A PHENOMENOLOGICAL APPROACH TO THE THEORY OF THE NUCLEUS

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A method for the calculation of nuclei is developed in which the observable quantities are expressed in terms of several constants which are the same for all nuclei (except the light ones) and for all types of transitions near the Fermi boundary. Equations are obtained which allow one to express the frequencies and rates of single-particle and collective transitions in terms of the constants of the theory.

1. INTRODUCTION

IN the theory of the nucleus it is impossible to obtain quantitative relations between observable quantities by restricting oneself, as is often done, to any select set of perturbation theory graphs, because the interaction between the nucleons is not small.

In the following we shall develop a method for the calculation of nuclear processes involving small excitation energies, which is based on the separation of sets of graphs that vary slowly near the Fermi boundary. These graphs can be replaced by constants for excitations with small energies. In contrast to the usual approach, we thus take many-particle collisions into account. A similar technique is used in the theory of elementary particles in the study of particle production near threshold. The constants introduced in this manner must be determined by experiment. They can be computed only if there is some small parameter in the problem.

For an infinite Fermi system without Cooper pairing correlations, such a program has been carried out by Landau in his theory of Fermi liquids.[1] It was shown that the spectrum of single particle excitations is characterized by a single constant, the effective mass. The spectrum of two-particle excitations (zero sound, spin waves) and the response of the system to an external field are determined by the function Γ^ω, the forward scattering amplitude at the Fermi surface, which depends only on the spin variables and on the angle between the momenta of the quasiparticles. For a system with Cooper correlations one must introduce yet another function of this angle, Γ^ξ.[2] The spherical harmonic of this function is connected with the width of the energy gap in the single-particle spectrum.

It is natural to expect that Γ^ω and Γ^ξ, as func-

tions of the angles between the momenta of the quasiparticles, are well described by two or three terms in the Legendre expansion. The coefficients of this expansion are the constants to be introduced into the theory.

The functions Γ^ω and Γ^ξ are determined by sets of graphs which depend mainly on integrations over regions far from the Fermi surface. Their values in a finite system therefore agree with the values for infinite nuclear matter with an accuracy determined by the ratio of the range of the nuclear forces over the radius of the nucleus. Since Γ^ω and Γ^ξ are insensitive to the level structure near the Fermi surface, they are identical for all nuclei with the same accuracy as the particle density of the nucleus is a constant. This remark applies to all quantities determined by integrals over regions far from the Fermi surface, such as the effective mass, the Green's function normalization, and the depth of the effective potential well.

The problem of finding the spectrum of single-particle excitations in the nucleus reduces[3] to the solution of the Schrödinger equation for a single particle with an effective mass in an effective potential well. The frequencies of the collective excitations and the transition rates in the nucleus are found by solving an equation which can be interpreted as a Schrödinger equation for a particle and a hole with interaction Γ^ω in a potential well.

2. SINGLE-PARTICLE EXCITATIONS

We quote without derivation earlier results on single-particle excitations in a finite system.[2,3]

It can be shown by exact inclusion of the internucleon force that the nucleus must have a proton and a neutron branch of single-particle excitations near the Fermi surface. The excitation energy is determined by the Schrödinger equation with the effective single-particle Hamiltonian

$$H\varphi_\lambda = (p^2/2m^* + U(r))\,\varphi_\lambda = \varepsilon_\lambda\varphi_\lambda. \tag{1}$$

The Hamiltonian H is defined by the condition that in the φ_λ representation the Green's function $G_{\lambda\lambda'}$ be diagonal in λ and have a pole at $\epsilon = \epsilon_\lambda$, where ϵ_λ is the energy of the single particle excitation. As follows from the Lehmann expansion of the Green's function, this condition implies that there is a single-particle level characterized by the quantum numbers λ. Owing to this condition the effective mass m* and the effective potential U(r) are expressed through the exact self-energy part Σ of the single-particle Green's function.

Since Σ can be found only in the form of a perturbation expansion, m* and U(r) cannot be calculated for a nucleus with strong interactions, but must be chosen (as is actually done in the shell model) such that the ϵ_λ coincide with the observed single-particle levels (or such that the magic nuclei have the correct position in the Mendeleev table). Thus the theory provides a rigorous foundation of the shell model.

In the φ_λ representation the Green's function G takes the form

$$G_{\lambda\lambda'} = a^{(\lambda)}\delta_{\lambda\lambda'}\,(\varepsilon - \varepsilon_\lambda + i\gamma\,\mathrm{sgn}\,\varepsilon)^{-1} + G^R_{\lambda\lambda'}(\varepsilon), \tag{2}$$

where $G^R_{\lambda\lambda'}(\epsilon)$ has no poles near $\epsilon = 0$ (ϵ and ϵ_λ are reckoned from the chemical potential), $a^{(\lambda)}$ is the renormalization of the Green's function and depends weakly on the state λ. In the following we shall omit the index λ of a.

Expression (2) is obtained as the result of an expansion of Σ in powers of ϵ, which can only be done if there is no Cooper pairing in the system (for example, in nuclei of the type magic core plus single particle). If there is Cooper pairing, the single-particle excitations are determined by a system of equations for the two Green's functions G and F. The meaning of the function F is that it describes the transition of a particle into a hole and a condensate pair. The function F will be represented graphically as

$$F = \longrightarrow\!\!-\!\!\longleftarrow.$$

The function F^+ describes the transition of a hole into a particle and a condensate pair and is given graphically by

$$F^+ = \longleftarrow\!\!-\!\!\longrightarrow.$$

For the pole terms of F and G we obtain the expressions

$$G_{\lambda\lambda'} = a\,\frac{\varepsilon + \varepsilon_\lambda}{\varepsilon^2 - \varepsilon_\lambda^2 - \Delta_\lambda^2 + i\gamma}\,\delta_{\lambda\lambda'},$$

$$F_{\lambda\lambda'} = -a\,\frac{\Delta_\lambda}{\varepsilon^2 - \varepsilon_\lambda^2 - \Delta_\lambda^2 + i\gamma}\,\delta_{\lambda\lambda'}. \tag{3}$$

The quantity Δ_λ, which may be called the pairing energy, is the average value of the quantity $\Delta(r)$:

$$\Delta_\lambda = \int \varphi_\lambda \Delta(r)\,\varphi_\lambda\,dr,$$

which is determined by the equation

$$\Delta(r) = -a^2\Gamma_0^\xi \int K(r, r')\,\Delta(r')\,dr', \tag{4}$$

$$K(r, r') = \sum_{\lambda_1\lambda_2}\theta_{\lambda_1\lambda_2}(\xi)\,\varphi^*_{\lambda_1}(r)\,\varphi_{\lambda_1}(r)\,\varphi_{\lambda_1}(r')\varphi^*_{\lambda_2}(r')$$

$$\times\left\{\delta_{\lambda_1\lambda_2}\frac{1}{2E_{\lambda_1}} + (1 - \delta_{\lambda_1\lambda_2})\frac{E_{\lambda_1}E_{\lambda_2} + \varepsilon_{\lambda_1}\varepsilon_{\lambda_2}}{2E_{\lambda_1}E_{\lambda_2}(E_{\lambda_1} + E_{\lambda_2})}\right\}. \tag{5}$$

