^{\infty} \sum_{A^{(q)}} T\{L(A^{(q)})\}. \qquad (5.1)$$

Let us denote by $\Sigma^{(c)}$ the contribution to the sum (5.1) of connected configurations:

$$\Sigma^{(c)} = \sum T\{L(A^{(1)})\}. \qquad (5.2)$$

In order to express the S matrix in terms of $\Sigma^{(c)}$ we define the operation of λ_0 product. Let F(A) be a certain operator dependent on the operators L(x) at the points of the configuration A, and let G(B) be an analogous operator defined for the configuration B. The operator

$$H(A + B) = \lambda_0(A, B)\, F(A)\, G(B) \qquad (5.3)$$

will be called the λ_0 product of the operators F and G. Equalities of the form of Eq. (5.3) will be abbreviated by

$$H = \lambda_0\{FG\}. \qquad (5.4)$$

The function $\lambda_0(A, B)$ in Eq. (5.3) is defined by condition (3.2) and is different from zero only if the configurations A and B are not causally connected. Starting from that fact it is easy to show that the λ_0 product is commutative and associative, i.e., possesses the properties of normal multiplication. We note that the λ_0 product has in fact already been used in the definition of the operator L(C) with the help of Eq. (3.21).

Recalling the definition of a disconnected configuration we see that the contribution of disconnected configurations $A^{(q)}$ $(q > 1)$ to the sum

(5.1) is expressed as the λ_0 product of contributions of connected configurations $A^{(1)}$. This fact is expressed by the relation

$$\sum T\{L(A^{(q)})\} = \frac{1}{q!}\lambda_0\{\Sigma^{(c)}\ldots\Sigma^{(c)}\}. \qquad (5.5)$$

On the right side of Eq. (5.5) enter q cofactors (5.2). The factor 1/q! compensates for the number of equal configurations.

With the help of the relation (5.5) we may formally sum the series (5.1) and represent the S matrix in the form

$$S = \lambda_0 \exp \Sigma^{(c)}. \qquad (5.6)$$

Equation (5.6) may turn out to be useful in the study of the general "causality" properties of the S matrix.

6. R-OPERATORS

Extracting from the integrands in Eq. (4.2) the dependence of the operators on the coordinates of the upper layer C^α, we may write the S matrix in the form

$$S = 1 + \sum_{\alpha=1}^{\infty} \frac{1}{\alpha!}\int R(C^\alpha)\, d\Gamma_\alpha, \qquad (6.1)$$

where $R(C^\alpha)$ are certain operators defined on the layer C^α. Keeping in mind the structure of the decomposition (4.2) we can derive a set of integral equations that must be satisfied by these operators:

$$R(C^\alpha) = L(C^\alpha)\, S(C^\alpha),$$
$$S(C^\alpha) = 1 + \sum_{\beta=1}^{\infty} \frac{1}{\beta!}\int \Lambda_+(C^\alpha, C^\beta)\, R(C^\beta)\, d\Gamma_\beta. \qquad (6.2)$$

We note that the Eqs. (6.2) may describe two different sets of operators depending on whether we classify the configurations according to the decomposition (2.7) or (2.9). In the first case we must set $\Lambda_+ = \Lambda_+^{I}$ and we get the operators R_I and S_I, in the second case we must set $\Lambda_+ = \Lambda_+^{II}$ and we get R_{II} and S_{II}. The use of the Λ_+ functions and of operators without roman subscripts is meant to indicate the validity of the relation for both cases.

Taking into account the condition $\Lambda_+^{I}(C^\alpha, C^0) = 1$ (see the table) and setting $R_I(C^0) = 1$ we may rewrite the expression for S_I in a more compact form:

$$S_I(C^\alpha) = \sum_{\beta=0}^{\infty} \frac{1}{\beta!}\int \Lambda_+^{I}(C^\alpha, C^\beta)\, R_I(C^\beta)\, d\Gamma_\beta. \qquad (6.3)$$

The analogous expression for the operator S_{II} is not valid.

