CONTRIBUTION TO THE THEORY OF OSCILLATIONS OF A WEAKLY INHOMOGENEOUS PLASMA

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The characteristic oscillations of a plasma which is weakly inhomogeneous in three dimensions are studied in the geometric optics approximation. The corresponding dispersion relations are derived from multi-dimensional quantization rules. High-frequency longitudinal and transverse oscillations of an electron plasma, low-frequency ion-acoustic oscillations of a quasi-isotropic electron-ion plasma, and low frequency oscillations of a cold magnetoactive plasma are considered by applying the dispersion relations.

1. Recently the method of geometric optics has been successfully applied to the theory of the oscillations of a weakly inhomogeneous plasma. However, all the results obtained in this approximation are limited to the case of a one-dimensional inhomogeneity. We shall set forth below certain results of the theory of oscillations of a weakly inhomogeneous plasma of more than one dimension.

The fundamental difference between the multidimensional quantization rules used below and the ordinary one-dimensional rules (see, for example, [1]) lies in the fact that the integration in them is carried out over the entire multi-dimensional space bounded by the turning points of the beam. This corresponds to the situation that the ray trajectory in the approximation of geometric optics almost completely fills the entire available ray space. It is not possible to speak here of a separation of variables, which is so necessary in the solution of problems in which the determination of the spectrum reduces to one-dimensional quantization rules. The fact that the trajectory of the ray nearly completely fills the available ray space leads to spectra, generally speaking, which are less anisotropic than those in the one-dimensional case. This situation appears, in particular, in the example considered below of acoustical oscillations of the plasma.

2. As is well known, there is a smooth dependence of the field in geometric optics, $\sim e^{i\psi}$, where ψ is the eikonal. Here the equation of the eikonal plays a fundamental role. For a medium with spatial dispersion, such as a plasma, the eikonal equation has the form [2]

$$|\omega^2\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{r}) - c^2k^2\delta_{ij} + c^2k_ik_j| = 0, \qquad (1)$$

where $\omega = -\partial\psi/\partial t$, $\mathbf{k} = \nabla\psi$. In Eq. (1), the dependence of the complex dielectric tensor on the coordinates corresponds to a slow dependence, brought about by the weak spatial inhomogeneity of the plasma.

The eikonal equation (1) is materially simplified when the velocity distribution of particles of the plasma is isotropic, and when there are no strong fields in the plasma. Then,

$$\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{r}) = (\delta_{ij} - k^{-2}k_ik_j)\,\varepsilon^{tr}(\omega, k, \mathbf{r}) + k^{-2}k_ik_j\varepsilon^{l}(\omega, k, \mathbf{r}). \qquad (2)$$

Here, locally transverse and longitudinal fields can be introduced and the eikonal equation splits into the equations

$$\omega^2\varepsilon^{tr}(\omega, k, \mathbf{r}) - c^2k^2 = 0, \qquad (3)$$

$$\varepsilon^{l}(\omega, k, \mathbf{r}) = 0. \qquad (4)$$

The simplest case is that of high-frequency transverse oscillations of an electron plasma. We begin our consideration with just this case. Here we can neglect spatial dispersion. Then

$$\varepsilon^{tr}(\omega, k, \mathbf{r}) = \varepsilon'(\omega, \mathbf{r}) \equiv 1 - \omega_{Le}^2(\mathbf{r})/\omega^2, \qquad (5)$$

where $\omega_{Le}^2(\mathbf{r}) = 4\pi e^2N_e/m$, while e and m are the charge and mass of the electron, and N_e is the number of electrons per unit volume.

Equations (3) and (5) make it possible to write down the following equation for the square of the wave number as a function of the frequency and of the coordinates:

$$k^2(\omega, \mathbf{r}) = c^{-2}[\omega^2 - \omega_{Le}^2(\mathbf{r})]. \qquad (6)$$

In all the following examples, we shall also determine the square of the wave number as a function of the frequency and of the coordinates. After such a determination, we can write down, approximately, the equivalent differential equation:

$$\Delta\varphi + k^2(\omega, \mathbf{r})\varphi = 0. \tag{7}$$

In the case considered of transverse oscillations, such an equation for the electric field is identical with the zeroth approximation equation of geometric optics, which follows from the exact field equations.