The spherical harmonic of the amplitude for the interaction of two particles Γ_0^ξ can be written in the form

$$a^2\frac{dn}{d\varepsilon_0}\Gamma_0^\xi = \frac{1}{\ln(\xi/c_0)}. \tag{6}$$

Furthermore,

$$\theta_{\lambda_1\lambda_2}(\xi) = \begin{cases} 1, \varepsilon_{\lambda_1}, & \varepsilon_{\lambda_2} < \xi \\ 0, \varepsilon_{\lambda_1}, & \varepsilon_{\lambda_2} > \xi \end{cases}$$

It can be verified that the auxiliary quantity ξ ($\Delta \ll \xi \ll \epsilon_0$) drops out of the final expressions in virtue of (6). c_0 is a constant introduced into the theory which is the same for all nuclei and all states. For deformed nuclei, $\Delta(r)$ depends weakly on r, and (5) simplifies to

$$1 = -\gamma_0^\xi\sum_\lambda\frac{1}{2E_\lambda}\theta_\lambda(\xi), \quad \gamma_0^\xi = a^2\Gamma_0^\xi\frac{dn}{d\varepsilon_0} = \frac{1}{\ln(\xi/c_0)}. \tag{7}$$

3. VERTEX PART

The behavior of the system in a weak external field is determined by a set of graphs with one line corresponding to the external field and two lines corresponding to the initial and final states of the particle:

$$\mathcal{J} = \quad\text{}\quad . \tag{8}$$

The interaction between the particles in the nucleus is not small. It is therefore not possible to restrict oneself to the calculation of the simplest graph in the determination of $\mathcal{J}$, but one has to sum over all graphs. Sets of graphs can be singled out which do not depend on the frequency of the external field or on the specific distribution of the levels near the Fermi surface. These graphs represent constants which are the same for all transitions and all nuclei.

As noted above, only the pole terms of the single-particle Green's functions are sensitive to

the level structure near the Fermi surface. There-
fore, all graphs which contain only the parts of the
Green's functions without poles can be considered
constant. Besides, many graphs which do contain
pole terms of the Green's functions are also in-
sensitive to the level structure near the Fermi
surface, since their singularities have been in-
tegrated out.

Indeed, let us consider, as part of $\mathscr{T}$, a block
which represents a set of two blocks joined by four
solid lines. For an estimate we may use the pole
terms of the Green's functions for an infinite sys-
tem, $G \sim (\epsilon - \epsilon_p)^{-1}$. Each Green's function in-
troduces one denominator and two integrations
over $d\epsilon\, d\epsilon_p$. The law of conservation of energy-
momentum reduces the total number of integra-
tions by two. As a result we find that the contri-
bution of this block is equal to an integral of the
type $\int \epsilon^{-4} d^6\epsilon$. The region of small ϵ, where the
level structure near the Fermi surface is impor-
tant, makes a small contribution to this integral.
This applies even more strongly to blocks with a
large number of lines.

The blocks sensitive to the Fermi surface are
those which contain only two Green's functions,
and only the pole terms of both need be retained.
Thus we obtain for the vertex part
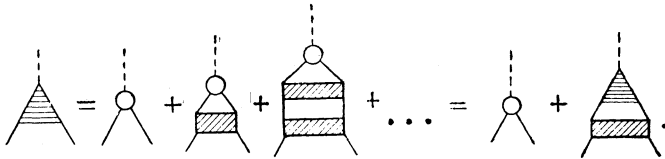
Here the solid lines denote the pole [1]) parts of the
single-particle Green's functions. The quadrangle
represents a block with four external lines which
does not contain parts joined by only two vertical
solid lines corresponding to the pole parts of the
Green's functions. Following Landau,[1] we shall
denote this block by Γ^ω. The circle represents
an analogous set of graphs for the vertex part,
which we denote by $\mathscr{T}^\omega$.

The quantities Γ^ω and $\mathscr{T}^\omega$ do not depend on the
dimensions of the system and are the same for all
transitions. More precisely they have the form
$\Gamma_A^\omega = \Gamma_\infty^\omega (1 + \gamma A^{-1/3})$, where $\gamma \sim 1$ and is identical
for all nuclei.

The quantity $\mathscr{T}^\omega$ satisfies the same relations
as the corresponding quantity for the infinite sys-

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1)More precisely, the solid lines correspond to the pole
parts of two Green's functions with opposite signs of the
imaginary parts.[4] If the signs of the imaginary parts are the
same, there will be an integration over regions far from the
Fermi surface, and such terms are included in the block Γ^ω.

tem. Thus, for a scalar field $\mathscr{T}^\omega$ is in the momen-
tum representation equal to

$$\mathscr{T}^\omega = \mathscr{T}_\infty^\omega = \partial G^{-1}/\partial\varepsilon = a^{-1},$$

i.e., is independent of the momentum of the par-
ticles and the field. Hence, in the coordinate rep-
resentation, $\mathscr{T}^\omega$ has the form

$$\mathscr{T}^\omega (r_1, r_2, r) = a^{-1} \delta (r_1 - r_2) \delta (r - r_1).$$

Going over to the φ_λ representation for r_1 and r_2
and to the momentum representation in the coordi-
nate of the field r, we obtain

$$\mathscr{T}_{\lambda\lambda'}^\omega [e^{iqx}] = a^{-1} (\varphi_\lambda e^{ikr}\varphi_{\lambda'}),$$

where $q = (k, \omega)$ is the wave vector of the field,
and $x = (r, t)$. Here and below the argument of
$\mathscr{T}_{\lambda\lambda'}^\omega [e^{iqx}]$ will contain an operator corresponding
to the "bare" vertex.

Equation (9) takes in the φ_λ representation the
form

$$\mathscr{T}_{\lambda\lambda'} = \mathscr{T}_{\lambda\lambda'}^\omega + \int \frac{d\varepsilon}{2\pi i} \sum_{\lambda_1\lambda_2} \Gamma_{\lambda\lambda'\lambda_1\lambda_2}^\omega G_{\lambda_1} G_{\lambda_2} \mathscr{T}_{\lambda_1\lambda_2}, \qquad (10)$$

where λ and λ' are the initial and final states of
the particles. $\Gamma_{\lambda\lambda'\lambda_1\lambda_2}$ is not small only if λ_1 and
λ_2 differ in their energies by the amount $\epsilon_0 A^{-1/3}$
$\ll \epsilon_0$. For states which are farther away the wave
functions φ_{λ_1} and φ_{λ_2} differ strongly in the num-
ber of nodes, and the matrix element is small.
Therefore (10) contains two close poles in ϵ and
smooth functions of ϵ; thus Γ^ω and $\mathscr{T}$ can be
taken out from under the integral sign in the inte-
gral over ϵ. The integration is then carried out
and one obtains the equation

$$\mathscr{T}_{\lambda\lambda'} = \mathscr{T}_{\lambda\lambda'}^\omega + \sum_{\lambda_1\lambda_2} a^2 \Gamma_{\lambda\lambda'\lambda_1\lambda_2}^\omega \frac{n_{\lambda_1} - n_{\lambda_2}}{\varepsilon_{\lambda_1} - \varepsilon_{\lambda_2} - \omega} \mathscr{T}_{\lambda_1\lambda_2}. \qquad (10')$$

Here

$$n_\lambda = \begin{cases} 1, & \varepsilon_\lambda < \varepsilon_0 \\ 0, & \varepsilon_\lambda > \varepsilon_0 \end{cases};$$

the states λ_1 and λ_2 lie on opposite sides of the
Fermi surface, and since they are close to one
another, they are also close to the Fermi surface.
Therefore the vertex $\mathscr{T}$, in contrast to $\mathscr{T}^\omega$, de-
pends on the level structure of the given nucleus
near the Fermi boundary.