If we extract from the integrands in the series (4.2) the dependence on the coordinates of the lowest layer we shall obtain a different representation of the S matrix:

$$S = 1 + \sum_{\alpha=1}^{\infty} \frac{1}{\alpha!}\int R'(C^\alpha)\, d\Gamma_\alpha, \qquad (6.4)$$

with the operators R' satisfying the set of equations

$$R'(C^\alpha) = S'(C^\alpha)\, L(C^\alpha),$$
$$S'(C^\alpha) = 1 + \sum_{\beta=1}^{\infty} \frac{1}{\beta!}\int R'(C^\beta)\, \Lambda_+(C^\beta, C^\alpha)\, d\Gamma_\beta. \qquad (6.5)$$

It is clear from Eq. (6.2) that the operator $R(C^\alpha)$ may be expressed in the form of a configurational sum

$$R(C^\alpha) = \sum T\{L(A(C^\alpha))\}. \qquad (6.6)$$

The summation in Eq. (6.6) extends over all configurations $A(C^\alpha)$ which have a fixed layer C^α as their highest layer in the decomposition (2.7) or (2.9).

In the following we shall need a more detailed relation for the operators R ("double" relation). This relation is of the form

$$R_I(C^\alpha + C^\beta) = L(C^\alpha)\,\lambda_0(C^\alpha, C^\beta) \sum_{\lambda=0}^{\infty} \frac{1}{\lambda!}\int \Lambda_+^{I}(C^\alpha, C^\lambda)\, R_I(C^\beta + C^\lambda)\, d\Gamma_\lambda. \qquad (6.7)$$

We note that for $\beta = 0$ the relation (6.7) coincides with (6.2), and for $\alpha = 0$ it turns into an identity. In order to prove Eq. (6.7) we make use of a representation of the type (6.6) in which the upper layer is taken in the form of a sum of two parts. Let us denote by $A_{ab}(C^\alpha + C^\beta)$ configurations with the upper layer $C^\alpha + C^\beta$ having the property that the maximum length of a chain that starts with a point from C^α is equal to a, and the maximum length of a chain starting with a point from C^β is equal to b. In that case the sum (6.6) may be rewritten in the form

$$R_I(C^\alpha + C^\beta) = \sum_{a,\,b=1}^{\infty} T\{L(A_{ab}(C^\alpha + C^\beta))\}. \qquad (6.8)$$

Substituting Eq. (6.8) into the right side of Eq. (6.7) we get

$$\sum_{b=1}^{\infty} T\{L(A_{1,b}(C^\alpha + C^\beta))\} + \sum_{a,\,b=1}^{\infty} T\{L(A_{a+1,b}(C^\alpha + C^\beta))\} = \sum_{a,\,b=1}^{\infty} T\{L(A_{ab}(C^\alpha + C^\beta))\} = R_I(C^\alpha + C),$$

i.e., the left side of Eq. (6.7).

Relation (6.7) is valid only for the operators R^{I}. An analogous relation may be obtained for the operators R'_{II} satisfying the relations (6.5).

7. EQUIVALENCE WITH THE CONVENTIONAL METHOD OF CONSTRUCTION OF THE S MATRIX

We shall show that the S matrix defined by the relations (4.2) or (6.1) is identical with the S matrix of quantum field theory obtained by conventional means. To this end we note that all previous relations remain valid if integration over all of space is replaced by integration over a region bounded by a fixed space-like hypersurface σ. We can define the operator

$$S[\sigma] = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \int^{\sigma} R_I(C^{\alpha})\, d\Gamma_{\alpha}. \tag{7.1}$$