The asymptotic dispersion equations which describe the spectrum of oscillations in accord with Eq. (7) have the form

$$4\pi n = \int dS k_\perp^2(\omega, \mathbf{r}), \tag{8}$$

$$6\pi^2 n = \int dV k^3(\omega, \mathbf{r}). \tag{9}$$

Here n is an integer considerably larger than unity, $k_\perp^2 = k^2 - k_\parallel^2$, where $k_\parallel$ is the projection of k (independent of the coordinates) on the homogeneity direction. In the case of a two-dimensional inhomogeneity, to which the dispersion equation (8) corresponds, the integration is carried out over the surface on which the possible trajectories of the ray lie. In the case of a three-dimensional inhomogeneity, the integration is carried out over the corresponding volume.

A proof of Eqs. (8), (9) for positive k^2 (or $k_\perp^2$) and different boundary conditions is given, for example, in the book by Courant and Hilbert.[3] In that case, however, the case of vanishing k^2 is not considered. This case is of special interest, inasmuch as it corresponds to the possibility of a locking of the rays inside the plasma. To be precise, for real k^2, it is obvious that the rays will be enclosed in the region $k^2 > 0$ if this region is surrounded by a closed curve in the case of a two-dimensional inhomogeneity or by a closed surface in the case of a three-dimensional inhomogeneity. It is clear that, because of the exponentially steep decay of the field in the region $k^2 < 0$, the error in the determination of the spectrum that is obtained by use, for example, of the boundary condition of the vanishing of the field along the turning curve (or surface) of the ray ($k^2 = 0$) in comparison with the exact result of the exact joining of the solutions on the turning curve, can lead only to relatively small corrections on the left hand sides of Eqs. (8), (9). This is also evident for the case in which k^2 has a small imaginary part. In the latter case, by using $\omega = \omega' + i\omega''$, one can write Eqs. (8) and (9) in the following form (compare [1]):

$$4\pi n = \int dS \operatorname{Re} k_\perp^2(\omega', \mathbf{r}),$$

$$\omega'' \int dS \frac{\partial}{\partial\omega'} \operatorname{Re} k_\perp^2(\omega', \mathbf{r}) + \int dS \operatorname{Im} k_\perp^2(\omega', \mathbf{r}) = 0; \tag{10}$$

$$6\pi^2 n = \int dV \operatorname{Re} k^3(\omega', \mathbf{r}),$$

$$\omega'' \int dV \frac{\partial}{\partial\omega'} \operatorname{Re} k^3(\omega', \mathbf{r}) + \int dV \operatorname{Im} k^3(\omega', \mathbf{r}) = 0. \tag{11}$$

The integration for enclosed rays in these formulas must be carried out over the region $\operatorname{Re} k^2(\omega', \mathbf{r}) > 0$ or $\operatorname{Re} k_\perp^2 > 0$.

As is well known, for the case of a not very small imaginary part, the boundaries of the turning region (if they exist) lie in the complex plane. In this case the principal problem which must be solved is this—the determination of the turning boundary. For a case of a multi-dimensional inhomogeneity, one can make use of the results of Langer,[4] which were obtained for the one-dimensional equation of second order by reducing the multi-dimensional problem to a one-dimensional one. This is possible for the reason that, close to the turning boundary, one can frequently isolate the direction along which the asymptotic behavior of the solution corresponds to an exponentially rapid decay behind the turning boundary. Then the analysis of asymptotic solutions as functions only of the distance in this direction allows one to determine the complex coordinates of the turning boundary. In this report we limit our observations to consideration of the case of a small imaginary part, for which Eqs. (10), (11) are valid.