4. INTERACTION BETWEEN QUASIPARTICLES

In the momentum representation, Γ^ω is a
smooth function of p in the region $p \sim p_0$; there-
fore, in the calculation of the matrix element
$\Gamma_{\lambda\lambda'\lambda_1\lambda_2}^\omega = (\varphi_\lambda\varphi_{\lambda'}\Gamma^\omega\varphi_{\lambda_1}\varphi_{\lambda_2})$, where all states

are near the Fermi surface, all moduli of the momenta can be replaced by p_0. Γ^ω will only depend on the angle between the momenta and can be represented in the form of an expansion in spherical harmonics:

$$\Gamma^\omega = \sum_l \Gamma_l^\omega P_l (\mathbf{p}\mathbf{p}_1 / p_0^2).$$

With increasing l the harmonics Γ_l^ω fall off rapidly, and we can restrict ourselves to the first few harmonics. As a result we obtain

$$\Gamma_{\lambda\lambda'\lambda_1\lambda_2}^\omega = \Gamma_0^\omega \int \varphi_\lambda (\mathbf{r}) \varphi_{\lambda'} (\mathbf{r}) \varphi_{\lambda_1} (\mathbf{r}) \varphi_{\lambda_2} (\mathbf{r}) \, d\mathbf{r}$$

$$+ \Gamma_1^\omega p_0^{-2} \int j_{\lambda\lambda'} (\mathbf{r}) j_{\lambda_1\lambda_2} (\mathbf{r}) \, d\mathbf{r} + \ldots,$$

where

$$j_{\lambda\lambda'} = -\tfrac{1}{2} i \left(\varphi_\lambda(\mathbf{r}) \nabla\varphi_{\lambda'}(\mathbf{r}) - \varphi_{\lambda'}(\mathbf{r}) \nabla\varphi_\lambda(\mathbf{r}) \right).$$

The function Γ^ω is a matrix in the spin and isospin variables. The spin-orbit splitting is small in comparison with the Fermi energy, and to this accuracy we can therefore neglect the spin-orbit term in the expression for Γ^ω. Moreover, we can assume with an accuracy $(N-Z)/A$ that Γ^ω must have an invariant form in isospace. Instead of Γ^ω it is convenient to introduce the dimensionless quantity $a^2 \Gamma^\omega dn/d\mu$. As a result we obtain

$$\Gamma^\omega a^2 \, dn/d\mu = f + g\sigma\sigma + f'\tau\tau + g' (\sigma\sigma)(\tau\tau). \quad (11)$$

The quanties f, g, f', and g' must be determined by a comparison of theory and experiment.

We have not included in (11) terms of the form $(\mathbf{p}\sigma)(\mathbf{p}'\sigma')$, which occur as relativistic corrections to the interaction of the nucleons and which, for nucleon energies close to the Fermi energy, are small as compared to the remaining terms of (11). With this understanding, (11) represents the most general expression for the scattering amplitude for small momentum transfer.

Let us show the connection of the constants f, g, f', and g' with the scattering amplitudes for two identical and two nonidentical particles introduced in the paper of one of the authors: [5]

$$f_{nn} = f_{pp} = f + f', \quad f_{np} = f - f',$$
$$g_{nn} = g_{pp} = g + g', \quad g_{np} = g - g'.$$

We note that in the case of β decay, where charge is transferred in the momentum transfer channel, the nondiagonal matrix elements of the operators $\tau \rightarrow \tau_x \pm i\tau_y$ play a role, and only f_{np} and g_{np} enter in the equation for the vertex. The values of the zeroth harmonics f_{nn} and f_{np} are expressed in [5] in terms of the rigidity of nuclear matter against changes of the density and concentration. From the empirical values of these rigidities one obtains

$$(f_{nn})_0 \approx 1, \quad (f_{np})_0 \approx -0.5.$$

5. INTERACTION BETWEEN QUASIPARTICLES VIA THE SURFACE

Besides the local interaction determined by the quantity Γ^ω, there exists another type of interaction between the quasiparticles which has no analog in the infinite system. Indeed, the energy of the deformation of the nuclear surface due to the change in the states of two nucleons is not equal to the sum of the energies due to the change of the state of each nucleon separately. The contribution of this interaction to the blocks containing the sums over graphs is small, since the statistical weight of the surface degrees of freedom is small in comparison with the statistical weight of the single-particle states. This circumstance manifests itself in the fact that the laws of conservation of angular momentum and its projection decrease the number of terms in the sums containing an interaction via the surface as compared to the sums of the single-particle interactions.

The interaction via the surface must be included only in graphs of the form

$$(12)$$
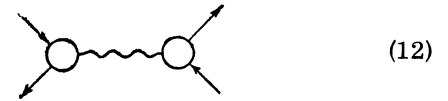
where the wavy line corresponds to a deformation of the surface. Let us consider more closely the interaction between quasiparticles coupled with the quadrupole deformations of the nucleus. We shall give a perspicuous but not rigorous proof that the interaction via the surface in spherical nuclei is comparable with the local interaction, whereas it plays a minor role in deformed nuclei.

The deformation of the nucleus corresponds to a change in the effective potential at the surface of the nucleus. It is convenient to go over to a changed metric of the Hamiltonian by a coordinate transformation which takes the new form of the nucleus back into the old one: x' = ax, y' = ay, z' = bz. This gives an additional term in the Hamiltonian of a single quasiparticle of the form

$$H' = (b-a)(p^2 - 3p_z^2)/3(b+a) \equiv \beta (p^2 - 3p_z^2)/3m^*. \quad (13)$$

Such a modification of the Hamiltonian of a single quasiparticle implies the addition to the Hamiltonian of the system of quasiparticles of a term of the form

$$b^+_{\lambda'}b_\lambda \beta \varepsilon_0 q_{\lambda\lambda'}, \qquad q = \tfrac{2}{3}\, p_0^{-2}\,(p^2 - 3p_z^2),$$

where b_λ and b^+_λ are the annihilation and creation operators for the quasiparticles. If the energy shift for the core is equal to

$$E = C_2\beta^2/2,$$

the complete Hamiltonian for the system of quasiparticles has the form

$$H = \Sigma b^+_\lambda b_\lambda \varepsilon_\lambda + \tfrac{1}{2}\Sigma b^+_{\lambda'}b_{\lambda_2}b^+_{\lambda'}b_{\lambda''}\, a^2\Gamma^+_{\lambda_1\lambda_2\lambda'\lambda''}$$

$$+\,\Sigma b^+_{\lambda'}b_{\lambda''}\,\beta\varepsilon_0 q_{\lambda'\lambda''} + \tfrac{1}{2}C_2\beta^2. \qquad (14)$$