We show that the operator (7.1) satisfies the Tomonaga-Schwinger equation.[1,2] To this end we first establish the following important property of the operator $R_I(C^{\alpha})$: if the layer C^{α} lies in its entirety below the surface σ then the operator $R_I(C^{\alpha})$ is independent of σ. In other words

$$\delta R_I(C^{\alpha})/\delta\sigma(x) = 0. \tag{7.2}$$

The validity of relation (7.2) follows from the representation of the operator $R_I(C^{\alpha})$ in the form of the configurational sum (6.6). If the layer C^{α} lies below the surface σ then this result is valid also for any point of the configuration $A(C^{\alpha})$, therefore the region within which points of configurations of the type $A(C^{\alpha})$ may vary is determined by the layer C^{α} itself and does not depend on the surface σ. From here (7.2) follows immediately. Taking Eq. (7.2) into account we may write the variational derivative of the operator $S[\sigma]$ in the form

$$\frac{\delta S[\sigma]}{\delta\sigma(x)} = \sum_{\beta=0}^{\infty} \frac{1}{\beta!} \int^{\sigma} R_I(x + C^{\beta})\, d\Gamma_{\beta}. \tag{7.3}$$

We now use the equality (6.7), summing its two parts over all layers C^{β} that lie below the surface σ. We get the relation

$$\sum_{\beta=0}^{\infty} \frac{1}{\beta!} \int^{\sigma} R_I(C^{\alpha} + C^{\beta})\, d\Gamma_{\beta} = L(C^{\alpha}) \sum_{\tau=0}^{\infty} \frac{1}{\tau!} \int^{\sigma} \Lambda(C^{\alpha}, C^{\tau})\, R_I(C^{\tau})\, d\Gamma_{\tau} \ldots \tag{7.4}$$

[in the derivation of (7.4) use was made of (3.15)].

If in Eq. (7.4) the layer C^{α} consists of one point x, and that point lies on σ, then it follows from the property of the Λ function, $\Lambda(x, C^{\tau}) = 1$, that

$$\sum_{\beta=0}^{\infty} \frac{1}{\beta!} \int^{\sigma} R_I(x + C^{\beta})\, d\Gamma_{\beta} = L(x)\, S[\sigma]. \tag{7.5}$$

From relations (7.3) and (7.5) follows the Tomonaga-Schwinger equation:

$$\delta S[\sigma]/\delta\sigma(x) = L(x)\, S[\sigma]. \tag{7.6}$$

This proves the equivalence of the method proposed here for the construction of the S matrix with the generally accepted method.

8. BILINEAR RELATION BETWEEN THE OPERATORS R^+ AND R

In this section we obtain a relation between the operators R^+ and R that does not involve the L operators. One of the consequences of this relation is the unitarity of the S matrix; consequently it may be useful in attempts to construct a field theory without the use of a Lagrangian L.

We write the relation (6.7) in a form analogous to (6.2):

$$R_I(C^{\alpha} + C^{\beta}) = L(C^{\alpha})\, P(C^{\alpha}, C^{\beta}),$$

$$P(C^{\alpha}, C^{\beta}) = \lambda_0(C^{\alpha}, C^{\beta}) \sum_{\lambda=0}^{\infty} \frac{1}{\lambda!} \int \Lambda_+^I(C^{\alpha}, C^{\lambda})\, R_I(C^{\beta} + C^{\lambda})\, d\Gamma_{\lambda}. \tag{8.1}$$

The hermitian conjugate operator R_I^+ satisfies the relation

$$R_I^+(C^{\alpha} + C^{\beta}) = P^+(C^{\alpha}, C^{\beta})\, L^+(C^{\alpha}). \tag{8.2}$$

Since $L^+(C^{\alpha}) = (-1)^{\alpha} L(C^{\alpha})$ we may eliminate from Eqs. (8.1) and (8.2) the operator $L(C^{\alpha})$ and arrive at the equality

$$R_I^+(C^{\alpha} + C^{\beta})\, P(C^{\alpha}, C^{\gamma}) = (-1)^{\alpha} P^+(C^{\alpha}, C^{\beta})\, R_I(C^{\alpha} + C^{\gamma}). \tag{8.3}$$

The relation (8.3) is bilinear with respect to the operators R_I^+ and R_I.