For a two-dimensional inhomogeneity, Eqs. (6) and (8) lead to the following spectrum of transverse oscillations:

$$\omega^2 = \langle \omega_{Le}^2(\mathbf{r}) \rangle_S + c^2(k_\parallel^2 + 4\pi n/S), \tag{12}$$

where S denotes the area of the surface reached by the rays, while $\langle \ldots \rangle$ denotes averaging over this surface. Similarly, for the three-dimensional inhomogeneity, we have, from (9) and (6),

$$\langle [\omega^2 - \omega_{Le}^2(\mathbf{r})]^{3/2} \rangle_V = c^3 \cdot 6\pi^2 n/V, \tag{13}$$

where V is the volume filled by the rays, and $\langle \ldots \rangle_V$ denotes averaging over this volume. Equations (12) and (13) describe the transverse oscillations of a weakly inhomogéneous plasma.

3. We now proceed to a consideration of longitudinal, weakly attenuating oscillations. In the consideration of electron Langmuir vibrations, one can limit oneself to the following approximate expression for the longitudinal complex dielectric constant:

$$\varepsilon^l(\omega, k) = \varepsilon'(\omega, \mathbf{r}) - 3\alpha(\omega, \mathbf{r}) k^2 - i\frac{4\pi^2 e^2}{k^2}\omega \int d\mathbf{p}\, \delta(\omega - \mathbf{kv}) \frac{\partial f_0(\mathbf{r}, E)}{\partial E}. \tag{14}$$

Here, ϵ' is determined by Eq. (5), $f_0(\mathbf{r}, \mathbf{E})$ is the electron distribution function, which depends on the coordinates and on the energy E of the electron; finally,

$$\alpha(\omega, \mathbf{r}) = -\frac{4\pi e^2}{\omega^4}\int d\mathbf{p}\, \frac{v^4}{15}\frac{\partial f_0(\mathbf{r}, E)}{\partial E}. \tag{15}$$

Substituting the expression (14) in the eikonal equation (4), and keeping in mind the assumption of the smallness of the absorption, which is satisfied under conditions for which the wavelength is significantly greater than the radius of the Debye screening, we get

$$k^2(\omega, \mathbf{r}) = \frac{\varepsilon'(\omega, \mathbf{r})}{3\alpha(\omega, \mathbf{r})} - i\,\frac{4\pi^2 e^2\omega}{\varepsilon'(\omega, \mathbf{r})}\int d\mathbf{p}\,\delta(\omega - \mathbf{k}_0\mathbf{v})\,\frac{\partial f_0(\mathbf{r}, E)}{\partial E}, \tag{16}$$

where $k^2(\omega, \mathbf{r}) = \epsilon'/3\alpha$. Owing to the isotropy of the velocity distribution, the direction of the vector $\mathbf{k}_0$ does not play a role.

For a Maxwellian distribution, in which $\alpha(\omega, \mathbf{r}) = (\kappa T/m)\omega_{Le}^2/\omega^4$, under the assumption of spatial homogeneity of the temperature of the electrons, we get from Eqs. (16) and (10) the following expressions for the frequency and the damping decrement of the longitudinal electron oscillations of a two-dimensional inhomogeneous plasma:

$$\omega'^2 = \left(\left\langle\frac{1}{\omega_{Le}^2(r)}\right\rangle_S\right)^{-1} + \frac{3\kappa T}{m}\left(k_\parallel^2 + \frac{4\pi n}{S}\right), \tag{17}$$

$$\omega'' = -\sqrt{\frac{\pi}{2}}\left\langle\frac{\omega_{Le}^2(\mathbf{r})\,(m/\kappa T)^{3/2}}{k_0(\omega', \mathbf{r})\,\varepsilon'(\omega', \mathbf{r})} \times \exp\left\{-\frac{m\omega'^2}{2\kappa T k_0^2(\omega', \mathbf{r})}\right\}\right\rangle_S \left(\left\langle\frac{\partial k_0^2(\omega', \mathbf{r})}{\partial\omega'}\right\rangle_S\right)^{-1}. \tag{18}$$