Indeed, it was shown by Landau[1] that the quantity $a^2\Gamma^\omega$ plays the role of a "residual" interaction between the quasiparticles. Let us make the change of variables

$$\beta = \beta' - \frac{1}{C_2}\Sigma b^+_{\lambda'}b_{\lambda''}\,\varepsilon_0\, q_{\lambda'\lambda''}.$$

Then the Hamiltonian is written in the form

$$H = \Sigma b^+_\lambda b_\lambda \varepsilon_\lambda + \tfrac{1}{2}\Sigma b^+_{\lambda'}b_{\lambda''}\, b^+_{\lambda_1}b_{\lambda_2}a^2\Gamma^\omega_{\lambda_1\lambda_2\lambda'\lambda''}$$

$$-\,(\varepsilon_0^2/C_2)\, q_{\lambda_1\lambda_2}q_{\lambda'\lambda''} + \tfrac{1}{2}C_2\beta'^2. \qquad (15)$$

To take account of the interaction via the surface we must therefore replace $a^2\Gamma^\omega$ by $a^2\Gamma^\omega + \Gamma^S_2$ in all expressions, where

$$(\Gamma^S_2)^{\lambda'\lambda''}_{\lambda_1\lambda_2} = -\varkappa_2 q_{\lambda_1\lambda_2}q_{\lambda'\lambda''}, \quad \varkappa_2 = \varepsilon_0^2/C_2. \qquad (16)$$

We have assumed in the derivation of (16) that the transition energies of the participating particles are sufficiently small so that one can use the static expression for the deformation rigidity C_2. This means that the frequencies of the processes considered with the help of (16) must be small compared with $\omega_1 \sim 7$ to 10 MeV. It follows from (16) that in the region of magic nuclei, where C_2 is estimated as $C_2 \sim \varepsilon_0 A$, the quantity $(\Gamma^S_2)^{\lambda'\lambda''}_{\lambda_1\lambda_2}$ is of the same order as $(a^2\Gamma^\omega)^{\lambda'\lambda''}_{\lambda_1\lambda_2}$: $\Gamma^S_2 \sim a^2\Gamma^\omega \sim \varepsilon_0 A^{-1}$.

In an analogous manner one can obtain the interaction via the surface caused by the octupole deformations of the nuclear shape:

$$(\Gamma^S_3)^{\lambda'\lambda''}_{\lambda_1\lambda_2} = -\varkappa v_{\lambda_1\lambda_2}v_{\lambda'\lambda''}, \qquad (17)$$

where $v = 5p_z^3 - 3p_z p_0^2$. The interaction Γ^S_3 is of the same order as Γ^S_2. The interaction Γ^S_2 is added to Γ^ω in the vertices for the quadrupole transitions, whereas Γ^S_3 enters only in the octupole transitions. For monopole, dipole, and magnetic transitions there is no addition to Γ^ω.

We have considered the interaction via the surface in a system of quasiparticles without Cooper pairing. The direct interaction between the quasiparticles found by us depends weakly on the pairing, since it is connected with what is usually called the interaction via the "core" of the nucleus. Since the pairing changes only the pole terms in the Green's functions, it has no noticeable effect on the magnitude of the direct interaction Γ^ω and Γ^S. The pairing affects only the interaction between a given pair of particles mediated by other quasiparticles.

In analogy to this one must also for deformed nuclei first find the direct interaction via the surface without account of pairing and then solve the problem of pairing using the known interaction.

The quantities $\varkappa_2$ and $\varkappa_3$ introduced above are the same for all spherical nuclei in the magic regions, although the rigidity of the nucleus against shape deformations decreases on account of pairing. For deformed nuclei the rigidities C_2 and C_3 against quadrupole and octupole deformations about the equilibrium shape are estimated as $C_{2,3} \sim \varepsilon_0 A^{2/3}$ and can be found from the energy surface given by the Weizsäcker formula.

We show that (16) and (17) are not applicable to deformed nuclei. Let us look at (13) and consider the diagonal matrix element. Then (13) will give the following energy shift of the quasiparticles for a change of the deformation, β, from the equilibrium deformation β_0:

$$\delta\varepsilon_\lambda = \tfrac{1}{3}\beta q_{\lambda\lambda}. \qquad (18)$$

This formula is only correct in the case when the change of the deformation does not lead to a crossing of levels. Since the level spacing in a deformed nucleus is of order ε_0/A, (13) is valid only for small changes of the deformation $\beta \lesssim A^{-1}$. The quantity β given by (14) is of order $\beta \sim A^{-2/3}$ for $C \sim \varepsilon_0 A^{2/3}$.

The character of the change of energy of the quasiparticles for large changes of the deformation ($\beta > A^{-1}$) is determined by the circumstance that at each crossing of levels there occurs a redistribution of the quasiparticles such that all levels below the chemical potential are filled. This fact is illustrated by the figure. The energy shift of the quasiparticles due to a change of the deformation is not determined by (18) but by the sawtooth-shaped curve (heavy line in the figure) such that $\delta\varepsilon_\lambda$ is of the order $\delta\varepsilon_\lambda \lesssim \varepsilon_0 A^{-1}$ even for
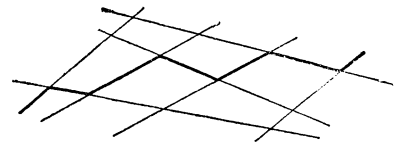
large deformations. The change of the deformation caused by the change of state of a single particle can lead to an increase as well as a decrease of the energy of the other particle depending on whether it is located on a rising or falling slope of the sawtooth-shaped curve. Thus the magnitude and the sign of the interaction via the surface in deformed nuclei depend irregularly on the equilibrium deformation and on the state of the particles. In expressions which contain a summation over the states of one of the particles, as for example in the equation for $\mathscr{T}$, the interaction via the surface makes a small contribution as compared with the local interaction, owing to the irregular variation of the sign.

6. VERTEX IN THE CASE OF PAIRING

The equations for the vertex in the case of pairing in an infinite system have been found earlier.[2] These equations can be taken over to the finite system if the pole terms in the functions G and F are replaced by the expressions (3) for the pole terms of the functions G and F for a finite system. Together with the vertex of the type $\mathscr{T}$ describing the creation of particles and holes by an external field we must, in the case of pairing, also introduce the vertex $\widetilde{\mathscr{T}}$ which describes the creation of two particles (holes) accompanied by the vanishing of a Cooper pair (creation of a Cooper pair). As is seen from the graph,

$$\widetilde{\mathscr{T}} = ---\!\!\!\!\bigcirc\!\!\!\!\!\nwarrow \quad , \tag{19}$$

$\widetilde{\mathscr{T}}$ is directly related to the change of Δ in the external field. It can be verified graphically[2] that $\mathscr{T}$ and $\widetilde{\mathscr{T}}$ are connected by the equations

$$\mathscr{T} = \mathscr{T}^\omega + \Gamma^\omega [GG - FF\hat{P}]\mathscr{T} + \Gamma^\omega [GF - FG]\,\widetilde{\mathscr{T}},$$
$$\widetilde{\mathscr{T}} = \Gamma^\xi [GG\theta_\xi - FF]\,\widetilde{\mathscr{T}} - \Gamma^\xi [GF - FG^-\hat{p}]\mathscr{T}. \tag{20}$$

Here G and F are the pole terms of the Green's functions with pairing given by (3), and

$$\hat{P}\mathscr{T}(p) = \pm \mathscr{T}(-p), \tag{21}$$

where p is the momentum operator. The plus sign corresponds to vertices not containing spin operators, the minus sign to vertices $\sim \sigma$.