The unitarity of the S matrix is proved by means of the following calculation. We multiply both sides of (8.3) by the function $\Lambda_+^I(C^{\gamma}, C^{\beta})$ and sum over all layers C^{α}, C^{β}, C^{γ}. The summing of the right side yields

$$\sum \frac{1}{\alpha!\,\beta!\,\gamma!\,\lambda!} \int R_I^+(C^{\alpha} + C^{\beta})\, R_I(C^{\gamma} + C^{\lambda})\, \lambda_0(C^{\alpha}, C^{\gamma})\, \Lambda_+^I(C^{\alpha}, C^{\lambda}) \times \Lambda_+^I(C^{\gamma}, C^{\beta})\, d\Gamma_{\alpha+\beta+\gamma+\lambda} = \sum \frac{1}{\rho!\,\sigma!} \int R_I^+(C^{\rho})\, R_I(C^{\sigma}) \times \Big\{ \sum_{C^{\alpha}+C^{\beta}=C^{\rho}} \sum_{C^{\gamma}+C^{\lambda}=C^{\sigma}} \Lambda_+^I(C^{\alpha}, C^{\lambda})\, \Lambda_+^I(C^{\gamma}, C^{\beta})\, \lambda_0(C^{\alpha}, C^{\gamma}) \Big\}\, d\Gamma_{\rho+\sigma} = \sum \frac{1}{\rho!\,\sigma!} \int R_I^+(C^{\rho})\, R_I(C^{\sigma})\, d\Gamma_{\rho}\, d\Gamma_{\sigma} = S^+S.$$

Here we have used relation (3.23). Summing the left side yields

$$\sum \frac{1}{\alpha!\,\beta!\,\gamma!\,\lambda!} \int R_I^+ (C^\beta + C^\lambda) R_I (C^\alpha + C^\gamma) (-1)^\alpha$$

$$\times \lambda_0 (C^\alpha, C^\beta) \Lambda_+^I (C^\alpha, C^\lambda) \Lambda_+^I (C^\gamma, C^\beta) d\Gamma_{\alpha+\beta+\gamma+\lambda}$$

$$= \sum \frac{1}{\sigma!\,\rho!} \int R_I^+ (C^\sigma) R_I (C^\rho)$$

$$\times \Big\{ \sum_{C^\alpha + C^\gamma = C^\rho} \sum_{C^\beta + C^\lambda = C^\sigma} (-1)^\alpha \Lambda_+^I (C^\alpha, C^\lambda) \Lambda_+^I$$

$$\times (C^\gamma, C^\beta) \lambda_0 (C^\alpha, C^\beta) \Big\} d\Gamma_{\sigma+\rho}$$

$$= \sum \frac{\delta_{\sigma 0} \delta_{\rho 0}}{\sigma!\,\rho!} \int R_I^+ (C^\sigma) R_I (C^\rho) d\Gamma_\sigma d\Gamma_\rho = 1.$$

Here we have used relation (3.24).

9. CONCLUDING REMARKS

Let us indicate possible ways in which the methods and results of this work may be extended. They may be divided in three groups.

1. The analysis of the configuration structure (Sec. 2) and the resultant representations of the S matrix (Secs. 4 and 5) and the R operators (Sec. 6) are in substance based on relativistic causality in the field theory. Since the causality conditions are first formulated in terms of space-time it is natural to construct the theory in the x-representation. As is known, causal properties of the theory in the x-representation find their reflection in analyticity properties of the corresponding quantities in the p-representation. Consequently one of the main problems consists of the transcription of our results into the p-representation. It is to be expected that in this way one will obtain more complete information than has been hitherto possible on the S-matrix analytic properties that are due to causality.

2. The proposed formulation of the theory of the scattering matrix opens up new possibilities for the construction of generalized field theories. For example, Eq. (8.3) may be taken as one of the basic relations in attempts to construct a theory of a unitary S matrix without use of Lagrangians.

3. Lastly, the method of configurational sums widely used here may also be utilized for the construction of approximate theories. To this end the summation would be extended over some subgroup of configurations instead of all of them.

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[1] S. Tomonaga, Progr. Theor. Phys. 1, 27 (1946).

[2] J. Schwinger, Phys. Rev. 74, 1439 (1948).

Translated by A. M. Bincer
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