For a three-dimensional inhomogeneous plasma, we have in place of Eqs. (17) and (18),

$$\left[\left\langle\left(\frac{\omega'^2}{\omega_{Le}^2} - 1\right)^{3/2}\right\rangle_V\right]^{2/3} = \frac{3\kappa T}{m\omega'^2}\left(\frac{6\pi^2 n}{V}\right)^{2/3}, \tag{19}$$

$$\omega'' = -\sqrt{\frac{\pi}{2}}\left\langle\frac{\omega_{Le}^2(\mathbf{r})}{\varepsilon'(\omega', \mathbf{r})}\left(\frac{m}{\kappa T}\right)^{3/2} \times \exp\left(-\frac{m\omega'^2}{2\kappa T k_0^2(\omega', \mathbf{r})}\right)\right\rangle_V \left(\left\langle\frac{2}{3}\frac{\partial k_0^3(\omega', \mathbf{r})}{\partial\omega'}\right\rangle_V\right)^{-1}. \tag{20}$$

The difference of these formulas from the corresponding expressions for a spatially homogeneous plasma is contained in the appearance of space-averaged expressions. Here, as we shall see, the laws of space averaging are actually shown to be different for two-dimensional and three-dimensional inhomogeneous distributions of the particles of the plasma.

We now write out the expressions determining the spectrum of low frequency ion-acoustic waves. Such waves are weakly damped under conditions for which the temperature of the electrons significantly exceeds the ion temperature. Then, under the assumption that the phase velocity of the waves on the one hand is small in comparison with the thermal velocity of the electrons, and on the other hand, large in comparison with the thermal velocity of the ions, we have the following expression for the longitudinal dielectric constant:

$$\varepsilon^l(\omega, k, \mathbf{r}) = 1 - \frac{\omega_{Li}^2(\mathbf{r})}{\omega^2} + \frac{1}{k^2 r_{\mathrm{scr}}^2} - \frac{i\omega}{k^2}\sum 4\pi^2 e^2\int d\mathbf{p}\,\delta(\omega - \mathbf{k}\mathbf{v})\,\frac{\partial f_0(\mathbf{r}, E)}{\partial E}. \tag{21}$$

Here the summation is carried out over the electrons and the ions, $\omega_{Li}(\mathbf{r}) = \sqrt{4\pi e_i^2 N_i/M}$ is the Langmuir frequency of the ions and, finally,

$$r_{\mathrm{scr}}^{-2}(\mathbf{r}) = -4\pi e^2\int d\mathbf{p}\,\frac{\partial f_{0e}}{\partial E}. \tag{22}$$

For a Maxwell distribution, r_{scr} is the Debye radius of the electrons.

Substituting (21) in the eikonal equation, and keeping in mind the smallness of dissipation effects, which is satisfied in the region of phase velocities for which the equation (21) is valid, we have

$$k^2(\omega, \mathbf{r}) = \frac{\omega^2}{\omega_{Li}^2(\mathbf{r}) - \omega^2}\,\frac{1}{r_{\mathrm{scr}}^2(\mathbf{r})} + \frac{i\omega^3}{\omega_{Li}^2(\mathbf{r}) - \omega^2}\sum 4\pi^2 e^2\int d\mathbf{p}\,\delta(\omega - \mathbf{k}_0\mathbf{v})\,\frac{\partial f_0(\mathbf{r}, E)}{\partial E}, \tag{23}$$

or

$$k_0^2(\omega, \mathbf{r}) = \frac{\omega^2}{\omega_{Li}^2(\mathbf{r}) - \omega^2}\, r_{\mathrm{scr}}^{-2}(\mathbf{r}).$$