To explain the symbolic equation (20) we write one of the terms explicitly:

$$(\Gamma^\omega G_s G_s \mathscr{T})_{\lambda\lambda'} = a^2 \int \frac{d\varepsilon}{2\pi i} \sum_{\lambda_1 \lambda_2} (\varphi_\lambda \varphi_{\lambda'} \Gamma^\omega \varphi_{\lambda_1} \varphi_{\lambda_2})$$
$$\times \frac{\varepsilon + \omega/2 - \varepsilon_{\lambda_1}}{(\varepsilon + \omega/2)^2 - \Delta_{\lambda_1}^2 - \varepsilon_{\lambda_1}^2} \frac{-\varepsilon - \omega/2 - \varepsilon_{\lambda_2}}{(\varepsilon - \omega/2)^2 - \Delta_{\lambda_2}^2 - \varepsilon_{\lambda_2}^2} \mathscr{T}_{\lambda_1 \lambda_2}. \tag{22}$$

As in a system without pairing we can carry out the integration over $d\varepsilon'$ leaving under the integral sign only the rapidly varying functions of ε': GG, FF, and FG and evaluating the weakly ε' dependent terms Γ^ω, $\mathscr{T}$, and $\widetilde{\mathscr{T}}$ in the point $\varepsilon = \mu$.

We give a table of the corresponding integrals which can easily be calculated in the complex plane:

$$\int G_{\lambda_1}\left(\varepsilon + \frac{\omega}{2}\right) G_{\lambda_2}\left(\varepsilon - \frac{\omega}{2}\right)\frac{d\varepsilon}{2\pi i}$$
$$= -\left\{\frac{v_{\lambda_1}(1 - v_{\lambda_2})}{E_{\lambda_1} + E_{\lambda_2} - \omega} + \frac{v_{\lambda_2}(1 - v_{\lambda_1})}{E_{\lambda_1} + E_{\lambda_2} + \omega}\right\},$$

$$\int F_{\lambda_1}\left(\varepsilon + \frac{\omega}{2}\right) F_{\lambda_2}\left(\varepsilon - \frac{\omega}{2}\right)\frac{d\varepsilon}{2\pi i}$$
$$= -\left\{\frac{\Delta_{\lambda_1}\Delta_{\lambda_2}}{E_{\lambda_1} + E_{\lambda_2} - \omega} + \frac{\Delta_{\lambda_1}\Delta_{\lambda_2}}{E_{\lambda_1} + E_{\lambda_2} + \omega}\right\}\frac{1}{4E_{\lambda_1}E_{\lambda_2}}$$

$$\int G_{\lambda_1}\left(\varepsilon + \frac{\omega}{2}\right) F_{\lambda_2}\left(\varepsilon - \frac{\omega}{2}\right)\frac{d\varepsilon}{2\pi i}$$
$$= -\left\{\frac{1 - v_{\lambda_1}}{E_{\lambda_1} + E_{\lambda_2} - \omega} + \frac{v_{\lambda_1}}{E_{\lambda_1} + E_{\lambda_2} + \omega}\right\}\frac{\Delta_{\lambda_2}}{2E_{\lambda_2}},$$

$$\int G_{\lambda_1}\left(\varepsilon + \frac{\omega}{2}\right) G_{\lambda_2}^-\left(\varepsilon - \frac{\omega}{2}\right)\frac{d\varepsilon}{2\pi i}$$
$$= \left\{\frac{(1 - v_{\lambda_1})(1 - v_{\lambda_2})}{E_{\lambda_1} + E_{\lambda_2} - \omega} + \frac{v_{\lambda_1}v_{\lambda_2}}{E_{\lambda_1} + E_{\lambda_2} + \omega}\right\},$$

$$\int F_{\lambda_1}\left(\varepsilon + \frac{\omega}{2}\right) G_{\lambda_2}^-\left(\varepsilon - \frac{\omega}{2}\right)\frac{d\varepsilon}{2\pi i}$$
$$= -\left\{\frac{1 + v_{\lambda_2}}{E_{\lambda_1} + E_{\lambda_2} - \omega} - \frac{v_{\lambda_2}}{E_{\lambda_1} + E_{\lambda_2} + \omega}\right\}\frac{\Delta_{\lambda_1}}{2E_{\lambda_1}}. \tag{23}$$

Here $v_\lambda = (E_\lambda - \epsilon_\lambda)/2E_\lambda$, $E_\lambda^2 = \Delta_\lambda^2 + \epsilon_\lambda^2$. The integrals $\int FG\,d\epsilon$ and $\int G^-F\,d\epsilon$ are obtained from the ones quoted by making the replacements $\lambda_1 \leftrightarrow \lambda_2$, $\omega \rightarrow -\omega$.

$\mathscr{T}_{\lambda_1 \lambda_2}$ depends on the form of the external field and its frequency ω. We shall write in the argument of $\mathscr{T}$ and $\widetilde{\mathscr{T}}$ the bare vertex which determines the character of the external field. For example, a rotational (or magnetic) field introduces in the Hamiltonian of the system the term*

$$-\sum_{\lambda\lambda'} a_\lambda^+ a_{\lambda'}\,[\mathbf{rp}]_{\lambda\lambda'}\,\Omega(t),$$

so that the bare vertex is characterized by the operator $\mathbf{r} \times \mathbf{p}$. The corresponding exact vertex will be written in the form $\mathscr{T}[\mathbf{r} \times \mathbf{p}]$. The effect of the operator $\hat{P}$ is a change of sign in front of such a vertex. The vertex for a field with uniform strength has the form $\mathscr{T}_{\lambda\lambda'}[\mathbf{r}]$ and $\hat{P}\mathscr{T}_{\lambda\lambda'}[\mathbf{r}] = \mathscr{T}_{\lambda\lambda'}[\mathbf{r}]$.

The summation over λ_1 and λ_2 implied in (20) can be simplified by using the known properties of the quasiclassical matrix elements: only the diagonal elements or the elements corresponding to

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*$[\mathbf{rp}] = \mathbf{r} \times \mathbf{p}$.

an energy difference $|\epsilon_{\lambda_1} - \epsilon_{\lambda_2}| \sim \epsilon_0 A^{-1/3}$ are appreciably different from zero. As we have noted, the only exception are the matrix elements of the angular momentum operator in a deformed nucleus where states with close-lying energies are combined. This case will be considered later.

We convince ourselves that the pairing, as a rule, is significant only for the diagonal matrix elements. First of all, we see from (23) that only the diagonal elements in the terms of the form FF and FG in (20) must be retained. Indeed, the nondiagonal elements contain at least one large denominator ($\sim \epsilon_0 A^{-1/3}$). On the other hand, it is easily seen that the nondiagonal elements in the terms of the form GG can be replaced by their values without pairing. As a result we obtain instead of (20)

$$\mathcal{T}_{\lambda_1\lambda_2} = \mathcal{T}_{\lambda_1\lambda_2}^{\omega} + \sum_{\lambda'\lambda''} \Gamma_{\lambda_1\lambda_2\lambda'\lambda''}^{\omega} \frac{n_{\lambda'} - n_{\lambda''}}{\epsilon_{\lambda'} - \epsilon_{\lambda''} + \omega} \mathcal{T}_{\lambda'\lambda''} + \sum_{\lambda'} \Gamma_{\lambda_1\lambda_2\lambda'\lambda'}^{\omega}$$

$$\times \left\{ \frac{\Delta_{\lambda'}^2}{E_{\lambda_1}} \frac{1 + \hat{P}}{4E_{\lambda'}^2 - \omega^2} \mathcal{T}_{\lambda'\lambda'} - \frac{\Delta_{\lambda'}\omega}{E_{\lambda'}(4E_{\lambda'}^2 - \omega^2)} \widetilde{\mathcal{T}}_{\lambda'\lambda'} \right\},$$

$$\widetilde{\mathcal{T}}_{\lambda_1\lambda_2} = \sum_{\lambda'\lambda''} \Gamma_{\lambda_1\lambda_2\lambda'\lambda''}^{\xi}$$

$$\times \left\{ \left[\ln\frac{2\xi}{\Delta} + \frac{\omega^2}{4E_{\lambda'}(4E_{\lambda'}^2 - \omega^2]} \right] \widetilde{\mathcal{T}}_{\lambda'\lambda'} - \frac{\Delta_{\lambda'}\omega(1 + \hat{P})}{E_{\lambda'}(4E_{\lambda'}^2 - \omega^2)} \mathcal{T}_{\lambda'\lambda'} \right\}. \quad (24)$$

From (24) we obtain immediately the important result: the Cooper pairing does not affect the vertices without diagonal elements, i.e., the matrix elements of the dipole and octupole transitions can be computed without taking pairing into account, using (10'), since $\mathcal{T}_{\lambda\lambda} = 0$ and hence $\widetilde{\mathcal{T}}_{\lambda\lambda} = 0$ for such transitions and (24) goes over into (10'). The same result is obtained for the spinor vertex $\mathcal{T}[\sigma_\alpha]$. The quantity $\mathcal{T}[\sigma_\alpha]$ drops out of the second Eq. (24), as can be easily seen, and hence $\widetilde{\mathcal{T}}[\sigma_\alpha] = 0$. Since $(1 + \hat{P})\mathcal{T}[\sigma_\alpha] = 0$, (24) goes over into (10'). If a given nucleus contains close-lying combining levels, additional terms will appear in (24).

As mentioned earlier, the vertices of the form $\mathbf{r} \times \mathbf{p}$ in deformed nuclei contain nondiagonal matrix elements with energies $|\epsilon_{\lambda'} - \epsilon_{\lambda''}| \sim \beta_0 A^{1/3} \sim \Delta$. However, in the case of deformed nuclei the expression (20) can be simplified without discarding the nondiagonal matrix elements in the terms of the form FF and GF, using the fact that several levels fit into the width Δ and that values of $\epsilon_{\lambda'}$ and $\epsilon_{\lambda''}$ close to the Fermi surface are important in terms of this type. This allows us to carry out the summation over one of the indices λ' or λ''. An analogous calculation has been carried out in [6] (see also [7]). We quote the resulting equations:

$$\mathcal{T}_{\lambda_1\lambda_2} = \mathcal{T}_{\lambda_1\lambda_2}^{\omega}$$

$$+ a^2 \frac{dn}{d\mu} \sum_{\lambda'\lambda''} (\lambda_1\lambda_2 | \Gamma^\omega | \lambda'\lambda'') \{ L_{\lambda'\lambda''} \mathcal{T}_{\lambda'\lambda''} + M_{\lambda'\lambda''} \widetilde{\mathcal{T}}_{\lambda'\lambda''} \},$$

$$\widetilde{\mathcal{T}} = a^2 \frac{dn}{d\mu} \sum_{\lambda'\lambda''} (\lambda_1\lambda_2 | \Gamma^\omega | \lambda'\lambda'')$$

$$\times \left\{ \left[\ln\frac{2\xi}{\Delta} + N_{\lambda'\lambda''} \right] \widetilde{\mathcal{T}}_{\lambda'\lambda''} + O_{\lambda'\lambda''} \mathcal{T}_{\lambda'\lambda''} \right\}. \quad (25)$$

Here

$$L_{\lambda'\lambda''} = \frac{\epsilon_{\lambda'} - \epsilon_{\lambda''}}{\epsilon_{\lambda'} - \epsilon_{\lambda''} + \omega} (1 - g(x)) - g(x) \frac{1 + \hat{P}}{2},$$

$$M_{\lambda'\lambda''} = \frac{\omega + \epsilon_{\lambda'} - \epsilon_{\lambda''}}{2\Delta} g(x), \quad N_{\lambda'\lambda''} = \frac{\omega^2 - (\epsilon_{\lambda'} - \epsilon_{\lambda''})}{4\Delta^2} g(x)$$

$$Q_{\lambda'\lambda''} = \left(\frac{\omega + \epsilon_{\lambda'} - \epsilon_{\lambda''}}{4\Delta} + \frac{\omega - \epsilon_{\lambda'} + \epsilon_{\lambda''}}{4\Delta} \hat{P} \right) g(x)$$

$$(26)$$

The function $g(x)$ depends on the argument $x = \frac{1}{2} \Delta^{-1} \sqrt{(\epsilon_{\lambda'} - \epsilon_{\lambda''})^2 - \omega^2}$ and is equal to

$$g(x) = x^{-1}(1 + x^2)^{-1/2} \text{ arc sh } x. \quad (27)*$$

Owing to the fact that $g(x)$ falls off like x^{-2} as x increases, the summation in (26) goes over states close to the Fermi surface in a layer of a thickness of order Δ. Vertices which do not contain matrix elements with close-lying energies are not affected by the pairing [as in (24)].

Owing to the centrifugal potential, the zeroth harmonic plays the most important role in Γ^ξ. One easily obtains from the second Eq. (24)

$$\sum_{\lambda'\lambda''} (N_{\lambda'\lambda''}\widetilde{\mathcal{T}}_{\lambda'\lambda'} + O_{\lambda'\lambda''}\mathcal{T}_{\lambda'\lambda''})\varphi_{\lambda'}^*(\mathbf{r})\varphi_{\lambda''}(\mathbf{r}) = 0. \quad (28)$$

The error of this relation is of order $\Gamma_1^\xi/3\Gamma_0^\xi \times \ln(\epsilon_0/\Delta)$, where Γ_0^ξ and Γ_1^ξ are the coefficients of the zeroth and first harmonic in the expansion of Γ^ξ in spherical harmonics. Equation (28) is equivalent to (17) and (33) of [6] which connect the change of Δ in the external field with the matrix elements of this field.