Assuming $\omega_{Li}^2 > 0$ and therefore considering the ion-acoustic waves in a plasma with walls, we shall take the integration in the dispersion relations to be over the region of the plasma bounded by the walls. For a Maxwellian distribution with spatially inhomogeneous temperature, we have, in the two-dimensional case,

$$\left\langle\frac{\omega'^2}{\omega_{Li}^2(\mathbf{r}) - \omega'^2}\,\frac{1}{r_{\mathrm{scr}}^2}\right\rangle_S = k_\parallel^2 + \frac{4\pi n}{S}, \tag{24}$$

$$\omega'' = -\left\langle\frac{\omega'^3}{\omega_{Li}^2(\mathbf{r}) - \omega'^2}\sum\sqrt{\frac{\pi}{2}}\,\frac{4\pi e^2 N}{m k_0}\left(\frac{m}{\kappa T}\right)^{3/2} \times \exp\left(-\frac{m\omega'^2}{2\kappa T k_0^2}\right)\right\rangle_S \left(\left\langle\frac{\partial k_0^2}{\partial\omega'}\right\rangle_S\right). \tag{25}$$

For a three-dimensional inhomogeneity, we write down the corresponding dispersion relations under the conditions when $\omega_{Li}^2 \gg \omega^2$, that is, for the acoustic spectrum. Then we have for the frequency

$$\omega' = (6\pi^2 n/V)^{1/3} (\langle v_s^{-3}(\mathbf{r})\rangle_V)^{1/3}. \quad (26)$$

Here $v_s^2(\mathbf{r}) = Z\kappa T(\mathbf{r})/M$, where Z is the ratio of the ion charge to the electron charge. In the corresponding expression for the damping decrement, we take into account only the absorption brought about by the electrons. Then

$$\omega'' = -\sqrt{\pi Zm/8M}\,\omega'. \quad (27)$$

This means that the electronic absorption leads to such an expression for the damping decrement that the ratio of the latter to the frequency is independent of the inhomogeneity. For a one-dimensional inhomogeneity, such a result was obtained earlier.[5] In comparing Eq. (26) and the analogous formula that follows from (24), with the formula for the acoustic spectrum

$$\int dx \sqrt{\omega^2/v_s^2(x) - k_y^2 - k_z^2} = \pi n, \quad (28)$$

obtained in [5] for the case of a one-dimensional inhomogeneity, one must emphasize the directional independence of the sound velocity in the multi-dimensional inhomogeneous plasma. More precisely, it should be pointed out that the approximation of geometric optics does not permit us to make clear the anisotropy of the sound velocity in the multi-dimensional plasma with anisotropic velocity distribution of the particles.

4. Equation (2) which holds in the case of an anisotropic velocity distribution of the particles and in the absence of strong fields in the plasma materially simplifies the complicated analysis of the spectrum of the oscillations of a weakly inhomogeneous plasma. We shall consider below a single case of a spectrum of a weakly inhomogeneous anisotropic plasma. More particularly, we shall consider the spectrum of a cold magnetoactive plasma in the region of frequencies much less than the Langmuir frequency of the ions. As is well known, the dielectric tensor of the plasma is diagonal in this case:[6]

$$\varepsilon_{ij}(\omega, \mathbf{r}) = (\delta_{ij} - B^{-2}B_iB_j)\,\varepsilon_1 + B^{-2}B_iB_j\varepsilon_2,$$

$$\varepsilon_1 = 1 + 4\pi N_i Mc^2/B^2, \quad \varepsilon_2 = 1 - 4\pi e^2 N_e/m\omega^2, \quad (29)$$

where **B** is the constant magnetic field.