Equations (24) [or, for deformed nuclei, also (25)] can be used for the determination of the frequencies and the intensities of collective transitions in nuclei. A collective excitation corresponds to a pole in the vertex $\mathcal{T}$ with the appropriate symmetry. The residues at these poles are simply connected with the probability for the excitation of the collective level.

It follows from this that the collective excitations of, for example, dipole type have the same frequency as in a system without pairing. In general, the pairing has no effect on the frequencies of the collective vibrations if the field associated with the vibration has no large matrix elements with energies of order Δ.

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*sh = sinh

7. GAUGE INVARIANCE

The requirement of gauge invariance leads to a relation between the vertices for the vector and scalar fields. If a field

$$A_i = (A_\alpha, \varphi) = A_i (q) e^{iqx}$$

is applied to the nucleus, the Hamiltonian of the system is changed to

$$H' = \sum_{\lambda\lambda'} a_\lambda^+ a_\lambda (e^{iqx} p_i)_{\lambda'\lambda} A_i (q).$$

Thus the "bare" vertex is

$$\mathscr{T}^0_{\lambda\lambda'} [e^{iqx} p_i] = (e^{iqx} p_i)_{\lambda\lambda'}.$$

It follows from the graphic definition of the vertex $\mathscr{T}$ that it is connected with the change of the Green's function in a weak external field by the relation

$$(\delta G^{-1})_{\lambda\lambda'} = \mathscr{T}_{\lambda\lambda'} [e^{iqx} p_i] A_i (q).$$

The argument of the exact vertex $\mathscr{T}_{\lambda\lambda'}$ contains the operator of the corresponding "bare" vertex.

On the other hand, if the fictitious field

$$A_i = \partial f/\partial x_i, \qquad A_i (q) = q_i f (q^2),$$

is applied, the system suffers no physical changes whatsoever, and the Green's function is altered only on account of the gauge transformation of the quantum operators ψ: $\psi' = e^{if}\psi$. It is easy to see that

$$(\delta G^{-1})_{\lambda\lambda'} = \mathscr{T}_{\lambda\lambda'} [e^{iqx} p_i] q_i$$

$$= \sum_{\lambda_1} \left\{ (e^{ikr})_{\lambda\lambda_1} G^{-1}_{\lambda_1\lambda'} \left(\varepsilon + \frac{\omega}{2}\right) - G^{-1}_{\lambda\lambda_1} \left(\varepsilon - \frac{\omega}{2}\right) (e^{ikr})_{\lambda_1\lambda'} \right\}. \tag{29}$$

If the quantity ε is close to the Fermi energy and $\omega_0 \ll \epsilon_0$, then $G^{-1}_{\lambda\lambda_1}$ is diagonal and (2) gives together with (29)

$$\mathscr{T}_{\lambda\lambda'} [e^{iqx} p_i] q_i = (e^{ikr})_{\lambda\lambda'} (\omega + \varepsilon_\lambda - \varepsilon_{\lambda'})/a. \tag{30}$$

We note that the transition amplitude in the fictitious field $A_i = q_i f$ vanishes for $\omega = \epsilon_{\lambda'} - \epsilon_\lambda$ and has no poles; this should be expected, since no real processes are involved.

In the case $k \ll k^{-1}$ we expand the exponent and obtain

$$- \mathscr{T}_{\lambda\lambda'} [p_\alpha] + \omega \mathscr{T}_{\lambda\lambda'} [r_\alpha] = (r_\alpha)_{\lambda\lambda'} (\omega + \varepsilon_\lambda - \varepsilon_{\lambda'})/a. \tag{31}$$

Keeping the next term in the expansion, one may also get

$$- \frac{1}{2} \mathscr{T}_{\lambda\lambda'} [p_\alpha r_\beta + p_\beta r_\alpha] + \omega \mathscr{T}_{\lambda\lambda'} [r_\alpha r_\beta]$$

$$= (r_\alpha r_\beta)_{\lambda\lambda'} (\omega + \varepsilon_\lambda - \varepsilon_{\lambda'})/a. \tag{32}$$

For $\omega = \epsilon_{\lambda'} - \epsilon_\lambda$

$$\mathscr{T}_{\lambda\lambda'} [p_\alpha] = (\varepsilon_{\lambda'} - \varepsilon_\lambda) \mathscr{T}_{\lambda\lambda'} [r_\alpha], \tag{33}$$

which allows one to compute the scalar vertex instead of the corresponding vector vertex.

The requirement of gauge invariance as formulated above is valid for the proton as well as neutron Green's functions. Therefore, the above-given relations apply to the proton as well as neutron vertices with arbitrary ratio of the neutron and proton "charges" with regard to the field.

It follows from (30) that the scalar vertex has for $\omega \gg | \epsilon_{\lambda_1} - \epsilon_{\lambda_2} |$ the form

$$\mathscr{T}_{\lambda_1\lambda_2} |_{\omega \gg | \varepsilon_{\lambda_1} - \varepsilon_{\lambda_2}|} = \mathscr{T}^\omega_{\lambda_1\lambda_2} [e^{iqx}] = (e^{ikr})_{\lambda_1\lambda_2} a^{-1}. \tag{34}$$

Since the quantity $a\mathscr{T}^\omega$ enters in the transition probabilities, this result can be interpreted as the nonrenormalizability of the scalar vertices for frequencies $\epsilon_0 \gg \omega \gg | \epsilon_{\lambda_1} - \epsilon_{\lambda_2} |$.

As to the case of pairing, it was shown in [2] that the condition of gauge invariance takes the form

$$q_i \partial G/\partial p_i = (GG - FF) \mathscr{T}_i q_i + (GF - FG) \widetilde{\mathscr{T}}_i q_i, \tag{35}$$

from where one easily obtains a relation analogous to the one found in [2] for an infinite system:

$$\widetilde{\mathscr{T}}_{\lambda\lambda'} [p_i] q_i = 2\Delta_{\lambda\lambda'}/a. \tag{36}$$

This relation is very useful as a check of calculations.

8. TRANSITION PROBABILITIES

The knowledge of the vertex parts permits a quantitative computation of the rates of electromagnetic and β transitions. The operator of the interaction of the nucleon with an external field has the form

$$\hat{A} = \sum_{\lambda\lambda'} a_\lambda^+ a_{\lambda'} A_{\lambda\lambda'}, \tag{37}$$

where $A_{\lambda\lambda'}$ is the unperturbed vertex. For a scalar electromagnetic field $A_{\lambda\lambda'} = (e^{ikr})_{\lambda\lambda'}$, for a vector field $A_{\lambda\lambda'} = (-i\partial e^{ikr}/\partial r)_{\lambda\lambda'}$ and for the β decay $A \sim \tau^x \pm i\tau^y$, where τ are the isotopic spin matrices.

The transition probability corresponding to the interaction (37) is found by the usual formula

$$W = 2\pi \sum_s | (A)_{0s} |^2 \delta (E_s - \omega), \tag{38}$$

where ω is the frequency of the external field. The summation goes over all exact states of the nucleus with energy $E_s = \omega$. For definiteness we consider the rate of absorption of a nucleus in the ground state.

Let us convince ourselves that this rate is proportional to the imaginary part of the Fourier component of the polarization operator

$$\mathscr{P} = \langle T\hat{A}(t_1)\hat{A}(t_2)\rangle. \qquad (39)$$

Here $\langle \ldots \rangle$ denotes the expectation value in the ground state of the nucleus; the operators $\hat{A}(t)$ are given in terms of the interaction operator (37) and the complete Hamiltonian of the system, H:

$$\hat{A}(t) = e^{-iHt}\hat{A}e^{iHt}; \qquad (40)$$

the symbol T is the time ordering operator:

$$\mathscr{P} = \begin{cases} \langle \hat{A}(t_1)\hat{A}(t_2)\rangle \\ \langle \hat{A}(t_2)\hat{A}(t_1)\rangle \end{cases} = \sum_s |(A)_{0s}|^2 \begin{cases} \exp[iE_s(t_1-t_2)], \ t_1 > t_2 \\ \exp[iE_s(t_2-t_1)], \ t_2 > t_1 \end{cases}.$$

Making a Fourier transformation in the variable $t_1 - t_2$, we obtain

$$\mathscr{P}(\omega) = \sum_s |\hat{A}_{0s}|^2 \left(\frac{1}{E_s - \omega - i\delta} + \frac{1}{E_s + \omega + i\delta}\right). \qquad (41)$$

Using $\text{Im}(E_s - \omega - i\delta)^{-1} = \pi\delta(\omega - E_s)$ and comparing (38) and (41), we find for the transition rate

$$W = 2\,\text{Im}\,\mathscr{P}(\omega). \qquad (42)$$

The polarization operator is given graphically by the set of all graphs with four external lines whose ends are joined pairwise in one point:

$$\mathscr{P} = \text{} . \qquad (43)$$

From this graphic definition the connection of this operator with the vertex is

$$\mathscr{P} = \mathscr{T}_0 GG\mathscr{T}. \qquad (44)$$

If the vertex has a pole corresponding to collective excitations, the polarization operator will have an imaginary part proportional to the residue at this pole. Formula (42) then gives the transition probability to the collective level.

For single-particle transitions the formula for the transition rate can be written in a simpler form directly in terms of the vertex. It can be shown that (44) and Eq. (10) for the vertex lead to the following expression for the imaginary part of the polarization operator in a system without pairing:

$$\text{Im}\,\mathscr{P} = \text{Im} \sum \mathscr{T}_{\lambda\lambda'} \frac{n_\lambda - n_{\lambda'}}{\omega + \varepsilon_\lambda - \varepsilon_{\lambda'}} \mathscr{T}_{\lambda'\lambda}. \qquad (45)$$

From this we obtain for the transition probability with creation of a particle in state λ_2 and a hole in state λ_1

$$W = 2\pi |a\mathscr{T}_{\lambda_1\lambda_2}|^2 n_{\lambda_1}(1 - n_{\lambda_2})\delta(\omega - E_{\lambda_1\lambda_2}). \qquad (46)$$

The quantity $a\mathscr{T}$ plays the role of a transition amplitude for the quasiparticle. An analogous formula can be derived for a system with pairing.

The amplitude for a single-particle transition can also be found by a different method using the fact that the matrix element for a single-particle transition differs from the corresponding vertex only by factors characterizing the admixture of the particle in the quasiparticle state and given by the matrix elements $\langle s|a_\lambda^+|0\rangle$ and $\langle s|a_\lambda|0\rangle$. This matrix element is equal to the root of the coefficient of the pole term of the single-particle Green's function corresponding to the single-quasiparticle state.

Indeed, we have for $G_{\lambda\lambda'}(\varepsilon)$, in analogy to (41),

$$G_{\lambda\lambda'}(\varepsilon) = \sum_s \left\{\frac{(a_\lambda)_{0s}(a_{\lambda'}^+)_{s0}}{\varepsilon - E_{s0} + i\delta} + \frac{(a_{\lambda'}^+)_{0s}(a_\lambda)_{s0}}{\varepsilon - E_{s0} - i\delta}\right\}. \qquad (47)$$

The state s corresponding to the quasiparticle λ' or the quasihole λ gives rise to an isolated pole of the Green's function $G_{\lambda\lambda'}(\varepsilon)$. Comparing with (2), we find

$$|\langle s|a_\lambda|0\rangle|^2 = an_\lambda, \qquad |\langle s|a_\lambda^+|0\rangle|^2 = a(1 - n_\lambda).$$

Thus the transition amplitude becomes

$$A_{0s} = \mathscr{T}_{\lambda_1\lambda_2}\langle s|a_{\lambda_1}|0\rangle\langle s|a_{\lambda_2}^+|0\rangle = a\mathscr{T}_{\lambda_1\lambda_2}\sqrt{n_{\lambda_1}(1 - n_{\lambda_2})},$$

which leads to (46).

In the case of pairing the state with a single quasiparticle can be found by acting on the ground state with the operator a_λ^+ as well as the operator $a_{-\lambda}$:

$$\alpha_\lambda^+ = ua_\lambda^+ va_{-\lambda}. \qquad (48)$$

Let us first consider the case when the paired particle makes the transition. Then the residue at the pole of G for the particle and the hole is given by the two formulas (3):

$$|\langle\lambda|a_\lambda^+|0\rangle|^2 = a(E_\lambda + \varepsilon_\lambda)/2E_\lambda = au^2,$$
$$|\langle\lambda|a_{-\lambda}|0\rangle|^2 = a(E_\lambda - \varepsilon_\lambda)/2E_\lambda = av^2. \qquad (49)$$

This corresponds to Bogolyubov's description in terms of the quasiparticle creation operator $\alpha_\lambda^+ = ua_\lambda^+ + va_{-\lambda}$. For the matrix element we obtain

$$A_{0s} = a[\mathscr{T}_{\lambda_1\lambda_2}v_{\lambda_1}u_{\lambda_2} + \mathscr{T}_{-\lambda_2,-\lambda_1}u_{\lambda_1}v_{\lambda_2} + \tilde{\mathscr{T}}_{\lambda_1\lambda_2}(v_{\lambda_1}v_{\lambda_2} - u_{\lambda_1}u_{\lambda_2})]. \qquad (50)$$

The transition energy is equal to $E_{\lambda_1} + E_{\lambda_2}$. In odd nuclei transitions with energies less than 2Δ are possible if they are effected by the odd particle originally in the state λ_0. To determine the matrix elements of the operators a, a^+ one must use the Green's functions for an odd nucleus.[8] The matrix element for such a transition is equal to

$$A_{0s} = a[\mathscr{T}_{\lambda_0\lambda}u_{\lambda_0}u_\lambda + \mathscr{T}_{-\lambda,-\lambda_0}v_{\lambda_0}v_\lambda + \tilde{\mathscr{T}}_{\lambda_0\lambda}(u_{\lambda_0}v_\lambda - v_{\lambda_0}u_\lambda). \qquad (51)$$

For a weak interaction, when $\tilde{\mathscr{T}} = 0$ and $\mathscr{T}$ coincides with the bare vertex, formula (51) has been obtained by Grin'[9] and Urin.[10]

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