The eikonal equation (1), after substitution into it of Eq. (29), leads to the following two relations:

$$k^2 = (\omega^2/c^2)\,\varepsilon_1, \quad (30)$$

$$\{\varepsilon_1(\delta_{ij} - B^{-2}B_iB_j) + \varepsilon_2 B^{-2}B_iB_j\}\,k_ik_j = c^{-2}\omega^2\varepsilon_1\varepsilon_2. \quad (31)$$

Equation (30) leads to the approximately equivalent differential equation (7). Therefore, by making use of the relations (8) and (9), we get the following dispersion relations:

$$\omega^2 \langle c^{-2} + v_A^{-2}(\mathbf{r})\rangle_S = k_\parallel^2 + 4\pi n/S, \quad (32)$$

$$\omega^3 \langle (c^{-2} + v_A^{-2}(\mathbf{r}))^{3/2}\rangle_V = 6\pi^2 n/V, \quad (33)$$

where $v_A(\mathbf{r}) = B/\sqrt{4\pi N_i/M}$ is the Alfvén velocity. Equations (32) and (33) describe the quasiclassical spectrum of Alfvén waves in two-dimensional and three-dimensional inhomogeneous plasma.

In contrast with (30), the relation (31) does not give equations of type (7) directly. However, it can be reduced to such an equation by a change of scale of the spatial coordinates. Assuming $\epsilon_2 > 0$, we get

$$6\pi^2 n/V = c^{-3}\omega^3 \langle \varepsilon_2 \sqrt{\varepsilon_1}\rangle_V \quad (34)$$

or

$$\omega^3 - \omega\frac{\langle \omega_{Le}^2 \sqrt{\varepsilon_1}\rangle_V}{\langle \sqrt{\varepsilon_1}\rangle_V} - \frac{6\pi^2 nc^3}{V\langle \sqrt{\varepsilon_1}\rangle_V} = 0. \quad (35)$$

For trapped waves, the turning boundary is defined here by the equation $\epsilon_2 = 0$. It should be noted that an approximately equivalent differential equation corresponding to the eikonal equation (31), is related to the class studied by Carleman.[7] 1)

For a two-dimensional inhomogeneous plasma, by considering the consequences of Eq. (31), we limit ourselves to the case of an inhomogeneity transverse to the magnetic field. Then the dispersion equation can obviously be written in the form

$$4\pi n/S = \langle \varepsilon_2(\omega^2/c^2 - k_\parallel^2/\varepsilon_1)\rangle_S. \quad (36)$$

We then get

$$\omega^2 = \frac{1}{2}\left\{\frac{4\pi c^2}{S} + \left\langle\frac{k_\parallel^2 c^2}{\varepsilon_1}\right\rangle_S + \langle\omega_{Le}^2\rangle_S \pm \left[\left(\frac{4\pi c^2}{S} + \left\langle\frac{k_\parallel^2 c^2}{\varepsilon_1}\right\rangle_S + \langle\omega_{Le}^2\rangle_S\right)^2 - 4k_\parallel^2 c^2\left\langle\frac{\omega_{Le}^2}{\varepsilon_1}\right\rangle\right]^{1/2}\right\}. \quad (37)$$

The results obtained above are closely related to the possibility of using an approximate equivalent second-order differential equation. For an isotropic plasma, this is practically always possible.2) This also holds in the case considered

1)According to Carleman, for the eigenvalue spectrum of the equation $p_{ik}\partial^2\phi/\partial x_i\partial x_k + \lambda\phi = 0$, where p_{ik} is a real matrix which yields a positive definite quadratic form; in the three-dimensional case the following relation holds

$$6\pi^2 n\lambda_n^{-3/2} = \int dV\,|p_{ik}|^{-1/2},$$

where $|p_{ik}|$ is the determinant of the matrix p. Equation (34) is an obvious consequence of this relation.

2)That this is so is already evident in the case of the equation $\Delta\Delta\phi = k^4\phi$ for which the dispersion equations (8) and (9) hold in the approximation of geometrical optics.

above of low frequency oscillations of a cold magnetoactive plasma. However, in an anisotropic plasma, one can point out a whole series of examples in which this is not the case and where it is necessary to study the asymptotic spectra of differential equations of higher orders.

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176